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On some rotundity and smoothness properties of Banach spaces

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CONTENTS

Introduction

Following A. L. Garkavi ([16], p. 97), we will call a real normed linear space X uniformly rotund in the direction $z \in X$, $\|z\| = 1$ if $x_n, y_n \in X$, $\|x_n\| = \|y_n\| = 1$, $x_n - y_n = \lambda_n z$, λ_n real, $\|\frac{1}{2}(x_n + y_n)\| \to 1$ imply $\lambda_n \to 0$. This notion is useful in the theory of Chebyshev sets, Chebyshev center [16] and may be used in the theory of fixed points, too [45].

In Chapter 1 of this paper we derive some necessary and sufficient conditions for this property and a characterization of such directions in some spaces.

In Chapter 2 we investigate a dual property and a connection to Chebyshev property and to a question of norm-preserving extensions of linear functionals.

In Chapter 3 we consider some equivalent renorming results concerning this notion.

Chapter 4 contains some remarks concerning Banach spaces with uniformly Gâteaux differentiable norm and some counterexamples to various notions of rotundity.

Chapter 5 contains an application of the theory of spaces uniformly rotund in each directions to a nonlinear fixed point problem.

In Chapter 6 there is proved a uniformization of one Mazur's theorem concerning the separation of a point and a convex set by a ball.

We consider only normed linear spaces over the reals. $x_n \to x$ respectively $x_n \xrightarrow{w} x$ in X or $f_n \xrightarrow{w^*} f$ in X^* (the dual space of X) mean norm respectively weak convergence in X or pointwise convergence in X^*.

N denotes the set of all positive integers, R the set of all real numbers. For $r > 0$,

$$S_r = \{x \in X;\ \|x\| = r\}, \qquad K_r = \{x \in X;\ \|x\| \leqslant r\}.$$

S_r^* and K_r^* are defined analogously in X^*. For $K \subset X$, $\delta_x(K)$ denotes the norm boundary of K. If $x, y \in X$, $x \neq y$, (x, y) respectively $[x, y]$ denote the line through x, y respectively the closed segment between x, y. By a subspace of a Banach space X we always mean a nonempty closed linear subspace of X with the norm induced from X. For a subspace $P \subset X$, the deficiency of P means the dimension of X/P, $P^{\perp}$ means the set $\{f \in X^*; f(x) = 0 \text{ for } x \in P\}$, for convex $K \subset X$, $\operatorname{ext} K$ denotes the set of all extreme points of K. Rx is one dimensional subspace generated by $x \in X$, $x \neq 0$. Under isomorphism T of X, Y/X, Y are normed linear spaces we mean a linear continuous one-to-one mapping of X onto Y such that T^{-1} is continuous.

CHAPTER 1

Uniformly rotund directions

First, we will need a simple geometrical lemma:

LEMMA 1. *Let X be a normed linear space. Suppose $x_n, y_n \in S_1 \subset X$, $z_n \in [x_n, y_n]$ such that $\|z_n\| \to 1$ and there exists $\delta > 0$ such that* $\min(\|x_n - z_n\|, \|y_n - z_n\|) > \delta$, $n \in N$. *Than* $\inf_{t \in \langle 0,1 \rangle} \|tx_n + (1-t)y_n\| \to 1$ *as* $n \to \infty$.

Proof. There exist $\delta_1 > 0$, $\delta_2 > 0$ such that $z_n = t_n x_n + (1-t_n) y_n$ where $0 < \delta_1 \leqslant t_n \leqslant \delta_2 < 1$ for every $n \in N$. For every $n \in N$ choose $f_n \in X^*$ such that $f_n(z_n) = \|z_n\|$, $\|f_n\| = 1$. Then $f_n(z_n) = t_n f_n(x_n) + (1-t_n) f_n(y_n)$. Hence for every $\varepsilon > 0$ there exists $n_0' \in N$ such that for $n \geqslant n_0'$, $n \in N$

$$f_n(x_n) = \frac{f_n(z_n) - (1-t_n) f_n(y_n)}{t_n} \geqslant 1 - \frac{\varepsilon}{t_n} \geqslant 1 - \frac{\varepsilon}{\delta_1}$$

and similarly

$$f_n(y_n) \geqslant \frac{1 - \varepsilon - t_n}{1 - t_n} \geqslant 1 - \frac{\varepsilon}{1 - \delta_2}.$$

Therefore, for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for $n \in N$, $n \geqslant n_0$, $\min(f_n(x_n), f_n(y_n)) \geqslant 1 - \varepsilon$. Consider an arbitrary $n \geqslant n_0$ and $t \in [x_n, y_n]$. Then $t = a x_n + (1-a) y_n$ for some $a \in \langle 0, 1 \rangle$, and

$$f_n(t) \geqslant a(1-\varepsilon) + (1-a)(1-\varepsilon) = 1 - \varepsilon.$$

Therefore $\|t\| \geqslant f_n(t) \geqslant 1 - \varepsilon$.

PROPOSITION 1. *Let $z \in S_1 \subset X$, X is a Banach space, $c > 0$ be an arbitrary fixed number.*

Then the following conditions are equivalent:

(1) *X is uniformly rotund in the direction z;*

(2) $x_n, y_n \in S_1$, $x_n - y_n = \lambda z$, $0 \leqslant \lambda \leqslant c$, $\|\frac{1}{2}(x_n + y_n)\| \to 1$ *imply* $\lambda = 0$;

(3) $x_n, y_n \in K_1$, $x_n - y_n = \lambda_n z$, $\|\frac{1}{2}(x_n + y_n)\| \to 1$ *imply* $\lambda_n \to 0$;

(4) $x_n, y_n \in X$, $\|x_n\| - \|y_n\| \to 0$, $\{x_n\}$ *bounded*, $x_n - y_n = \lambda_n z$, $\|x_n\| + \|y_n\| - \|x_n + y_n\| \to 0$ *imply* $\lambda_n \to 0$;

(5) $\|x_n\| \to 1$, $\|x_n + \lambda_n z\| \to 1$, $\|x_n + 2\lambda_n z\| \to 1$ *imply* $\lambda_n \to 0$;

(6) *there exists no $\{x_n\}$ bounded, such that* $\|x_n + z\| - \|x_n\| \to 0$, $\|x_n + z\| + \|x_n\| - \|2x_n + z\| \to 0$;

(7) *there exists no* $\{x_n\}$ *bounded, such that* $2(\|x_n+z\|^2+\|x_n\|^2)-\|2x_n+ +z\|^2\to 0$;

(8) *for every* $\varepsilon>0$

$$\inf_{\substack{x,y\in S_1\\ x-y\in Rz\\ \|x-y\|\geqslant\varepsilon}}\big(1-\|\tfrac{1}{2}(x+y)\|\big)>0;$$

(9) *for every* $\varepsilon>0$

$$\inf_{\substack{x,y\in S_1\\ x-y\in Rz\\ \|x-y\|\geqslant\varepsilon}}\big(1-\inf_{t\in\langle 0,1\rangle}\|tx+(1-t)y\|\big)>0;$$

(10) *for every* $\varepsilon>0$

$$\inf_{\substack{x,y\in S_1\\ x-y=\varepsilon z}}\left(1-\left\|\frac{x+y}{2}\right\|\right)>0;$$

(11) *for every* $\varepsilon>0$

$$\inf_{\substack{x,y\in S_1\\ x-y=\varepsilon z}}\big(1-\inf_{t\in\langle 0,1\rangle}\|tx+(1-t)y\|\big)>0.$$

Proof. (1) implies (2): obvious.

(2) implies (1): Suppose (1) is not satisfied. Then there exist x_n, $y_n\in S_1$, $x_n-y_n=\lambda_n z$, $|\lambda_n|\geqslant\delta>0$, $\|x_n+y_n\|\to 2$. By Lemma 1 $\inf_{t\in[x_n,y_n]}\|t\| =1-\varepsilon_n\to 1$. Therefore for $n\geqslant n_0$, $n\in N$, $1-\varepsilon_n>\frac{1}{2}$. Take $n\geqslant n_0$ an arbitrary number. If $0\in(x_n,y_n)$, $0\notin[x_n,y_n]$ we would obtain a simple contradiction with convexity since $x_n,y_n\in S_1$ (see for instance Th.1.10b, of [43]). If $0\in[x_n,y_n]$ then $\|0\|>\frac{1}{2}$ — a contradiction. Therefore we may denote by P^n the plane through $0, x_n, y_n$ in X and $K^n=K_1\cap P^n$. Then $\delta_{P^n}(K^n)=S_1\cap P^n$ — which is easily seen by a characterization of boundary points of a convex body through the values of its Minkowski functional. By our assumptions $\|x_n-y_n\|\geqslant\min(\delta,c)$. Denote $\min(\delta,c) =c^1$. If $[x_n,y_n]\subset S_1$ then we may simply choose $\mathring{x}_n,\mathring{y}_n\in[x_n,y_n]$ so that $\mathring{x}_n-\mathring{y}_n=c^1z$ and $\|t\mathring{x}_n+(1-t)\mathring{y}_n\|=1$ for all $t\in\langle 0,1\rangle$. Now suppose $[x_n,y_n]$ contains some interior point of K^n. Denote H the closed half-space determined by (x_n,y_n) and $0\notin H$. Define $\tilde{K}^n=H\cap K^n$. Then $\tilde{K}^n$ is a compact convex body. There exists a supporting line $l\neq(x,y)$ in P^n to $\tilde{K}^n$ in the direction z. Since $l\cap\tilde{K}^n\neq\emptyset$ denote $V=l\cap\tilde{K}^n$. Consider an arbitrary element x of $\tilde{K}^n$. Then there obviously exists $a\in(0,1\rangle$ such that $ax\in(x_n,y_n)$. Suppose that $ax\notin[x_n,y_n]$. Than since $[x_n,y_n]$ contains some interior point z_n of K^n, $z_n\neq x_n$, $z_n\neq y_n$ and since $ax\in K^n$ since K^n is a convex set and $0\in K^n$, we see that in this case x_n or y_n would not be a boundary point of K^n again by a simple consideration involving con-

vexity of K^n. Therefore $ax \epsilon [x_n, y_n]$ and thus $\|x\| \geqslant \|ax\| \geqslant 1-\varepsilon_n$ by our hypotheses. Now choose an arbitrary element v of V. Then again the line $(0, v)$ crosses $[x_n, y_n]$ in some point u. Consider now the Banach space B on $(0, v)$ with the norm from X. Consider a coordinate system $0, (0, v), (x_n', y_n')$ in P^n where (x_n', y_n') denotes the line through 0 parallel to (x_n, y_n), with the measure units on $(0, v), (x_n', y_n')$ obtained from X and an orientation such that v has a positive $(0, v)$ — coordinate.

Now, for every $t \epsilon [u, v]$, $p(t)$ denotes the line through t parallel to (x_n, y_n). Denote $q(t)$ a function defined on $[u, v]$:

$$q(t) = \sup_{x,y \epsilon p(t) \cap \tilde{K}^n} \|x-y\|.$$

Then it is easy to see that $q(t)$ is well defined, bounded. We check that $q(t)$ may be regarded as a sum of two well defined functions $f_1(-f_2)$ on $[u, v]$:

$$f_1(t) = \sup_{\{t_1, x_2\} \epsilon \tilde{K}^n} x_2, \quad f_2(t) = \inf_{\{t_1, x_2\} \epsilon \tilde{K}^n} x_2, \quad t = \{t_1, 0\}$$

where $\{x_1, x_2\}$ denotes the coordinates of $x \epsilon P^n$ in our coordinate system. Further, since we are in a compact set, these suprema and infima are always attained. f_1 (f_2) is a concave (convex) function. For it, take for instance f_1. Suppose $t = at^1 + (1-a)t^2$, $a \epsilon \langle 0, 1 \rangle$, $t^1, t^2 \epsilon [u, v]$. Denote $f_1(t^1) = \beta_1$, $f_1(t^2) = \beta_2$. It means $\{t_1^1, \beta_1\} \epsilon \tilde{K}^n$, $\{t_1^2, \beta_2\} \epsilon \tilde{K}^n$. Therefore $\{at_1^1 + (1-a)t_1^2, \ a\beta_1 + (1-a)\beta_2\} \epsilon \tilde{K}^n$ since $\tilde{K}^n$ is a convex set. Thus $\{t_1, a\beta_1 + (1-a)\beta_2\} \epsilon \tilde{K}^n$ and therefore $f_1(t) \geqslant a\beta_1 + (1-a)\beta_2 = af_1(t^1) + (1-a)f_1(t^2)$. Using the continuity theorem for convex functions ([4b, Ch.2; 5]), we see that these functions f_1, f_2 are continuous on $[u, v]$ excetp perhaps u, v. We may, of course, prove continuity of both of these functions at u and v with respect to $[u, v]$. Take for example f_2 and u. We are to prove $\lim_{t_1 \to u_1^+} f_2(t) = f_2(u)$ $(t = \{t_1, 0\})$. Obviously, $\limsup_{t_1 \to u_1^+} f_2(t) \leqslant f_2(u)$, $\liminf_{t_1 \to u_1^+} f_2(t) \leqslant f_2(u)$ otherwise we get a simple contradiction with convexity of f_2. If $\liminf_{t_1 \to u_1^+} f_2(t) < f_2(u)$ denote $a = \liminf_{t_1 \to u_1^+} f_2(t)$. Take $t^n \epsilon [u, v]$ so that $t_1^n \to u_1^+$ and $f_2(t^n) \to a$. It means $\{t_1^n, f_2(t^n)\} \to \{u_1, a\} \epsilon \tilde{K}^n$, $a < f_2(u)$ — a contradiction with definition of $f_2(u)$. Thus $q(t)$ is continuous on $[u, v]$. Now, if $q(v) \geqslant c^1$ it simply implies that there exists a line segment on S_1 in the direction z of the length c^1. So, in this case we may again choose $\mathring{x}_n, \mathring{y}_n \epsilon S_1$ such that

$$\mathring{x}_n - \mathring{y}_n = c^1 z, \quad \|t\mathring{x}_n + (1-t)\mathring{y}_n\| = 1 \quad \text{for all } t \epsilon \langle 0, 1 \rangle.$$

If $q(v) < c^1$ then since $q(u) \geqslant c^1$ and $[u, v]$ is a connected set of B we obtain by well known Darboux property of continuous functions again there exist $\mathring{x}_n, \mathring{y}_n \in S_1 \cap \tilde{K}^n$, $\mathring{x}_n - \mathring{y}_n = c^1 z$. Since for $t \in \langle 0, 1 \rangle$, $t\mathring{x}_n + (1-t)\mathring{y}_n \in \tilde{K}^n$ we have already proved $\|t\mathring{x}_n + (1-t)\mathring{y}_n\| \geqslant 1 - \varepsilon_n$. Therefore for every $n \geqslant n_0$, $n \in N$ we have found $\mathring{x}_n \in S_1$, $\mathring{y}_n \in S_1$ such that $\mathring{x}_n - \mathring{y}_n = c^1 z$ where $c^1 \leqslant c$, $\inf_{t \in \langle 0,1 \rangle} \|t\mathring{x}_n + (1-t)\mathring{y}_n\| \geqslant 1 - \varepsilon_n \to 1$. Thus (2) is not valid.

(1) implies (3): Suppose there exist $x_n', y_n' \in K_1$, $x_n' - y_n' = \lambda_n z$, $|\lambda_n| \geqslant \delta > 0$, $\|\frac{1}{2}(x_n' + y_n')\| \to 1$. Take $n \in N$. Then there are only two possibilities. $[x_n', y_n']$ contains some interior point of K_1 or not. In the second case $x_n', y_n' \in S_1$ and therefore we take $x_n = x_n'$, $y_n = y_n'$. Otherwise it is a simple consequence of convexity that $(x_n', y_n') \cap S_1$ are exactly two points which we denote x_n, y_n. Let $z_n = \frac{1}{2}(x_n' + y_n')$. Then $\|x_n - z_n\| \geqslant \frac{1}{2}\delta$, $\|y_n - z_n\| \geqslant \frac{1}{2}\delta$, $\|z_n\| \to 1$ by our hypotheses. Thus by Lemma 1 $\|\frac{1}{2}(x_n + y_n)\| \to 1$. Of course, $x_n - y_n = \tilde{\lambda}_n z$, where $|\tilde{\lambda}_n| \geqslant |\lambda_n| \geqslant \delta$. Therefore (1) is not satisfied.

From the proof of the rest of our proposition we mention only a few parts. The others are easily deduced from these ones. We prove the equivalences from (1) up to (11).

(3) implies (4): Suppose (4) is not satisfied. Then there exist $\varepsilon_0 > 0$ and $\{x_n\}$ bounded $\{y_n\} \subset X$, $\|x_n\| - \|y_n\| \to 0$, $x_n - y_n = \lambda_n z$, $\|x_n\| + \|y_n\| - \|x_n + y_n\| \to 0$, and $|\lambda_n| \geqslant \varepsilon_0 > 0$. Assume without loss of generality $\|x_n\| \to 1$ $(x_n \not\to 0)$. Then $\|y_n\| \to 1$. Take $x_n' = (\max(\|x_n\|, \|y_n\|))^{-1} x_n$, $y_n' = (\max(\|x_n\|, \|y_n\|))^{-1} y_n$. Then $x_n' - y_n' = \lambda_n' z$ where $|\lambda_n'| \geqslant \frac{1}{2}\varepsilon_0 > 0$ for $n \geqslant n_0$, $x_n', y_n' \in K_1$ $\|x_n' + y_n'\| \to 2$. Therefore (3) is not satisfied.

(6) implies (7): Suppose $\{x_n\}$ is a bounded sequence of X such that $2(\|x_n + z\|^2 + \|x_n\|^2) - \|2x_n + z\|^2 \to 0$. If for some $\delta > 0$ and subsequence n_k, $(\|x_{n_k} + z\| - \|x_{n_k}\|)^2 \geqslant \delta$ we would have

$$2(\|x_{n_k} + z\|^2 + \|x_{n_k}\|^2) - \|2x_{n_k} + z\|^2 \geqslant (\|x_{n_k} + z\| - \|x_{n_k}\|)^2 \geqslant \delta > 0,$$

a contradiction with our hypotheses. Therefore $\|x_n + z\| - \|x_n\| \to 0$. Then

$$(\|x_n + z\| + \|x_n\|)^2 - \|2x_n + z\|^2 =$$
$$= 2(\|x_n + z\|^2 + \|x_n\|^2) - \|2x_n + z\|^2 - (\|x_n + z\| - \|x_n\|)^2 \to 0.$$

Thus

$$A_n = (\|x_n + z\| + \|x_n\| - \|2x_n + z\|)(\|x_n + z\| + \|x_n\| + \|2x_n + z\|) \to 0.$$

Now, if for some subsequence n_k, $x_{n_k} \to 0$, then from $2(\|x_{n_k} + z\|^2 + \|x_{n_k}\|^2) - \|2x_{n_k} + z\|^2 \to 0$ we have (limiting $k \to \infty$) $2\|z\|^2 - \|z\|^2 = 0$, i.e. $z = 0$, a contradiction. Otherwise there exist $\delta > 0$ and $n_0 \in N$ such that $\|x_n\| \geqslant \delta$ for $n \geqslant n_0$. Then the second member of A_n is not less than δ for $n \geqslant n_0$. Therefore $\|x_n + z\| + \|x_n\| - \|2x_n + z\| \to 0$. Thus we have a contradiction with (6).

A characterization of uniformly rotund directions in some spaces

Let S be some nonempty set, $B(S)$ denote the Banach space of all bounded real valued functions on S with supremum norm and if S is a topological space, $C(S)$ be the Banach space of all bounded real valued continuous functions on S with supremum norm. $c_0(S)$ denotes the Banach space of all real valued functions f on S with the property that for every $\varepsilon > 0$ the set $\{s \in S;\ |f(s)| \geqslant \varepsilon\}$ is finite, with supremum norm. Let (T, Σ, μ) mean a measure space, i.e. a set T, σ-field Σ of subsets of T and a nonnegative countably additive real valued measure on Σ. If we denote $L_1(T, \Sigma, \mu)$ the Banach space of all integrable (i.e. absolutely integrable) functions (of course, "classes"), we suppose T is σ-finite since $L_\infty(T, \Sigma, \mu)$ — the Banach space of all essentially bounded real valued functions with the norm ess sup is then isometrically isomorphic to $L_1^*(T, \Sigma, \mu)$. $l_1(S)$ is the space of all absolutely summable real valued functions on S with its usual norm $\sum_{s \in S} |x(s)|$, $c(N)$ the Banach space of all real valued convergent sequences with the supremum norm.

For all these spaces see [11], [14].

For $a, b \in R$, $a \neq b$, $]a, b[$ denotes the open line segment between a, b.

PROPOSITION 2. *Assume K is some subspace of $B(S)$. Suppose $g \in K$, $\|g\| = 1$ is such that the following conditions are satisfied:*

For every $\varepsilon > 0$ there exists $t_\varepsilon \in S$ and $f_\varepsilon \in K$, $\|f_\varepsilon\| = 1$ such that the following conditions are satisfied simultaneously:

1) $|g(t_\varepsilon)| \leqslant \varepsilon$, $|f_\varepsilon(t_\varepsilon)| \geqslant 1-\varepsilon$,

2) $|g(t)| \geqslant \varepsilon$ *implies* $|f_\varepsilon(t)| \leqslant \varepsilon$.

Then K is not uniformly rotund in the direction g.

Proof. For $\varepsilon < \frac{1}{2}$ denote $h_\varepsilon = f_\varepsilon - g$. Then $\|h_\varepsilon\| \leqslant 1+\varepsilon$, $\|f_\varepsilon\| = 1$, $f_\varepsilon - h_\varepsilon = g$. Furthermore,

$$\|\tfrac{1}{2}(h_\varepsilon + f_\varepsilon)\| \geqslant |(f_\varepsilon - \tfrac{1}{2}g)(t_\varepsilon)| \geqslant |f_\varepsilon(t_\varepsilon)| - |\tfrac{1}{2}g(t_\varepsilon)| \geqslant 1-2\varepsilon.$$

On the other hand, $\|\frac{1}{2}(f_\varepsilon + h_\varepsilon)\| \leqslant \frac{1}{2}(\|h_\varepsilon\| + \|f_\varepsilon\|)$. Furthermore, $\|h_\varepsilon\| \geqslant 1-2\varepsilon$. Thus $\|h_\varepsilon\| \to 1$, $\|f_\varepsilon\| = 1$, $f_\varepsilon - h_\varepsilon = g$, $\|\frac{1}{2}(f_\varepsilon + h_\varepsilon)\| \to 1$ whenever $\varepsilon \to 0$. Hence K is not uniformly rotund in the direction g.

PROPOSITION 3. *Let $K \subset B(S)$ be a subspace of $B(S)$. Suppose K is not uniformly rotund in the direction g, $g \in K$, $\|g\| = 1$. Then there exist $t_n \in S$, such that $g(t_n) \to 0$.*

Proof. K is not uniformly rotund in the direction g means there exist $f_n \in K$, $g_n \in K$, $\|f_n\| = \|g_n\| = 1$, $f_n - g_n = \lambda_n g$, $|\lambda_n| \geqslant \delta > 0$, $\|\frac{1}{2}(f_n + g_n)\| = 1-\varepsilon_n' \to 1$. Take $\varepsilon_n = \varepsilon_n' + 1/n$. Then for every $n \in N$ there exists $x_n \in S$ such that $|\frac{1}{2}(f_n + g_n)(x_n)| \geqslant 1-\varepsilon_n$. It is easy to see that $|f_n(x_n) - g_n(x_n)| \leqslant 2\varepsilon_n$, $n \in N$, Since $|\lambda_n| \geqslant \delta > 0$ and $|\lambda_n g(x_n)| = |f_n(x_n) - g_n(x_n)| \to 0$, we have $g(x_n) \to 0$.

PROPOSITION 4. *Let $K \subset B(S)$ be a subspace of $B(S)$ such that*

(1) $f(s) \equiv 1$ on S is in K,

(2) $h \in K$ implies $|h| \in K$.

g be an arbitrary element of K, $\|g\| = 1$.

Then the following properties are equivalent:

(1) K is not uniformly rotund in the direction g.

(2) K has a nontrivial line segment on the boundary of its unit ball in the direction g.

(3) $\inf_{t \in S} |g(t)| = 0$.

Proof. Suppose K is not uniformly rotund in the direction g. Then, by Proposition 3 there exist $s_n \in S$ such that $g(s_n) \to 0$. Thus (1) implies (3). Also, of course, (2) implies (1). Therefore it suffices to prove (3) implies (2). Suppose (3) is satisfied. Take $f(t) = 1 - |g(t)| \in K$. Then $0 \leqslant f \leqslant 1$, $\|f\| = 1$ and for an arbitrary $a \in]0, 1[$, $0 \leqslant |g(t)| - ag(t) \leqslant 1 + a$. Thus $1 \geqslant 1 - - |g(t)| + ag(t) \geqslant -a$. Therefore $\|f + ag\| \leqslant 1$. On the other hand, $g(t_n) \to 0$ for some $t_n \in S$ and thus $|1 - |g(t_n)| + ag(t_n)| \to 1$. Therefore $\|f + ag\| = 1$ for $a \in]0, 1[$. Now consider $h_1 = f$, $h_2 = f + \frac{1}{2}g$ to obtain the desired results.

COROLLARY. *We may take in Proposition 4 $K = B(S)$, $K = C(S)$ or $K = c(N)$.*

In the following, we will use the following simple and trivial fact:

Remark 1. Let X, Y be Banach spaces, T a linear isometry of X onto Y. Then X is uniformly rotund in the direction z, $\|z\| = 1$ iff Y is uniformly rotund in the direction Tz.

From the results of R. R. Phelps (Th. 3.6 of [34]) we have a corollary that $c_0(S)$ does not contain any nontrivial line segment on its S_1 in the direction $z \in S_1$ iff z does not take on zero. Thus $c_0(S)$, S uncountable has a nontrivial line segment in every direction.

We now give in the following proposition an example of the space and its direction such that

(1) the space is not uniformly rotund in this direction;

(2) the space does not contain any nontrivial line segment of its S_1 in this direction.

PROPOSITION 5. *$c_0(N)$ is uniformly rotund in no direction.*

Proof. Take $z \in c_0(N)$, $\|z\| = 1$ an arbitrary element. We will show the assumptions of Proposition 2 are satisfied for z. Let $\varepsilon > 0$ be an arbitrary number. Then there exists an integer n_ε such that $|z(n_\varepsilon)| < \varepsilon$. Take f_ε as follows: $f_\varepsilon(n_\varepsilon) = 1$, $f_\varepsilon(n) = 0$ for $n \neq n_\varepsilon$. Then $f_\varepsilon \in c_0(N)$, $\|f_\varepsilon\| = 1$. Then the assumptions of Proposition 2 are satisfied and therefore $c_0(N)$ is not uniformly rotund in the direction z.

Now consider an arbitrary Banach space X and let E denote a topological Hausdorff space of all extremal points of K_1^* with relativized w^* topology. Consider a mapping T of X into $C(E)$ defined as follows: For $x \in X$, $Tx(g) = g(x)$, $g \in E$. T is obviously linear. Consider $x \in S_1 \subset X$. Denote F the set of all elements f of S_1^* so that $f(x) = 1$. F is a nonempty convex, w^* compact subset of X^*. Therefore, by Krein-Milman theorem it has an extremal point h. h is an extremal point of K_1^*, too. For it suppose $h = \frac{1}{2}(h_1 + h_2)$, h_1, $h_2 \in K_1^*$, $h_1 \neq h_2$. Then $h_i(x) = 1$, $i = 1, 2$. Thus $h_i \in F$, $i = 1, 2$, $h_1 \neq h_2$, $h = \frac{1}{2}(h_1 + h_2)$, a contradiction with extremality of h in F. Therefore T is a linear isometry of X onto some subspace of $C(E)$. Using these well known arguments we may immediately derive the following:

PROPOSITION 6. *Let X be an arbitrary Banach space. If X is not uniformly round in the direction $z \in S_1 \subset X$ then there exists a sequence $\{g_n\}$ of extremal points of K_1^* such that $g_n(z) \to 0$.*

Proof. If X is not uniformly rotund in the direction z, then by Remark 1 $T(X)$ is not uniformly rotund in the direction Tz, where T means the natural isometry of X into $C(E)$, mentioned above. Therefore by Proposition 3 there exists $\{g_n\} \subset E$ such that $Tz(g_n) = g_n(z) \to 0$.

In the space $L_1\langle 0, 1\rangle$ of all Lebesgue integrable functions on $\langle 0, 1\rangle$ there exists a nontrivial line segment on its S_1 in each direction [1]. The same result is proved in [34], Th. 2,5 for the space $L_1(T, \Sigma, \mu)$ where μ contains no atoms.

If we consider $l_1(S)$ for an arbitrary nonempty set S — the space of absolutely summable real valued functions on S, we have the following:

PROPOSITION 7. *Assume $l_1(S)$ is not uniformly rotund in the direction $z \in S_1 \subset l_1(S)$. Then there exists a nontrivial line segment on $S_1 \subset l_1(S)$ in the direction z.*

Proof. By Proposition 6 there exist $g_n \in \operatorname{ext} K_1^*$ such that $g_n(z) \to 0$. Denote T a well known natural linear isometry of $l_1^*(S)$ and $m(S)$, namely, for $f \in l_1^*(S)$, $(Tf)(s) = f(e_s)$ for $s \in S$ where e_s is an element of $l_1(S)$ defined as follows: $e_s(t) = 0$ for $t \neq s$, $e_s(s) = 1$.

By the Alaoglu Theorem there exists a subnet $g_{n_\nu} \xrightarrow{w^*} g \in K_1^*$. Then $Tg_{n_\nu} \to Tg$ pointwise in $m(S)$. Since T is an linear isometry, Tg_{n_ν} are extreme points of $K_1 \subset m(S)$, i.e., by the characterization of extreme points of $K_1 \subset m(S)$, $|Tg_{n_\nu}(s)| = 1$ for $s \in S$. Therefore $|Tg(s)| = 1$ for $s \in S$ and Tg is hence an extreme point of $K_1 \subset m(S)$. Thus g is an extreme point of $K_1^* \subset l_1^*(S)$, $g(z) = 0$. Denote $Tg = \{\varepsilon(s)\}_{s \in S}$. Take $x(s) = \varepsilon(s)|z(s)|$, where $z = \{z(s)\}_{s \in S}$. Then $\{x(s)\} \in l_1(S)$ and it is easy to see that for $\lambda \in]0, 1[$ $\sum_{s \in S} |x(s) + \lambda z(s)| = \sum_{s \in S} |x(s)| + \lambda \sum_{s \in S} \varepsilon(s) z(s) = 1$, since $g(z) = \sum_{s \in S} \varepsilon(s) z(s) = 0$.

Take, for example, $h_1 = \{x(s)\}_{s \in S}$, $h_2 = \{x(s) + \frac{1}{2}z(s)\}_{s \in S}$ to obtain the desired conclusion.

As for a finite dimensional case, it is obvious (compactness) that the notion of uniformly convex direction coincides with nonexistence of nontrivial line segments of S_1 in this direction.

In this connection, Professor V. L. Klee settled in [25], p. 419, a problem, one part of which asks whether a direction for a compact n dimensional convex body K must exist such that there is no nontrivial line segment on $\delta(K)$ in this direction. This problem was solved positively by T. Mc. Minn ([32], p. 944) and A. S. Besicovitch ([4], p. 24) for $n = 3$ and by Professor V. Klee and Professor B. Grünbaum for $(n-2)$-smooth compact convex n dimensional bodies, ([26], p. 408). Professor V. Klee kindly communicated to the author a preprint by W. D. Pepe where the problem had been solved positively for $n = 4$.

CHAPTER 2

Duality properties

For $x, y \in X$ (a normed linear space) $\varrho(x, y) = \|x-y\|$.

Definition 1. $P \subset X$ be a one-dimensional subspace of a Banach space X. Then P is said to have *S property* if, whenever $\varrho(x_n, P) - \varrho(x_n, y_n) \to 0$, $\varrho(x_n, P) - \varrho(x_n, z_n) \to 0$, $\{x_n\} \subset X$ bounded, $\{y_n\} \subset P$ $\{z_n\} \subset P$ imply $y_n - z_n \to 0$.

Remark 2. Obviously, S property is not weaker than the Chebyshev (Motzkin) property (i.e. each point of the space has a unique nearest point in a set) and these notions coincide in finite dimensional case since the distance is a continuous function of a point.

From the compactness argument we immediately have:

Proposition 8. $P \subset X$, $\dim P = 1$ *has S property iff, whenever* $\{x_n\} \subset X$, $y \in P$, $z \in P$, $\{x_n\}$ *bounded*, $\varrho(x_n, P) - \varrho(x_n, y) \to 0$, $\varrho(x_n, P) - \varrho(x_n, z) \to 0$, *then* $y = z$.

Proposition 9. *Let* $P \subset X$ *be a one-dimensional subspace of the Banach space* X. *Then* P *has S property iff* X *is uniformly rotund in* $z \in S_1 \cap P$.

Proof. Suppose X is uniformly rotund in z. Assume P does not have S property. Then there exist sequences $\{x_n\} \subset X$, $\{y_n\}, \{z_n\} \subset P$, $\{x_n\}$ bounded such that $\varrho(x_n, y_n) - \varrho(x_n, P) \to 0$, $\varrho(x_n, z_n) - \varrho(x_n, P) \to 0$ and $\|y_n - z_n\| \geqslant \varepsilon > 0$.

If for some n_k, $\varrho(x_{n_k}, P) \to 0$, then $\varrho(y_{n_k}, z_{n_k}) \leqslant \varrho(y_{n_k}, x_{n_k}) + \varrho(x_{n_k}, z_{n_k}) \to 0$, a contradiction. Because of the boundedness of $\{x_n\}$ we thus, without loss of generality, suppose $\varrho(x_n, P) \to k \neq 0$. Take $s_n = x_n - y_n$, $t_n = x_n - z_n$. Then $\|s_n\| \to k$, $\|t_n\| \to k$, $\|\frac{1}{2}(s_n + t_n)\| = \|x_n - \frac{1}{2}(y_n + z_n)\|$. Thus $\|\frac{1}{2}(s_n + t_n)\| \geqslant \varrho(x_n, P)$. We have $\|\frac{1}{2}(s_n + t_n)\| \to k$, $s_n - t_n \in P$, $\|s_n - t_n\| \geqslant \varepsilon > 0$, a contradiction with the uniform rotundity in z (Proposition 1, (5)). On the other hand, suppose P has S property. If X were not uniformly rotund in $z \in S_1 \cap P$, there would exist $\{x_n\}$, $\{y_n\} \subset S_1 \subset X$, $x_n - y_n \in P$, such that $\|x_n - y_n\| \geqslant \varepsilon > 0$, $\inf_{t \in R} \|tx_n + (1-t)y_n\| = 1 - \varepsilon_n \to 1$ (see Proposition 1; (9)). Consider $p_n = \frac{1}{2}(y_n - x_n)$, $q_n = \frac{1}{2}(x_n - y_n)$ $(n = 1, 2, \ldots)$, $r_n = \frac{1}{2}(x_n + y_n)$. Then $p_n \in P$, $q_n \in P$, $\|r_n - p_n\| = \|r_n - q_n\| = 1$, $\|r_n\| \leqslant 1$. We have $\varrho(r_n, P) \leqslant \|r_n\|$. Furthermore, if $z \in P$, there exists $a_n \in R$ such that $z = a_n p_n + (1 - a_n) q_n$. Then $z - r_n = -\{a_n x_n + (1 - a_n) y_n\}$. Therefore

$\|r_n - z\| \geqslant 1-\varepsilon_n$ and thus $1 \geqslant \varrho(r_n, P) \geqslant 1-\varepsilon_n$. Thus we have $\varrho(r_n, P) - \varrho(r_n, p_n) \to 0$, $\varrho(r_n, P) - \varrho(r_n, q_n) \to 0$, $p_n, q_n \in P$, $\|p_n - q_n\| = \|x_n - y_n\| \geqslant \varepsilon > 0$, $\|r_n\| \leqslant 1$. This is the required contradiction with S property of P.

Now we may study a dual property to the property S. We will use the notation $\|f\|_P = \sup_{x \in S_1 \cap P} |f(x)|$.

DEFINITION 2. $P \subset X$ be a subspace of X, deficiency of P equal to 1. We say P has *$\tilde{U}$ property* if the following condition is satisfied: whenever $f_n, g_n \in X^*$, $f_n - g_n \in P^\perp$, $\|f_n\| \to 1$, $\|g_n\| \to 1$, $\|f_n\|_P \to 1$, then $f_n - g_n \to 0$.

Remark. It is obvious that $\tilde{U}$ property is not weaker than the unique extension property with preserving the norm which is studied in [34], and coincides with it in finite dimensional case.

The following is a very simple argument using the compactness considerations in $P^\perp$.

PROPOSITION 10. *Let P be a subspace of X, deficiency of P is equal to* 1. *Then the following three conditions are equivalent*:

(1) *P has $\tilde{U}$ property,*

(2) *whenever f_n, $g_n \in X^*$, $f_n - g_n \in P^\perp$, $\|f_n\| \to 1$, $\|g_n\| \to 1$, $\|f_n\|_P = 1$ then $f_n - g_n \to 0$,*

(3) *whenever $f_n, g_n \in X^*$, $f_n - g_n = h \in P^\perp$, $\|f_n\|_P = 1$, $\|f_n\| \to 1$, $\|g_n\| \to 1$, then $h = 0$.*

PROPOSITION 11. *P has $\tilde{U}$ property iff $P^\perp$ has S property* ($\dim X|P = 1$).

Proof. Suppose $P^\perp$ does not have S property. Then there exist $\{f_n\} \subset X^*$ bounded, $\{h_n\}$, $\{g_n\} \subset P^\perp$ so that $\varrho(f_n, P^\perp) - \varrho(f_n, h_n) \to 0$, $\varrho(f_n, P^\perp) - \varrho(f_n, g_n) \to 0$, $\|h_n - g_n\| \geqslant \varepsilon > 0$. As it was pointed out in [34], for every $f \in X^*$ and every subspace $Y \subset X$, $\varrho(f, Y^\perp) = \|f\|_Y$. Indeed, if $g \in Y^\perp$, $\|f\|_Y = \sup_{x \in Y \cap S_1} |(f-g)(x)| \leqslant \|f-g\|$ so that $\|f\|_Y \leqslant \varrho(f, Y^\perp)$. On the other hand, choose $h \in X^*$, $h = f$ on Y and $\|h\| = \|f\|_Y$. Then $f - h \in Y^\perp$ and therefore $\varrho(f, Y^\perp) \leqslant \|f-(f-h)\| = \|f\|_Y$.

Again, without loss of generality suppose $\|f_n\|_P \to a \neq 0$. We have $\|f_n - h_n\| - \|f_n\|_P \to 0$, $\|f_n - g_n\| - \|f_n\|_P \to 0$, $h_n, g_n \in P^\perp$. Denote $F_n = f_n - h_n$, $G_n = f_n - g_n$. Then $\|F_n\| \to a$, $\|G_n\| \to a$, $\|F_n\|_P \to a$, $F_n - G_n = g_n - h_n \in P^\perp$, $\|F_n - G_n\| = \|h_n - g_n\| \geqslant \varepsilon > 0$. Therefore P would not have $\tilde{U}$ property.

On the other hand, suppose P does not have $\tilde{U}$ property. Then there exist $F_n, G_n \in X^*$, $F_n - G_n = h_n \in P^\perp$, $\|F_n - G_n\| \geqslant \varepsilon > 0$, $\|F_n\| \to 1$, $\|G_n\| \to 1$, $\|F_n\|_P \to 1$. Then $\varrho(F_n, P^\perp) = \|F_n\|_P \to 1$, $\varrho(G_n, P^\perp) = \|G_n\|_P \to 1$, $\{F_n\}$ bounded. This means $\varrho(F_n, P^\perp) - \varrho(F_n, 0) \to 0$, $\varrho(F_n, P^\perp) - \varrho(F_n, h_n) \to 0$, $\{F_n\}$ bounded, $h_n \in P^\perp$, $\|h_n\| \geqslant \varepsilon > 0$, i.e. $P^\perp$ does not have S property.

PROPOSITION 12. *Suppose X is a reflexive Banach space, P a one dimensional subspace of X. Then P has S property iff $P^\perp$ has $\tilde{U}$ property.*

Proof. By Remark 1, P has S property in X iff $\pi(P)$ has S property in $\pi(X) = X^{**}$ where π is the canonical isometry of X onto X^{**}. $\pi(P) = (P^{\perp})^{\perp}$ and by Proposition 11, $(P^{\perp})^{\perp}$ has S property in X^{**} iff $P^{\perp}$ has $\tilde{U}$ property in X^*.

PROPOSITION 13. *Let $P \subset L_1(T, \Sigma, \mu)$, T σ-finite, deficiency of P equal to 1. Let $P^{\perp} = Rx$, $\|x\| = 1$, $x \in L_1^*(T, \Sigma, \mu)$. Then the following properties are equivalent*:

(1) *P has the unique norm-preserving extension property*;

(2) *P has $\tilde{U}$ property*;

(3) $\operatorname{ess\,inf}|x| > 0$.

Proof. The equivalence of (1) and (3) was proved by R. R. Phelps in [34, p. 251]. Evidently, (2) implies (1). It suffices, therefore, to show (1) implies (2). By a result of R. R. Phelps, [34, p. 240], (1) is equivalent to the fact that $P^{\perp}$ has Chebyshev property. By Proposition 11, (2) is equivalent to the saying that $P^{\perp}$ has S property. This means, by Proposition 9, $L_1^*(T, \Sigma, \mu)$ is uniformly rotund in the direction x. By Theorem V. 8. 11 of [14] there exist a compact Hausdorff space S_1 and a linear isometry of $L_\infty(T, \Sigma, \mu)$ and $C(S_1)$. Denote T_1 the linear isometry between $L_1^*(T, \Sigma, \mu)$ and $L_\infty(T, \Sigma, \mu)$, T_2 the linear isometry of $L_\infty(T, \Sigma, \mu)$ and $C(S_1)$. Then T_2T_1 is a linear isometry of $L_1^*(T, \Sigma, \mu)$ onto $C(S_1)$. By Remark 1 we see that our assumptions imply $C(S_1)$ is not uniformly rotund in the direction T_2T_1x. By Proposition 4 it means that there exists a nontrivial line segment on the boundary of the unit ball of $C(S_1)$ in the direction T_2T_1x. That means there exists a nontrivial line segment on the boundary of $L_1^*(T, \Sigma, \mu)$ in the direction x. From this fact follows immediately that $P^{\perp}$ is not Chebyshev subspace of $L_1^*(T, \Sigma, \mu)$.

CHAPTER 3

Some renorming results

PROPOSITION 14. *Suppose X, Y are Banach spaces, T is a linear continuous one-to-one mapping of X into Y. Assume Y is uniformly rotund in every direction $\|Tz\|^{-1}$. Tz, $z \in S_1 \subset X$. Then X has an equivalent norm which is uniformly rotund in every direction.*

Proof. Denote $|||x||| = \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}$ an equivalent norm of X. Take $z \in S_1 \subset X$. Suppose $2(|||x_n+z|||^2 + |||x_n|||^2) - |||2x_n+z|||^2 \to 0$ where $\{x_n\}$ is bounded. Then

$$2(\|x_n+z\|^2 + \|x_n\|^2) - \|2x_n+z\|^2 + 2(\|Tx_n+Tz\|^2 + \|Tx_n\|^2) - \|2Tx_n+Tz\|^2 \to 0.$$

This is a sum of two nonnegative members. Therefore both of them converge to zero. Further, $\{Tx_n\}$ is bounded in Y. We have

$$2\left(\left\|\|Tz\|^{-1}\cdot(Tx_n+Tz)\right\|^2 + (\|Tz\|^{-1}\cdot\|Tx_n\|)\right)^2 - \left\|\ \|Tz\|^{-1}\cdot(2Tx_n+Tz)\right\|^2 \to 0.$$

Since Y is uniformly rotund in the direction $\|Tz\|^{-1}Tz$ we have that this is not possible.

COROLLARY. *Every Banach space which has a countable total subset of X^* has an equivalent norm which is uniformly rotund at every direction.*

Proof. Denote by $\{f_i\}_{i\in N} \subset S_1^*$ this countable total subset of X^*. Define a mapping T of X into $l_2(N)$ as follows:

$$Tx = \left\{\frac{f_i(x)}{2^i}\right\}_{i\in N} \in l_2(N).$$

Then T is obviously continuous, linear, one-to-one mapping of X into $l_2(N)$. $l_2(N)$ is, as a unifomly rotund space trivially uniformly rotund at each direction. Now apply considerations of Proposition 14.

Now, we would like to remark that an excellent averaging procedure of E. Asplund ([2]) works in this case, too.

For this, suppose $\|x\|_1$, $\|x\|_2$ are two equivalent norms on a Banach space X, so that $\frac{1}{2}\|x\|_2^2 \leqslant \frac{1}{2}\|x\|_1^2 \leqslant \frac{1+C}{2}\|x\|_2^2$. Denote $f_0(x) = \frac{1}{2}\|x\|_1^2$,

$g_0(x) = \frac{1}{2}\|x\|_2^2$. Then E. Asplund constructs another equivalent norm $\|x\|_3$ such that if $h(x) = \frac{1}{2}\|x\|_3^2$ then

$$h(x)+h(y)-2h\left(\frac{x+y}{2}\right) \geqslant \frac{1}{2^n}\left[f_0(x)+f_0(y)-2f_0\left(\frac{x+y}{2}\right)-\right.$$
$$\left.-\frac{C}{2^n}\left(f_0(x)+f_0(y)\right)\right], \qquad x, y \in X,$$

for every $n \in N$ where $C > 0$ is our constant from the equivalence of $\|x\|_1$ and $\|x\|_2$.

An analogous assertion holds for h and g_0 only with change of C to some other constant.

Now suppose $\|x\|_1$ (or $\|x\|_2$) is uniformly rotund at z, $\|z\|_3 = 1$. It is easy to see that by Proposition 1 (7) it means

$$a = \inf_{\substack{x,y \\ x \in B, y = x-z}} \left[f_0(x)+f_0(y)-2f_0\left(\frac{x+y}{2}\right)\right] > 0$$

$$\left(\inf_{\substack{x,y \\ x \in B, y = x-z}} \left[g_0(x)+g_0(y)-2g_0\left(\frac{x+y}{2}\right)\right] > 0\right),$$

for every bounded subset B of X. Suppose now there exists B_0 bounded in X such that

$$(*) \qquad \inf_{\substack{x,y \\ x \in B_0, y = x-z}} \left[h(x)+h(y)-2h\left(\frac{x+y}{2}\right)\right] = 0.$$

Then, obviously, in this infimum, y runs a bounded subset $B_1 = B-z$. Take $n_0 \in N$ so that

$$\frac{C}{2^{n_0}}\left[f_0(x)+f_0(y)\right] < \frac{a}{2} \quad \text{for} \quad x \in B_0, y \in B_1.$$

Then

$$h(x)+h(y)-2h\left(\frac{x+y}{2}\right) \geqslant \frac{1}{2^{n_0}} \cdot \frac{a}{2} > 0$$

for every x, y, $x \in B_0$, $y = x-z$, a contradiction with $(*)$. Thus $\|x\|_3$ is uniformly rotund in the direction z.

DEFINITION 3. A Banach space X is said to be *locally uniformly rotund* (LUR) if $x_n, x_0 \in S_1 \subset X$, $\|\frac{1}{2}(x_n+x_0)\| \to 1$, $n = 1, 2, \ldots$, imply $x_n \to x$.

A Banach space X is called *weakly uniformly rotund* (WUR), if x_n, $y_n \in S_1 \subset X$, $\|\frac{1}{2}(x_n+y_n)\| \to 1$ imply $x_n - y_n \xrightarrow{w} 0$. Similarly for the case (W*UR) — with respect to the w^*-topology of X^*.

A Banach space X is said to have *uniformly Gâteaux* (UG) (*uniformly Fréchet* (UF)) differentiable norm if the limit $\lim_{t\to 0}\frac{\|x+th\|-\|x\|}{t} = D\|\cdot\|(x,h)$ is uniform on $x\in S_1$ for each $h\in S_1$ (is uniform on $\{x,h\}\in S_1\times S_1$).

PROPOSITION 15. *Every separable Banach space has an equivalent norm which is locally uniformly rotund, uniformly rotund in each direction and uniformly Gâteaux differentiable.*

Proof. M. I. Kadec ([19], p. 54) proved there exists in our case an equivalent norm of X which is locally uniformly rotund. We proved ([45], p. 199) there exists in our case an equivalent norm $\|x\|_1$ being locally uniformly rotund and whose dual norm in X^* is (W*UR). (= *a* direct consequence of the methods of E. Asplund mentioned above). By the definition, (W*UR) means the following: $f_n, g_n\in S_1^*$, $\|\frac{1}{2}(f_n+g_n)\|\to 1$, then $f_n-g_n\xrightarrow{w^*}0$. This property is dual to Gâteaux differentiability of a norm ([30], p. 646). The corollary to Proposition 14 gives an equivalent norm $\|x\|_2$ of X which is uniformly rotund at every direction. The Asplund's method applied to $\|x\|_1$ and $\|x\|_2$ gives the desired result.

Another method how to obtain spaces which are uniformly rotund at every direction, is, for example, the following

PROPOSITION 16. $X = l_2(B_i)$, $i\in N$, *is uniformly rotund in every direction if so are all* B_i.

Proof. Using Proposition 1 (7), take $\{x_n\}$ bounded, $z\in S_1\subset X$ so that $2(\|x_n+z\|^2+\|x_n\|^2)-\|2x_n+z\|^2\to 0$. Then

$$\sum_{i=1}^{\infty}[2(\|x_n^i+z^i\|_{B_i}^2+\|x_n^i\|_{B_i}^2)-\|2x_n^i+z^i\|_{B_i}^2]\to 0.$$

Since all members of the sum are nonnegative, we have $2(\|x_n^i+z^i\|_{B_i}^2+\|x_n^i\|_{B_i}^2)-\|2x_n^i+z^i\|_{B_i}^2\to 0$ as $n\to\infty$ for every $i\in N$, $z=\{z^i\}\in S_1$ in X. Choose $i_0\in N$ so that $z^{i_0}\neq 0$. Then we have

$$2\left(\left\|\frac{x_n^{i_0}+z^{i_0}}{\|z^{i_0}\|_{B_{i_0}}}\right\|_{B_{i_0}}^2+\left\|\frac{x_n^{i_0}}{\|z^{i_0}\|_{B_{i_0}}}\right\|_{B_{i_0}}^2\right.-\left.\left\|\frac{2x_n^{i_0}+z^{i_0}}{\|z^{i_0}\|_{B_{i_0}}}\right\|_{B_{i_0}}^2\right)\to 0.$$

Since B_{i_0} is uniformly rotund at every direction, we obtain a contradiction.

CHAPTER 4

Some remarks to the theory of weakly uniformly rotund spaces

PROPOSITION 17. *X is (WUR) iff the following condition is satisfied:* $x_n, y_n \in X$, $2(\|x_n\|^2+\|y_n\|^2)-\|x_n+y_n\|^2 \to 0$, $\{x_n\}$ *bounded, then* $x_n - y_n \overset{w}{\to} 0$.

Similarly for the case of (W^*UR).

Proof. It depends only on the methods which were discussed in the proof of Proposition 1.

PROPOSITION 18. *Let X, Y be Banach spaces, T a linear continuous mapping of X onto a dense subset of Y. Suppose X is the space with uniformly Gâteaux differentiable norm. Then there exists an equivalent norm of Y which is uniformly Gâteaux differentiable.*

Proof. T^* is one-to-one linear continuous and w^*-w^* continuous mapping of Y^* into X^*. Take a new norm of Y^*:

$$|||f||| = \sqrt{\|f\|_{Y^*}^2 + \|T^* f\|_{X^*}^2}.$$

Then $|||f|||$ is an equivalent norm of Y^* which is w^*-lower semicontinuous functional on Y^*. Therefore $|||f|||$ is a dual norm of some equivalent norm of Y. Now we want to prove $|||f|||$ is (W^*UR). To this purpose suppose $f_n, g_n \in Y^*$, $2(|||f_n|||^2+|||g_n|||^2)-|||f_n+g_n|||^2 \to 0$, $\{f_n\}$ bounded. Then $\{\|T^* f_n\|_{X^*}\}$ is bounded and $2(\|T^* f_n\|_{X^*}^2+\|T^* g_n\|_{X^*}^2)-\|Tf_n^*+Tg_n^*\|_{X^*}^2 \to 0$. Since X^* is (W^*UR), by the duality theorem of V. L. Šmuljan ([41], a shorter proof see [10], p. 291), we have $T^*f_n - T^*g_n \overset{w^*}{\to} 0$ in X^*. Since $\{f_n\}$, $\{g_n\}$ are bounded in Y^*, there exists a closed ball $K_r^* \subset Y^*$ such that $\{f_n - g_n\} \subset K_r^*$. By the Alaoglu theorem K_r^* is weakly* compact. Since T^* is w^*-w^* continuous and one-to-one in K_r^*, it is a homeomorphism of K_r^* onto $T^*(K_r^*)$ and therefore from $T^*(f_n-g_n) \overset{w^*}{\to} 0$ we have $f_n - g_n \overset{w^*}{\to} 0$ in Y^*. Thus $|||f|||$ is (W^*UR).

Remark. Proposition 18 is a uniform analogy of the Theorem [7] Th. 5 of M. M. Day. The proof is also similar.

COROLLARY 1. *$c_0(S)$ has an equivalent (UG) norm for every nonempty S.*

Proof. Take $X = l_2(S)$ and T the "natural identity" mapping of $l_2(S)$ into $c_0(S)$ in Proposition 18.

COROLLARY 2. *$L_1(T, \Sigma, \mu)$ has an equivalent (UG) norm if T is finite or σ-finite.*

Proof. The proof is analogous to one of M. M. Day in [12] for an analogous corollary for the case of the Gâteaux differentiability. If T is finite, we simply put $X = L_2(T, \Sigma, \mu)$ and F the "identity mapping" of $L_2(T, \Sigma, \mu)$ into $L_1(T, \Sigma, \mu)$. If $T = \bigcup_{j=1}^{\infty} T_j$, $0 < \mu(T_j) < \infty$, then M. M. Day constructed the mapping F of $L_2(T, \Sigma, \mu)$ into $L_1(T, \Sigma, \mu)$ by the following: If Φ is the characteristic function of T_j in $T (j = 1, 2, \ldots)$, then for $x \in L_2(T, \Sigma, \mu)$, $Fx = \sum_{j=1}^{\infty} x \cdot \Phi_j \cdot \frac{1}{2^j} \cdot (\mu(T_j))^{-1/2}$. The mapping F satisfies hypotheses of Proposition 18.

Remark 3. M. M. Day proved [12] (see also [20]) that $l_1(\Gamma)$ has no equivalent Gâteaux differentiable norm if Γ is uncountable. This result implies immediately that $c_0(\Gamma)$ has no equivalent (WUR) norm if Γ is uncountable, see also [42], p. 95.

Similarly to Proposition 18 we obtain

PROPOSITION 19. *Suppose X is a reflexive Banach space T is a linear continuous one-to-one mapping of X into (WUR) space Y. Then X has an equivalent (WUR) norm.*

Proof. Take $|||x||| = \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}$. Analogously to Proposition 18 we may see $|||x|||$ is an equivalent (WUR) norm of X.

Another method for obtaining (WUR) spaces is again:

PROPOSITION 20. *$X = l_2(B_i)$, $i \in N$, is (WUR) if so are all B_i.*

Proof. Assume $x_n, y_n \in X$, $\{x_n\}$ bounded, $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \to 0$. It means

$$\sum_{j=1}^{\infty} [2(\|x_n^j\|_{B_j}^2 + \|y_n^j\|_{B_j}^2) - \|x_n^j + y_n^j\|_{B_j}^2] \to 0 \quad \text{as } n \to \infty.$$

It immediately follows from these facts that $\{y_n\}$ is bounded (see Proof of Proposition 1) and

$$2(\|x_n^j\|_{B_j}^2 + \|y_n^j\|_{B}^2) - \|x_n^j + y_n^j\|_{B_j}^2 \to 0 \quad \text{as } n \to \infty$$

for every $j \in N$. Therefore, since obviously $\{x_n^j\}$ is bounded in B_j for every $j \in N$, we have that $x_n^j - y_n^j \xrightarrow{w} 0$ in B_j for every $j \in N$ since B_j are (WUR). Now, if $f \in X^*$ then $f(x) = \sum_{j=1}^{\infty} f^j(x^j)$ where $f^j \in B_j^*$, $\sum_{j=1}^{\infty} \|f^j\|_{B_j^*}^2 < \infty$. Let us say $\|x_n - y_n\|_X \leqslant K < \infty$. We are to prove the following: For every $\varepsilon > 0$ and every $f \in X^*$ there exists $n_0 \in N$ such that for $n \geqslant n_0$, $n \in N$ $|f(x_n - y_n)| \leqslant \varepsilon$.

Take $f \in X^*$, $\varepsilon > 0$. Then there exists $j_0 \in N$ such that $\left(\sum_{j=j_0+1}^{\infty} \|f^j\|_{B_j^*}^2\right)^{1/2} < \frac{\varepsilon}{2k}$. Now choose $n_0 \in N$ so that for $j = 1, 2, \ldots, j_0$ $|f^j(x_n^j - y_n^j)| \leqslant \frac{\varepsilon}{2j_0}$ whenever $n \geqslant n_0$. Then

$$\begin{aligned}
|f(x_n - y_n)| &= \Big|\sum_{j=1}^{\infty} f^j(x_n^j - y_n^j)\Big| \\
&\leqslant \sum_{j=1}^{j_0} |f^j(x_n^j - y_n^j)| + \sum_{j=j_0+1}^{\infty} |f^j(x_n^j - y_n^j)| \\
&\leqslant \frac{\varepsilon}{2} + \sum_{j=j_0+1}^{\infty} \|f^j\|_{B_j^*} \cdot \|x_n^j - y_n^j\|_{B_j} \\
&\leqslant \frac{\varepsilon}{2} + \Big(\sum_{j=j_0+1}^{\infty} \|f^j\|_{B_j^*}^2\Big)^{1/2} \Big(\sum_{j=j_0+1}^{\infty} \|x_n^j - y_n^j\|_{B_j}^2\Big)^{1/2} \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2k} k = \varepsilon
\end{aligned}$$

whenever $n \geqslant n_0$.

Similarly we obtain

PROPOSITION 21. *$X = l_2(B_i)$, $i \in N$ is (UG) if so are all B_i.*

Proof. X^* is isometrically isomorphic to $l_2(B_i^*)$. B_i^* are (W*UR) ([41], p. 646) and the method of the proof of Proposition 20 may be used for the rest of this one.

Now, we shall proceed with some remarks concerning uniformly Gâteaux differentiable norms.

PROPOSITION 22. *X is (UG) in the direction $z \in S_1 \subset X$ iff $S_1 \ni x_n$, $y_n \in K_1$, $x_n - y_n = \lambda_n z$, $\lambda_n \neq 0$ imply $\frac{1 + \|y_n\| - \|x_n + y_n\|}{\lambda_n} \to 0$ whenever $\lambda_n \to 0$.*

Proof. The proof is similar to that of G. Köthe ([27], p. 363) for the case of uniformly Fréchet differentiable norms. Suppose X is (UG) in the direction z. Then it is easy to see by a well known argument ([27], p. 349) that this is equivalent to the following:

$$\frac{\|x + tz\| + \|x - tz\| - 2}{t} \to 0$$

whenever $t > 0$, $t \to 0$, uniformly on $x \in S_1$.

Assume we have $x_n \in S_1$, $y_n \in K_1$, $x_n - y_n = \lambda_n z$, $\lambda_n \neq 0$, $\lambda_n \to 0$, $1 + \|y_n\| - \|x_n + y_n\| \geqslant \varepsilon \|x_n - y_n\|$, $\varepsilon > 0$. Take $x_n + y_n = s_n$, $x_n - y_n = t_n$. Then $s_n + t_n = 2x_n$, $s_n - t_n = 2y_n$. Hence

$$\|x_n\| + \|y_n\| - \|x_n + y_n\| = \tfrac{1}{2}(\|s_n + t_n\| + \|s_n - t_n\|) - \|s_n\| \geqslant \varepsilon \|t_n\|.$$

If for some subsequence n_k, $x_{n_k}+y_{n_k} \to 0$ then since $x_n-y_n \to 0$ we would have $2x_{n_k} = x_{n_k}-y_{n_k}+x_{n_k}+y_{n_k} \to 0$, a contradiction with $x_n \in S_1$. Thus there exist $\delta > 0$ and $n_0 \in N$ such that $\|s_n\| \geq \delta$ for $n \geq n_0$. Take $s_n' = \frac{s_n}{\|s_n\|}$, $t_n' = \frac{t_n}{\|s_n\|}$. We have:

$$\|s_n'+t_n'\|+\|s_n'-t_n'\|-2\|s_n'\| \geq 2\varepsilon\|t_n'\|, \quad \|s_n'\| = 1, \quad t_n' = a_n z$$

where $a_n = \frac{1}{\|s_n\|}\lambda_n$. Therefore $a_n \to 0$. This gives a contradiction with (UG) in the direction z.

On the other hand, assume X satisfies the condition: Suppose X is not (UG) in the direction z. This means there exist $x_n \in S_1$, $y_n = a_n z$, $a_n \neq 0$, $a_n \to 0$ such that $\|x_n+y_n\|+\|x_n-y_n\| \geq 2+\varepsilon_0\|y_n\|$, $\varepsilon_0 > 0$. Take $x_n+y_n = v_n$, $x_n-y_n = w_n$. Then $\|v_n\|+\|w_n\| \geq 2+\frac{1}{2}\varepsilon_0\|v_n-w_n\|$. Further, $\|v_n+w_n\| = 2$ and therefore $\|v_n\|+\|w_n\|-\|v_n+w_n\| \geq \frac{1}{2}\varepsilon_0\|v_n-w_n\|$. Take

$$v_n' = \frac{v_n}{\max(\|v_n\|, \|w_n\|)}, \quad w_n' = \frac{w_n}{\max(\|v_n\|, \|w_n\|)}.$$

Then

$$\|v_n'\|+\|w_n'\|-\|v_n'+w_n'\| \geq \tfrac{1}{2}\varepsilon_0\|v_n'-w_n'\|, \; v_n', w_n' \in K_1 \text{ and } \max(\|v_n'\|,\|w_n'\|) = 1.$$

Let us say $\|v_n'\| = 1$, otherwise we rearrange v_n', w_n'. Further, since

$$\max(\|v_n\|, \|w_n\|) \to 1, \quad \|v_n'-w_n'\| = \|v_n-w_n\| \frac{1}{\max(\|v_n\|, \|w_n\|)} \to 0,$$

since $v_n-w_n = 2y_n \to 0$. Further, for $n \geq n_0$,

$$v_n'-w_n' = \frac{1}{\max(\|v_n\|, \|w_n\|)}(v_n-w_n) = \tilde{\lambda}_n z \quad \text{for some } \tilde{\lambda}_n \neq 0, \tilde{\lambda}_n \to 0.$$

Thus we have obtained a contradiction with the fact that X satisfies our condition.

Remark 4. We cannot, of course, suppose in the conditions of Proposition 22 $x_n, y_n \in S_1$. The simple counterexample is easily seen in P_2 with $\|x\| = \max(|x_1|, |x_2|)$ where $x = \{x_1, x_2\}$, in the direction $\{1, 0\}$.

We finish the chapter with some examples concerning the notions studied here.

1. An example of a Banach space which is uniformly rotund in every direction, but it is not weakly uniformly rotund.

Take $C\langle 0, 1\rangle$ with an equivalent norm

$$|||x||| = \sqrt{\|x\|^2_{C\langle 0,1\rangle}+\|Tx\|^2_{L_2\langle 0,1\rangle}},$$

where T is the natural "identity mapping" of $C\langle 0,1\rangle$ into $L_2\langle 0,1\rangle$. By the proof of Proposition 14, $|||x|||$ is uniformly rotund in each direction. But $C\langle 0,1\rangle$ does not have any equivalent (WUR) norm, since otherwise, by the universality of $C\langle 0,1\rangle$ for separable Banach spaces, $l_1(N)$ would have an equivalent (WUR) norm and since the weak and strong convergence of sequences in $l_1(N)$ coincide, $l_1(N)$ would have an equivalent norm which is uniformly rotund ($x_n, y_n \in S_1$, $\|\frac{1}{2}(x_n+y_n)\| \to 1$ imply $x_n - y_n \to 0$). But then by the well-known theorem of Milman-Pettis ([27], p. 354) asserting that each uniformly rotund Banach space is reflexive we would obtain a contradiction.

2. An example of a Banach space which is locally uniformly rotund, but it is not uniformly rotund in some directions.

Take the space $c_0(N)$. If $x \in c_0(N)$ then x has a countable support $E(x)$ which can be enumerated so that $|x(a_k)| \geqslant |x(a_{k+1})|$. The Day's equivalent norm on $c_0(N)$ is then defined as follows: Consider D a mapping of $c_0(N)$ into $l_2(N)$:

$$Dx = \begin{cases} \dfrac{x(a_k)}{2^k} & \text{on } E(x), \\ 0 & \text{outside } E(x). \end{cases}$$

Then $|||x||| = \|Dx\|_{l_2(N)}$ is an equivalent locally uniformly rotund norm on $c_0(N)$ ([3], p. 41, [37], p. 335).

However, it is not uniformly rotund in some directions, namely in $\{2, 0, 0, \ldots\}$ (use $a_k = \{0, 0, \ldots, \underset{k}{1}, \ldots, \underset{2k}{1}, 0, \ldots\}$ and Proposition 1, (1)-(7).

3. An example of a Banach space X^* which is (W*UR) and therefore uniformly rotund in every direction, but it is not locally uniformly rotund.

Take the space $l_1(N)$. Since it is separable, there exists an equivalent norm $|||x|||$ of $l_1(N)$ which is (UG) ([45], p. 199). Take the dual norm of $|||x|||$ in $l_1^*(N)$. This is, by the Šmuljan's theorem on duality of (W*UR) and (UG) ([42], p. 646) W*UR and therefore uniformly rotund in every direction. However, it is not locally uniformly rotund, since then, by the Lovaglia's Theorem on the relation between (LUR) and the Fréchet differentiability of the dual norm ([30]) the norm $|||x|||$ would be Fréchet differentiable which is not possible since the Theorem of V. L. Klee- G. Restrepo asserts: If X is a separable space which has an equivalent Fréchet differentiable norm, then X^* must be separable ([24], p. 27, [38], p. 413). For the Theorem of Lovaglia, we recall that the norm $\|x\|$ is Fréchet differentiable in $x \in X$ if the limit in the Definition 3 is for a given x uniform on $h \in S_1$.

4. An example of a space X^* which is rotund but not uniformly rotund in some directions.

Take $l_1(N)$ and denote by T a usual linear isometry of $l_1^*(N)$ onto

$m(N)$. Denote by A a mapping of $m(N)$ into $l_2(N)$ defined by $Ax = \{a_i x_i\}_{i\in N}$ where $x = \{x_i\}_{i\in N} \in m(N)$ and $\{a_i\}$ is a fixed sequence of positive real numbers such that $\sum_{i=1}^{\infty} a_i^2 = 1$.

Consider a new equivalent norm of $l_1^*(N)$:

$$|||f||| = \|Tf\|_{m(N)} + \|ATf\|_{l_2(N)}.$$

This is a rotund one, by a standard argument ([27], p. 362). But $|||f|||$ is not uniformly rotund in some directions. For it take $h \in l_1^*(N)$ such that $Th = \{\frac{1}{5}, \frac{1}{5}, \ldots\}$ analogously to Garkavi (see [17]). Further, let $f_n \in l_1^*(N)$ be so that $Tf_n = \{\frac{1}{5}, \ldots, \underset{n-1}{\frac{1}{5}}, \underset{n}{-\frac{4}{5}}, \underset{n+1}{\frac{1}{5}}, \frac{1}{5}, \ldots\}$. Then $|||h||| = \frac{2}{5}$, $|||f_n||| \to 1$, $|||f_n+h||| \to 1$, $|||f_n-h||| \to 1$. Denote $f_n+h = p_n, f_n-h = q_n$. Then $|||p_n||| \to 1$, $|||q_n||| \to 1$, $|||p_n+q_n||| = 2\,|||f_n||| \to 2$, $p_n - q_n = 2h$. Therefore $l_1^*(N)$ with $|||f|||$ is not uniformly rotund in the direction $h_0 = \dfrac{1}{|||h|||}h$. By the Fatou lemma the new ball in $l_1^*(N)$ is w^*-sequentially closed and since $l_1(N)$ is separable, it is also w^*-closed ([27], p. 273). Thus $|||f|||$ is a dual norm of some equivalent norm of $l_1(N)$, denoting by $|||x|||$.

5. An example of a subspace P of a Banach space which has a unique norm preserving extension property but not the property $\tilde{U}$ ($\dim X|P = 1$) (For the definition of $\tilde{U}$, see p. 18).

Take X, $|||\cdot|||$ from the example 4, L being a subspace of X^* generated by h_0. L is w^*-closed since one dimensional. Therefore there exists a subspace $P \subset X$, deficiency of P equal to 1 such that $L = P^{\perp}$.

Since X^* is rotund, L has the Chebyshev property, and therefore, using the Theorem of R. R. Phelps ([34], p. 240), asserting that the Chebyshev and extension propetries are in some sense dual, we immediately see that P has a unique norm preserving extension property. But since X^* is not uniformly rotund in the direction h_0, L does not have S property (Proposition 9) and this P does not have $\tilde{U}$ property (Proposition 11).

CHAPTER 5

An application to a fixed point theory

DEFINITION 4. Let C be a subset of a Banach space X. A mapping T: $C \to C$ is said to be *nonexpansive on* C if $\|Tx-Ty\| \leqslant \|x-y\|$ whenever $x, y \in C$.

DEFINITION 5. Let C be a bounded subset of a Banach space X, $\operatorname{diam} C$ denotes its diameter. The point $x \in C$ is said to be a *diametral point of* C if $\sup\limits_{y \in C} \|x-y\| = \operatorname{diam} C$.

DEFINITION 6. A convex subset $C \subset X$ is said to have *normal structure* if every bounded convex subset $C_1 \subset C$ which cointains more than one point contains a point which is not diametral of C_1.

THEOREM (W. A. Kirk ([21], p. 1004)). *Let X be a reflexive Banach space, C be a bounded closed convex subset of X which has normal structure. Then every nonexpansive mapping T of C into itself has a fixed point.*

It is easy to see that every convex subset of a (WUR) or (W^*UR) Banach space has normal structure ([44], p. 431). Moreover, we have.

PROPOSITION 23. *Suppose X is a Banach space which is uniformly rotund in every direction. Then every convex subset of X has normal structure.*

Proof. (Analogical to the case of a uniformly rotund space — see for example [44], p. 431). It is sufficient to prove that every bounded convex subset of X which contains more than one point contains a point being not diametral. Take $x, y \in C$, $x \neq y$. Consider $u = \frac{x+y}{2} \in C$. It is easy to see that u is not diametral. For it suppose $v_n \in C$ is a sequence such that $\|u-v_n\| \to \operatorname{diam} C$. We have $\|x-v_n\| \leqslant \operatorname{diam} C$, $\|y-v_n\| \leqslant \operatorname{diam} C$, $\|\frac{1}{2}(x-v_n+y-v_n)\| \to \operatorname{diam} C$, $x-v_n-(y-v_n) = x-y$. Thus $x = y$ since X is uniformly rotund in every direction (see Proposition 1).

CHAPTER 6

On one Mazur's theorem

In this chapter we prove a uniformization of one Mazur's result concerning the fact that in reflexive Banach space with Fréchet differentiable norm each closed convex bounded subset K is the intersection of all balls $B \supset K$.

This result was improved by R. R. Phelps ([34], p. 977). In the following, $K_r(x) = \{y \in X;\ \|y-x\| \leqslant r\}$. First of all, we will need the following simple and obvious fact:

LEMMA 2. *Suppose X is a Banach space, $K_r(x) \subset X$ some ball such that $\varrho = \varrho(K_r(x), 0) > 0$. Then for every $r' > r$ there exists a ball $K_{r'}(y) \supset K_r(x)$, $\varrho(K_{r'}(y), 0) = \varrho$.*

Proof. Denote $\delta(K_r(x)) \cap Rx = \{u, v\}$ where $\|u\| < \|v\|$. Then $\|x\| > r$ and $u = x\left(1 - \frac{r}{\|x\|}\right)$.

Now, if $p \in K_r(x)$ consider the line segment $[p, 0]$. Denote by q the Minkowski functional of $K_r(x)$ with respect to x. Since $q(0) > 1$, $q(p) \leqslant 1$, q is continuous on X, there exists $p' \in [p, 0]$ such that $q'(p') = 1$ which means $p' \in \delta(K_r(x))$. Since $p' \in [p, 0]$, we have $\|p'\| \leqslant \|p\|$. Therefore $\|x\| = \|u\| + r = \varrho(x, 0) \leqslant \varrho(x, p') + \varrho(p', 0) \leqslant r + \|p\|$. Thus $\|p\| \geqslant \|u\|$. Now take for an arbitrary $r' > r$ the ball $K_{r'}\left((x + (r'-r)\frac{x}{\|x\|}\right)$. Then if $\|y - x\| \leqslant r$ we have $\left\|y - x - (r'-r)\frac{x}{\|x\|}\right\| \leqslant r'$. Therefore $K_{r'}\left(x + (r'-r)\frac{x}{\|x\|}\right) \supset K_r(x)$. We have $\delta\left(K_{r'}\left(x + (r'-r)\frac{x}{\|x\|}\right)\right) \cap Rx = \left\{u, u + \frac{2r'}{\|x\|}x\right\}$, $\left\|u + \frac{2r'}{\|x\|}x\right\| > \|u\|$. Furthermore, as in the first part of our proof we may prove that $\varrho\left(K_{r'}\left(x + (r'-r)\frac{x}{\|x\|}\right), 0\right) = \|u\|$. Now we recall the following well known statement:

LEMMA 3. *If the norm of X is uniformly Gâteaux (uniformly Fréchet) differentiable, then the limit $\lim\limits_{t \to 0} \frac{\|x+th\| - \|x\|}{t} = D\|\cdot\|(x, h)$ is uniform on*

$(x, h) \in S_1 \times K$ *where* K *is an arbitrary norm compact* (*bounded*) *set of* X.

Proof. Suppose the norm of X is uniformly Gâteaux differentiable. Assume our limit is not uniform on some $S_1 \times K$ where K is a norm compact set of X. Wirte $\|x+th\| - \|x\| = D\|\cdot\|(x, th) + w(x, th)$ then the last fact means there exist $\varepsilon > 0$, $x_n \in S_1$, $h_n \in K$, $t_n \in R$, $t_n \neq 0$, $\lim t_n = 0$, such that $\frac{w(x_n, t_n h_n)}{t_n} \geqslant \varepsilon$. Suppose without loss of generality $h_n \to h \in K$. We have

$$\left|\frac{w(x_n, t_n h)}{t_n}\right| \geqslant \left|\frac{w(x_n, t_n h_n)}{t_n}\right| - \left(\left|\frac{\|x_n + t_n h\| - \|x_n + t_n h_n\|}{t_n}\right| + \right.$$

$$\left. + \left|D\|\cdot\|(x_n, h_n) - D\|\cdot\|(x_n, h)\right|\right)$$

$$\geqslant \left|\frac{w(x_n, t_n h_n)}{t_n}\right| - (\|h_n - h\| + \|h_n - h\|) \geqslant \frac{1}{2}\varepsilon \quad \text{if } n \geqslant n_0.$$

This gives a contradiction since we have $h \neq 0$ and

$$\lim_{n\to\infty}\left(\frac{\left\|x_n + t_n' \frac{h}{\|h\|}\right\| - \|x_n\|}{t_n'} - D\|\cdot\|\left(x_n, \frac{h}{\|h\|}\right)\right) = 0$$

whenever $t_n' \neq 0$, $\lim t_n' = 0$, by our assumption on the norm of X and therefore if we put $t_n' = \|h\| \cdot t_n$,

$$\lim_{n\to\infty}\left(\frac{\left\|x_n + \frac{t_n'}{\|h\|} h\right\| - \|x_n\|}{\frac{t_n'}{\|h\|}} - D\|\cdot\|(x_n, h)\right) = \lim_{n\to\infty}\frac{w(x_n, t_n h)}{t_n} = 0.$$

Now consider the case of uniformly Fréchet differentiability. If our conclusion were not satisfied for some bounded $M \subset X$ we would have there exist $x_n \in S_1$, $h_n \in M$, $h_n \neq 0$, $t_n \in R$, $t_n \neq 0$, $\lim t_n = 0$ such that $\left|\frac{w(x_n, t_n h_n)}{t_n}\right| \geqslant \varepsilon > 0$. But we have for our x_n, h_n and arbitrary $t_n' \to 0$

$$\frac{1}{t_n'} w\left(x_n, t_n', \frac{h_n}{\|h_n\|}\right) \to 0.$$

Therefore for arbitrary $t_n' \to 0$

$$\left[w\left(x_n, \frac{t_n'}{\|h_n\|} h_n\right)\right]\left(\frac{t_n'}{\|h_n\|}\right)^{-1} \to 0$$

since $\{h_n)$ is bounded. If we put $t_n' = \|h_n\| \cdot t_n$ we obtain a contradiction.

PROPOSITION 24. *Suppose X has uniformly Gâteaux differentiable norm. Let K to a norm compact subset of X, $a > 0$ an arbitrary positive number such that $\sup_{x \in K} \|x\| \geqslant a$. Then there exists $r > 0$ such that for every convex $M \subset K$, $\varrho(M, 0) \geqslant a$ there exists $x \in X$ such that $K_r(x) \supset M$, $0 \notin K_r(x)$.*

Proof. Denote by $\mathfrak{a}$ the system of all convex sets M with the properties that $M \subset K$ and $\varrho(0, M) \geqslant a$, for a given K, a. The statement will be proved if we prove:

There exists $p > \frac{1}{2}a$ such that for every $M \in \mathfrak{a}$ there exists $x \in S_1 \subset X$ such that for every $y \in M$ we have $\|px - y\| \leqslant p - \frac{1}{2}a$. Suppose this is not true. It means for every $p > \frac{1}{2}a$ there exists $M_p \in \mathfrak{a}$ such that for every $x \in S_1$ there exists $y_{p,x} \in M_p$ such that $\|px - y_{p,x}\| > p - \frac{1}{2}a$. Now take for every $p > \frac{1}{2}a$, $f'_p \in S_1^*$ such that $f'_p(y) \geqslant a$ for $y \in \bar{M}_p$ ($\bar{M}_p$ is the closure of M_p), by the well known theorem ([4], p. 345). Using the theorem of E. Bishop and R. R. Phelps on subreflexivity of every Banach space, [5], p. 97, take some $f_p \in S_1^*$ such that $f_p(y) \geqslant \frac{3}{4}a$ for $y \in \bar{M}_p$ and there exists $x_p \in S_1$ such that $f_p(x_p) = 1$.

Denote $z_p = -\dfrac{y_{p,x_p}}{p}$. Than we have

$$\|x_p + z_p\| - 1 = f_p(z_p) + w(x_p, z_p)$$

since $D\|\cdot\|(x_p, z_p) = f_p(z_p)$ by well known theorem of S. Mazur ([14], p. 84). Further

$$p\,w(x_p, z_p) = \|px_p - y_{p,x_p}\| - p + f(y_{p,x_p}) \geqslant p - \tfrac{1}{2}a - p + \tfrac{3}{4}a = \tfrac{1}{4}a.$$

Therefore we have

$$p\,w\left(x_p, \frac{-y_{p,x_p}}{p}\right) = p\,w(x_p, z_p) \geqslant \tfrac{1}{4}a > 0.$$

Since $y_{p,x_p} \in K$ and $x_p \in S_1 \subset X$, this fact gives a contradiction with uniform Gâteaux differentiability of the norm of X (Lemma 3).

Similarly we may prove the following statement:

PROPOSITION 25. *Suppose the Banach space X has uniformly Fréchet differentiable norm. Then for every two positive numbers $a \leqslant b$ there exists $r > 0$ with the following property:*

Whenever M is a convex set in X such that $\varrho(0, M) \geqslant a$ and $M \subset K_b(a)$ then there exists a ball $K_r(x)$ such that $M \subset K_r(x) \not\ni 0$.

COROLLARY 1. *Suppose the Banach space X has uniformly Fréchet differentiable norm. Then for every two positive numbers $a \leqslant b$ there exists $r > 0$ such that whenever a convex set M satisfies $\varrho(M, 0) \geqslant a$,* diam *$M \leqslant b$, there exists a ball $K_r(x)$ such that $M \subset K_r(x) \not\ni 0$.*

Proof. By Proposition 25 there exists $r' > 0$ for a given a, b such that whenever a convex set M satisfies $M \subset K_{3b}(0)$, $\varrho(M_1 0) \geqslant a$ then there exists a ball $K_{r'}(x')$ such that $K_{r'}(x') \subset M$, $K_{r'}(x') \not\ni 0$. Take $r = \max(r', b)$. Suppose M is an arbitrary convex set in X such that $\varrho(0, M) \geqslant a$, diam $M \leqslant b$. If $M \subset K_{3b}(0)$ then there exists a ball $K_{r'}(x')$ such that $K_{r'}(x') \supset M$ and $0 \notin K_{r'}(x')$. Now, using Lemma 2 we have there exists a ball $K_r(x)$ such that $K_r(x) \supset M, 0 \notin K_r(x)$. If $M \cap \big(X \dot{-} K_{3b}(0)\big) \neq \emptyset$, then take an arbitrary point z of $M \cap \big(X \dot{-} K_{3b}(0)\big)$ and the ball $K_b(z)$ has the property that $K_b(z) \supset M$, $0 \notin K_b(z)$. Now again using Lemma 2 there exists a ball $K_r(y) \supset M$, $0 \notin K_r(y)$.

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DISSERTATIONES
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ERRATA

Page line	For	Read
5_6	δ_x	δ_X
8_7	(x, y)	(x_n, y_n)
21_{13}	$Tf_n^* + Tg_n^*$	$T^* f_n + T^* g_n$
22^9	Fx	$F(x)$
25_{10}	$\{2, 0, 0, \ldots\}$	$\{1, 0, 0, \ldots\}$
25_{10}	(1) - (7)	(7)