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Approximation of functions possessing derivatives of positive orders

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Abstract. In this note we deduce some estimates for the best trigonometric approximation in the mean of $2\text{-}\pi$periodic functions f of one variable. Moreover, two theorems concerning the order of approximation of $f^{(r)}$ by suitable trigonometric polynomials are given.

1. Introduction. Let L^p $(1 \leqslant p \leqslant \infty)$ be the class of all 2π-periodic real-valued functions Lebesgue-integrable with p-th power [essentially bounded if $p = \infty$] over the interval $\langle -\pi, \pi\rangle$. Write L instead of L^1. Denote by $\omega_k(\delta, f)_{L^p}$ and $E_n(f)_{L^p}$, respectively, the k-th modulus of smoothness and the best trigonometric approximation of $f \in L^p$, in the L^p-norm:

$$\|f(\cdot)\|_{L^p} = \Big\{\int\limits_{-\pi}^{\pi} |f(x)|^p dx\Big\}^{1/p} \quad \text{if } 1 \leqslant p \leqslant \infty,$$

$$\|f(\cdot)\|_{L^\infty} = \operatorname*{ess\,sup}_{\langle -\pi,\pi\rangle} |f(x)|.$$

(see [6], p. 115, 41).

Suppose that

$$S[f] = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

is the Fourier series of $f \in L$ for which the integral over $\langle -\pi, \pi\rangle$ is zero, so that $c_0 = 0$. Given any $a > 0$, we define the a-th integral of f by the identity

$$I_a(x, f) = \sum_{k=-\infty}^{\infty}{}' c_k (ik)^{-a} e^{ikx},$$

where

$$(ik)^{-a} = |k|^{-a} \exp(-\tfrac{1}{2}\pi i a \operatorname{sign} k);$$

the dash $'$ indicates that the term $k = 0$ is omitted in summation. As is well known ([7], p. 134), $f_a(x) = I_a(x, f)$ exists possibly for almost every x, is Lebesgue-integrable and $S[f_a] = f_a(x)$ a.e.

If $0 < a < 1$, the derivative $f^{(a)}(x)$ of f is defined by the formula

$$f^{(a)}(x) = \frac{d}{dx} I_{1-a}(x, f),$$

provided the right side exists. We set

$$f^{(a+r)}(x) = \big(f^{(a)}(x)\big)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-a}(x, f)$$

for positive integers r.

Using the above notions, we shall present the L^p-analogues of two theorems given in [6], p. 316, 555.

The suitable positive constants, depending on the parameters $\gamma, \eta, \ldots$ only, will be signified by $C_j(\gamma, \eta, \ldots)$ $(j = 1, 2, \ldots)$.

2. Auxiliary results. Considering 2π-periodic functions $f \in L$ such that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

we shall prove the following

LEMMA. *Suppose that, for a certain positive $a < 1$, the function*

$$g(x) = I_{1-a}(x, f)$$

is of bounded variation over the interval $\langle -\pi, \pi \rangle$. Then

$$\omega_1(\delta, f)_L \leqslant C_1(a) M \delta^a \quad \textit{if } 0 < \delta \leqslant \pi,$$

where

$$M = \frac{2}{\pi} \operatorname*{var}_{-\pi \leqslant x \leqslant \pi} g(x).$$

Proof. As is well known ([3], p. 38),

$$\omega_1(\delta, g)_L \leqslant M\delta \quad \text{when } 0 < \delta \leqslant \pi.$$

Clearly,

$$g(x) = \sum_{k=-\infty}^{\infty}{}' c_k (ik)^{a-1} e^{ikx} \quad \text{for every } x$$

and

$$I_a(x, g) = \sum_{k=-\infty}^{\infty}{}' c_k (ik)^{a-1} (ik)^{-a} e^{ikx} = f_1(x)$$

uniformly in $x \in (-\infty, \infty)$. Hence

$$I_a(x, g) = f_1(x) \quad \text{for every } x.$$

Putting

$$F(x) = \int_0^x f(t)\,dt,$$

we have

$$F(x) = \sum_{k=-\infty}^{\infty}{}' c_k (ik)^{-1}(e^{ikx}-1) = f_1(x) - f_1(0)$$

uniformly in $x \in (-\infty, \infty)$. Consequently,

$$F(x) = I_a(x, g) - I_a(0, g) \quad \text{for all } x.$$

In view of the Lebesgue dominated convergence theorem,

$$\begin{aligned} I_a(x, g) &= \sum_{k=-\infty}^{\infty}{}' \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ikt} dt \cdot (ik)^{-a} e^{ikx} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty}{}' (ik)^{-a} \int_{-\pi}^{\pi} g(t) e^{ik(x-t)} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \Psi_a(x-t)\, dt \quad \text{for every } x, \end{aligned}$$

where

$$\Psi_a(u) = \sum_{k=-\infty}^{\infty}{}' (ik)^{-a} e^{iku} = 2 \sum_{k=1}^{\infty} \frac{\cos\left(ku - \frac{a\pi}{2}\right)}{k^a}.$$

Therefore, for all real x,

$$(1) \qquad F(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \Psi_a(x-t)\, dt - f_1(0), \qquad \int_{-\pi}^{\pi} \Psi_a(x-t)\, dt = 0.$$

Given positive numbers h, λ $(2\lambda < h \leqslant \pi)$, we have

$$\begin{aligned} J_\lambda &\equiv \int_{-\pi}^{\pi} \left| \frac{F(x+h+\lambda) - F(x+h-\lambda)}{2\lambda} - \frac{F(x+\lambda) - F(x-\lambda)}{2\lambda} \right| dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \{g(x+h-u) - g(x-u)\} \frac{\Psi_a(u+\lambda) - \Psi_a(u-\lambda)}{2\lambda} du \right| dx \\ &\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \int_{-\pi}^{\pi} |g(x+h-u) - g(x+h) - g(x-u) + g(x)|\, dx \times \right. \\ &\qquad\qquad \left. \times \left| \frac{\Psi_a(u+\lambda) - \Psi_a(u-\lambda)}{2\lambda} \right| \right\} du. \end{aligned}$$

By the mean-value theorem,

$$J_\lambda \leqslant \frac{1}{\pi} M \Bigg\{ \int_{|u|\leqslant 2\lambda} |u| \frac{|\Psi_a(u+\lambda)|+|\Psi_a(u-\lambda)|}{2\lambda}\, du +$$

$$+ \int_{2\lambda\leqslant|u|\leqslant h} |u|\, |\Psi_a^{(1)}(u-\lambda+2\vartheta\lambda)|\, du + \int_{h\leqslant|u|\leqslant\pi} h\, |\Psi_a^{(1)}(u-\lambda+2\vartheta\lambda)|\, du \Bigg\}$$

$$= \frac{1}{\pi} M\{P+Q+R\}, \quad \text{with some } \vartheta = \vartheta(u,\lambda) \quad (0<\vartheta<1).$$

It is known ([7], p. 136) that

$$|\Psi_a(t)| \leqslant C_2(a)|t|^{a-1}, \quad |\Psi_a^{(1)}(t)| \leqslant C_3(a)|t|^{a-2} \quad \text{if } 0<|t|\leqslant 3\pi/2.$$

Hence

$$P \leqslant C_2(a) \int_{|u|\leqslant 2\lambda} \{|u+\lambda|^{a-1}+|u-\lambda|^{a-1}\}\, du$$

$$= 2C_2(a)\left\{\frac{\lambda^a}{a}+\frac{(3\lambda)^a}{a}\right\} = 2C_2(a)\frac{1+3^a}{a}\lambda^a,$$

$$Q \leqslant C_3(a) \int_{2\lambda\leqslant|u|\leqslant h} |u|\, |u-\lambda+2\vartheta\lambda|^{a-2}\, du$$

$$\leqslant C_3(a)\left\{\int_{2\lambda}^{h} \frac{udu}{(u-\lambda)^{2-a}} - \int_{-h}^{-2\lambda} \frac{udu}{(-u-\lambda)^{2-a}}\right\}$$

$$\leqslant C_3(a)\left\{\int_{\lambda}^{h-\lambda} \frac{v+\lambda}{v^{2-a}}\, dv + \int_{\lambda}^{h-\lambda} \frac{w+\lambda}{w^{2-a}}\, dw\right\}$$

$$= C_3(a)\left\{2\frac{(h-\lambda)^a-\lambda^a}{a} + 2\lambda\frac{\lambda^{a-1}-(h-\lambda)^{a-1}}{1-a}\right\}$$

$$= 2C_3(a)\left\{\frac{(h-\lambda)^a-\lambda^a}{a}+\frac{\lambda^a}{1-a}\right\} \leqslant 2C_3(a)\left\{\frac{(h-\lambda)^a}{a}+\frac{\lambda^a}{1-a}\right\},$$

$$R \leqslant C_3(a) \int_{h\leqslant|u|\leqslant\pi} h\, |u-\lambda+2\vartheta\lambda|^{a-2}\, du = C_3(a)\left\{\int_h^\pi \frac{h}{|u-\lambda+2\vartheta\lambda|^{2-a}} du + \right.$$

$$\left. + \int_{-\pi}^{-h} \frac{h}{|u-\lambda+2\vartheta\lambda|^{2-a}}\, du\right\} \leqslant C_3(a)\left\{\int_h^\pi \frac{h}{(u-\lambda)^{2-a}}\, du + \int_{-\pi}^{-h} \frac{h}{(-u-\lambda)^{2-a}}\, du\right\}$$

$$= 2C_3(a)h\frac{(h-\lambda)^{a-1}-(\pi-\lambda)^{a-1}}{1-a} < 2C_3(a)\, h\, \frac{(h-\lambda)^{a-1}}{1-a}.$$

Consequently,

$$J_\lambda \leqslant \frac{2}{\pi} M\left\{C_2(a)\frac{1+3^a}{a}\lambda^a + C_3(a)\left(\frac{(h-\lambda)^a}{a}+\frac{\lambda^a}{1-a}\right)+C_3(a)\frac{h(h-\lambda)^{a-1}}{1-a}\right\}.$$

Applying the Fatou lemma, we obtain

$$\int_{-\pi}^{\pi} |F^{(1)}(x+h)-F^{(1)}(x)|\,dx \leqslant \lim_{\lambda\to 0+} I_\lambda \leqslant \frac{2}{\pi} MC_3(a)\left(\frac{h^a}{a}+\frac{h^a}{1-a}\right),$$

i.e.,

$$\int_{-\pi}^{\pi} |f(x+h)-f(x)|\,dx \leqslant \frac{2}{\pi}\left(\frac{1}{a}+\frac{1}{1-a}\right)C_3(a)Mh^a \qquad (0<h\leqslant\pi)$$

and the desired assertion follows.

Analogously we can show that if the measurable function

$$g(x) = I_{1-a}(x, f) \qquad \text{for some positive } a<1$$

is of bounded second variation in the sense of Kharšiladze [1] over $\langle -\pi, \pi\rangle$, then

$$\omega_2(\delta, f)_L = O(\delta^a) \qquad \text{as } \delta\to 0+.$$

Two corresponding results for the modulus $\omega_k(\delta, f)_{L^p}$ $(1<p<\infty$, $k=1,2)$ also hold whenever g is the integral of a function φ, Lebesgue-integrable over $\langle -\pi, \pi\rangle$ and such that

$$\text{(i)} \int_{-\pi}^{\pi} |\varphi(x)|^p dx < \infty, \qquad \text{(ii)} \operatorname*{ess\,sup}_{\langle 0,\pi\rangle} \int_{-\pi}^{\pi} |\varphi(x+u)-\varphi(x-u)|^p dx < \infty,$$

respectively.

3. Approximation of differentiable functions. Let us start with the following

THEOREM 1. *Suppose that the 2π-periodic function f possesses a derivative $f^{(r-1)}$ of non-negative integer order $r-1$, absolutely continuous in $\langle -\pi, \pi\rangle$ and such that $f^{(r)}\in L^p$ $(1\leqslant p\leqslant\infty)$. Then,*

$$E_n(f)_{L^p} \leqslant \frac{C_4(r)}{(n+1)^r} E_n(f^{(r)})_{L^p} \qquad \textit{for } n = 0, 1, 2, \ldots$$

Proof. Consider the case $r=1$. Write

$$S[f] = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \quad g(x) = I_1(x, f^{(1)}).$$

Clearly,

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = f(x) \quad \text{uniformly in } x,$$

$$S[f^{(1)}] = \sum_{k=-\infty}^{\infty}{}' c_k ike^{ikx}, \quad g(x) = \sum_{k=-\infty}^{\infty}{}' c_k e^{ikx} = f(x) - c_0$$

or every x. Hence

$$E_n(f)_{L^p} = E_n(g)_{L^p} \quad \text{for } n = 0, 1, 2, \ldots$$

and

$$f^{(1)}(x) = g^{(1)}(x) \quad \text{a.e.}$$

The partial integration gives

$$2\pi c_k = \int_{-\pi}^{\pi} f(t) e^{-ikt} dt = \frac{1}{ik} \int_{-\pi}^{\pi} g^{(1)}(t) e^{-ikt} dt.$$

Consequently,

$$g(x) = \sum_{k=-\infty}^{\infty}{}' c_k e^{ikx} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty}{}' \frac{1}{ik} \int_{-\pi}^{\pi} g^{(1)}(t) e^{ik(x-t)} dt$$

and by the Lebesgue theorem,

$$g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g^{(1)}(t) \Psi_1(x-t) dt,$$

where

$$\Psi_1(u) = \sum_{k=-\infty}^{\infty}{}' (ik)^{-1} e^{iku} = 2 \sum_{k=1}^{\infty} \frac{\sin ku}{k}.$$

As is well known ([6], p. 316), for each trigonometric polynomials $U_n(x)$, $V_n(x)$ of the order n at most, there is a trigonometric polynomial $T_n(x)$ such that

$$g(x) - T_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{g^{(1)}(t) - U_n(t)\}\{\Psi_1(x-t) - V_n(x-t)\} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{g^{(1)}(x-s) - U_n(x-s)\}\{\Psi_1(s) - V_n(s)\} ds.$$

In view of Minkowski's generalized inequality,

$$2\pi\Big\{\int_{-\pi}^{\pi}|g(x)-T_n(x)|^p dx\Big\}^{1/p}$$
$$\leqslant \int_{-\pi}^{\pi}|\Psi_1(s)-V_n(s)|\Big\{\int_{-\pi}^{\pi}|g^{(1)}(x-s)-U_n(x-s)|^p dx\Big\}^{1/p} ds$$
$$\leqslant \Big\{\int_{-\pi}^{\pi}|g^{(1)}(t)-U_n(t)|^p dt\Big\}^{1/p}\cdot\int_{-\pi}^{\pi}|\Psi_1(s)-V_n(s)|\,ds;$$

whence

$$E_n(g)_{L^p}\leqslant\frac{1}{2\pi}E_n(g^{(1)})_{L^p}\cdot E_n(\Psi_1)_L.$$

By a suitable theorem of Jackson's type ([2], p. 73–78),

$$E_n(\Psi_1)_L\leqslant 6\omega_1\Big(\frac{1}{n+1},\Psi_1\Big)_L \quad \text{for } n=0,1,2,\ldots$$

Since

$$\Psi_1(u)=\begin{cases}\pi-u & \text{if } 0<u<2\pi,\\ 0 & \text{if } u=0,\ u=2\pi,\end{cases}$$

i.e.,

$$\operatorname*{var}_{-\pi\leqslant u\leqslant\pi}\Psi_1(u)=\operatorname*{var}_{0\leqslant u\leqslant 2\pi}\Psi_1(u)=4\pi,$$

we have

$$E_n(\Psi_1)_L\leqslant 6\cdot\frac{2}{\pi}\cdot 4\pi\cdot\frac{1}{n+1}=\frac{48}{n+1}\qquad\text{for } n=0,1,2,\ldots$$

Thus,

$$E_n(g)_{L^p}\leqslant\frac{48}{2\pi(n+1)}E_n(g^{(1)})_{L^p}$$

and, for $r=1$, the proof is completed.

If $r=2,3,\ldots$, we apply the induction argument.

Evidently, the theorem of Jackson's type

$$E_n(f^{(r)})_{L^p}\leqslant C_5(k)\,\omega_k\Big(\frac{1}{n+1},f^{(r)}\Big)_{L^p}$$

([6], p. 274–275) and estimates for the modulus $\omega_k(\delta,f^{(r)})_{L^p}$ lead, by Theorem 1, to some special inequalities concerning $E_n(f)_{L^p}$.

THEOREM 2. *Given any p ($1\leqslant p\leqslant\infty$), let us consider the trigonometric polynomials $T_n(x)=T_n(x,f)$ of order n at most, such that*

(2) $$\|f(\cdot)-T_n(\cdot,f)\|_{L^p}\leqslant C_6(p)E_n(f)_{L^p}\quad(n=0,1,2,\ldots)$$

for all functions f satisfying the conditions of Theorem 1, with some positive integer r. Then, for these f's,

(3) $$\|f^{(r)}(\cdot)-T_n^{(r)}(\cdot,f)\|_{L^p}\leqslant C_7(p,r)E_n(f^{(r)})_{L^p}\quad (n=0,1,2,\ldots).$$

Proof. Let $S_\nu(x,f)$ denotes the ν-th partial sum of the Fourier series of f, and let $W_n(x,f)$ or $W_n(f)$ be the de la Vallée-Poussin means of this series, defined by the formula

$$W_n(x,f)=\frac{1}{n+1}\sum_{\nu=n}^{2n}S_\nu(x,f)\quad (n=0,1,2,\ldots).$$

As is well known, for any $f\in L$,

$$W_n(x,f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\Phi_n(t)\,dt,$$

where

$$\Phi_n(t)=\frac{1}{n+1}\sum_{\nu=n}^{2n}\frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t}=\frac{\sin(n+\frac{1}{2})t\cdot\sin(\frac{3}{2}n+\frac{1}{2})t}{2(n+1)(\sin\frac{1}{2}t)^2},$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\Phi_n(t)\,dt=1,\qquad \frac{1}{\pi}\int_{-\pi}^{\pi}|\Phi_n(t)|\,dt\leqslant 2\,\frac{4n+1}{2n+2}<4.$$

Moreover,

$$W_n(x,f^{(r)})=W_n^{(r)}(x,f)\quad\text{whenever } f,f^{(r)}\in L.$$

In view of Minkowski's inequality,

$$\|f^{(r)}(\cdot)-T_n^{(r)}(\cdot,f)\|_{L^p}\leqslant\|f^{(r)}(\cdot)-W_n(\cdot,f^{(r)})\|_{L^p}+$$
$$+\big\|T_n^{(r)}\big(\cdot,W_n(f)\big)-T_n^{(r)}(\cdot,f)\big\|_{L^p}+\big\|W_n^{(r)}(\cdot,f)-T_n^{(r)}\big(\cdot,W_n(f)\big)\big\|_{L^p}$$

for every $f\in L$ such that $f^{(r)}\in L^p$.

Denoting by $T_n^*(x,f)$ the trigonometric polynomial of best approximation of f in the L^p-metric, of the order n at most, we have

$$\begin{aligned}\|f^{(r)}(\cdot)-W_n(\cdot,f^{(r)})\|_{L^p}&\leqslant\|f^{(r)}(\cdot)-T_n^*(\cdot,f^{(r)})\|_{L^p}+\|T_n^*(\cdot,f^{(r)})-W_n(\cdot,f^{(r)})\|_{L^p}\\&=E_n(f^{(r)})_{L^p}+\big\|W_n\big(\cdot,T_n^*(f^{(r)})-f^{(r)}\big)\big\|_{L^p}\\&\leqslant 5\,E_n(f^{(r)})_{L^p}.\end{aligned}$$

The Zygmund inequality ([7], p. 11) gives

$$\big\|T_n^{(r)}\big(\cdot,W_n(f)\big)-T_n^{(r)}(\cdot,f)\big\|_{L^p}\leqslant n^r\big\|T_n\big(\cdot,W_n(f)\big)-T_n(\cdot,f)\big\|_{L^p},$$

$$\begin{aligned}\big\|W_n^{(r)}(\cdot,f)-T_n^{(r)}\big(\cdot,W_n(f)\big)\big\|_{L^p}&\leqslant(2n)^r\big\|W_n(\cdot,f)-T_n\big(\cdot,W_n(f)\big)\big\|_{L^p}\\&\leqslant C_6(p)\cdot(2n)^rE_n\big(W_n(f)\big)_{L^p}.\end{aligned}$$

Further,

$$\|T_n(\cdot, W_n(f)) - T_n(\cdot, f)\|_{L^p}$$
$$\leqslant \|T_n(\cdot, W_n(f)) - W_n(\cdot, f)\|_{L^p} + \|W_n(\cdot, f) - f(\cdot)\|_{L^p} + \|f(\cdot) - T_n(\cdot, f)\|_{L^p}$$
$$\leqslant C_6(p) E_n(W_n(f))_{L^p} + 5E_n(f)_{L^p} + C_6(p) E_n(f)_{L^p}$$

and

$$E_n(W_n(f))_{L^p} \leqslant 4E_n(f)_{L^p}.$$

Thus,

$$\|f^{(r)}(\cdot) - T_n^{(r)}(\cdot, f)\|_{L^p} \leqslant 5E_n(f^{(r)})_{L^p} + 5n^r(C_6(p)+1) E_n(f)_{L^p} + $$
$$+ 4C_6(p)(2n)^r E_n(f)_{L^p}$$

and, by Theorem 1, the desired result follows (cf. [5], p. 233–234; [6], p. 350–351).

4. Case of derivatives of non-integer orders. Assuming that

$$f \in L \quad \text{and} \quad \int_{-\pi}^{\pi} f(t)\,dt = 0,$$

an extension of Theorem 1 will now be given.

THEOREM 3. *Suppose that the function*

$$g(x) = I_{1-a}(x, f),$$

with a positive $a \in (0, 1)$, *is absolutely continuous in* $\langle -\pi, \pi\rangle$, *and that the derivative* $g^{(1)}(x) = f^{(a)}(x)$ *is of class* L^p $(1 \leqslant p \leqslant \infty)$. *Then*

$$E_n(f)_{L^p} \leqslant \frac{C_8(a)}{(n+1)^a} E_n(f^{(a)})_{L^p} \quad \textit{for } n = 0, 1, 2, \ldots \tag{4}$$

If, moreover, the derivatives $f^{(a)}(x), f^{(a+1)}(x), \ldots, f^{(a+\varrho-1)}(x)$, *with a positive integer* ϱ, *are absolutely continuous in* $\langle -\pi, \pi\rangle$ *and if* $f^{(a+\varrho)} \in L^p$ $(1 \leqslant p \leqslant \infty)$, *then*

$$E_n(f)_{L^p} \leqslant \frac{C_8(a) C_4(\varrho)}{(n+1)^{a+\varrho}} E_n(f^{(a+\varrho)})_{L^p} \quad \textit{for } n = 0, 1, 2, \ldots \tag{5}$$

Proof. We start with identities (1) and we observe that for each trigonometric polynomials $U_n(x)$, $V_n(x)$ of the order n at most, there is a trigonometric polynomial $T_n(x)$ such that

$$F(x) - T_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{g(x-s) - U_n(x-s)\}\{\Psi_a(s) - V_n(s)\}\,ds - I_a(0, g).$$

Assuming that $0 < \lambda \leqslant \pi/2$, we have

$$\left\{\int_{-\pi}^{\pi}\left|\frac{F(x+\lambda)-T_n(x+\lambda)-F(x-\lambda)+T_n(x-\lambda)}{2\lambda}\right|^p dx\right\}^{1/p}$$

$$=\frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}\frac{g(x+\lambda-s)-U_n(x+\lambda-s)-g(x-\lambda-s)+U_n(x-\lambda-s)}{2\lambda}\times\right.\right.$$

$$\left.\left.\times\{\Psi_a(s)-V_n(s)\}ds\right|^p dx\right\}^{1/p} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|\Psi_a(s)-V_n(s)|\times$$

$$\times\left\{\int_{-\pi}^{\pi}\left|\frac{g(x+\lambda-s)-g(x-\lambda-s)}{2\lambda}-\frac{U_n(x+\lambda-s)-U_n(x-\lambda-s)}{2\lambda}\right|^p dx\right\}^{1/p} ds.$$

Let $U_n(t)$, $V_n(s)$ be the trigonometric polynomials of best approximation of $g(t)$ and $\Psi_a(s)$ in L^p and L-metrics, respectively. Then,

$$2\pi\left\{\int_{-\pi}^{\pi}\left|\frac{F(x+\lambda)-F(x-\lambda)}{2\lambda}-\frac{T_n(x+\lambda)-T_n(x-\lambda)}{2\lambda}\right|^p dx\right\}^{1/p}$$

$$\leqslant\left\{\int_{-\pi}^{\pi}\left|\frac{g(t+\lambda)-g(t-\lambda)}{2\lambda}-\frac{U_n(t+\lambda)-U_n(t-\lambda)}{2\lambda}\right|^p dt\right\}^{1/p}\cdot E_n(\Psi_a)_L$$

and, by Fatou's lemma (see also Theorem 2),

$$2\pi\left\{\int_{-\pi}^{\pi}|F^{(1)}(x)-T_n^{(1)}(x)|^p dx\right\}^{1/p} \leqslant \left\{\int_{-\pi}^{\pi}|g^{(1)}(t)-U_n^{(1)}(t)|^p dt\right\}^{1/p}\cdot E_n(\Psi_a)_L$$

$$\leqslant C_9 E_n(g^{(1)})_{L^p}\cdot E_n(\Psi_a)_L.$$

Hence

$$2\pi\left\{\int_{-\pi}^{\pi}|f(x)-T_n^{(1)}(x)|^p dx\right\}^{1/p} \leqslant C_9 E_n(f^{(a)})_{L^p}\cdot E_n(\Psi_a)_L$$

and, consequently,

$$E_n(f)_{L^p} \leqslant \frac{1}{2\pi} C_9 E_n(f^{(a)})_{L^p} E_n(\Psi_a)_L.$$

Applying a suitable theorem of Jackson's type ([2]) and the above lemma, we get

$$E_n(f)_{L^p} \leqslant \frac{3}{\pi} C_9 E_n(f^{(a)})_{L^p}\,\omega_1\left(\frac{1}{n+1},\Psi_a\right)_L$$

$$\leqslant \frac{3}{\pi} C_9 C_1(a) M \frac{1}{(n+1)^a} E_n(f^{(a)})_{L^p},$$

where

$$M = \frac{2}{\pi} \operatorname*{var}_{-\pi \leqslant x \leqslant \pi} \Psi_1(x) = 8.$$

Thus, estimate (4) is established.

To obtain (5), we apply (4) and Theorem 1.

THEOREM 4. *Under the assumption of Theorem* 3, *inequality* (2) *implies* (3) *for a corresponding positive non-integer* $r = a + \varrho$ $(0 < a < 1$, $\varrho = 0, 1, 2, \ldots)$.

The proof is similar to that of Theorem 2. In this case, the inequality of the Bernstein–Civin type ([6], p. 266, [4], p. 49) and the assertion of Theorem 3 are used.

We note that the above results can easily be extended to the Orlicz spaces L^{φ}.

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