INDEPENDENCE WITH RESPECT TO A FAMILY OF MAPPINGS

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Introduction. In this note I consider a certain general notion of independence (announced in [5], p. 173) which contains, as special cases, some notions defined by Grätzer [1] (see also [2]), J. Schmidt [6], Świerczkowski [7] and myself [3] (see also [4] and [5]).

I adopt the definitions and notation of my quoted papers.

$\mathcal{H} = (A; F)$ is a fixed (but arbitrary) algebra, letters $a$ and $b$ (with or without indices) denote always arbitrary elements of the carrier $A$ and the letters $f$ and $g$ — arbitrary algebraic operations in $\mathcal{H}$.

1. Extending of mapping to homomorphisms. Let us recall a proposition concerning these extensions:

(i) Let $p$ be a mapping of a non-void set $S \subseteq A$ into $A$. Then the following conditions are equivalent:

(h) there exists an extension of $p$ to a homomorphism $h$ of $C(S)$ into $A$,

(a) if $a_1, \ldots, a_{m+n} \in S$, $f \in A^{(m)}$, $g \in A^{(n)}$ ($m, n = 1, 2, \ldots$) and

$$ f(a_1, \ldots, a_m) = g(a_{m+1}, \ldots, a_{m+n}) , $$

then

$$ f(p(a_1), \ldots, p(a_m)) = g(p(a_{m+1}), \ldots, p(a_{m+n})) ; $$

(a') if $a_1, \ldots, a_n \in S$, $f, g \in A^{(m)}$ ($n = 1, 2, \ldots$) and

(*)

$$ f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) , $$

then

(**)

$$ f(p(a_1), \ldots, p(a_n)) = g(p(a_1), \ldots, p(a_n)) ; $$

(a'') if $a_1, \ldots, a_n$ are distinct elements of $S$, $f, g \in A^{(m)}$ ($n = 1, 2, \ldots$) and (*), then (**).

For proofs see [4], p. 51-52.
2. Independence with respect to a family of mappings. Let us denote by $M$ the family of all mappings $p$, whose domain ($\text{dom } p$) and range ($\text{ran } p$) are contained in $A$. In other words,

$$M = \bigcup_{S \in A} A^S.$$ 

Further, denote by $H$ the family of all mappings $p \in M$ extendable to a homomorphism of $C(\text{dom } p)$ into $A$ or, in other words, of all mappings $p \in M$ satisfying (h).

For any subfamily $Q$ of $M$ a subset $I$ of $A$ is called $Q$-independent (in symbols $I \in \text{Ind}(A, Q)$ or, shortly, $I \in \text{Ind}(Q)$), whenever every mapping of $I$ belonging to $Q$ can be extended to a homomorphism of $C(I)$ into $A$. Further, we say that a set $S \subset A$ satisfies conditions (A), (A') or (A'') if any mapping $p$ of $S$ belonging to $Q$ satisfies (a), (a') or (a''), respectively. Proposition 1(i) implies proposition

(i) For any $S \subset A$ the conditions $S \in \text{Ind}(Q)$, (A), (A') and (A'') are equivalent.

The following propositions are easily deduced from definition of $\text{Ind}(Q)$:

(ii) If $Q_1 \subset Q_2 \subset M$, then $\text{Ind}(Q_2) \subset \text{Ind}(Q_1)$.

(iii) For any $Q \subset M$ we have

$$\text{Ind} = \text{Ind}(M) \subset \text{Ind}(Q) \subset \text{Ind}(H) = 2^A.$$ 

(iv) Let $Q \subset M$ and suppose that for every $S \subset T \subset A$ and every $p \in Q \cap A^T$ there is a $q \in Q \cap A^T$ such that $q|S = p$. Then the family $\text{Ind}(Q)$ is hereditary.

Now I shall prove proposition

(v) Let $Q \subset M$ and suppose that for every $S \subset T \subset A$ and every $q \in Q \cap A^T$ we have $q|S \in Q$. Then, if every finite subset of a set $I$ is $Q$-independent, then $I$ is $Q$-independent.

Let $p \in Q \cap A^I$. In view of 2(i) it suffices to prove that $p$ satisfies (a'). We suppose

$$S = \{a_1, \ldots, a_n\} \subset I$$

and

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n).$$

Hence, by hypothesis, $p|S \in Q$ and, consequently, by hypothesis on finite subsets of $I$, we have

$$f(p(a_1), \ldots, p(a_n)) = g(p(a_1), \ldots, p(a_n)), \quad \text{q.e.d.}$$

3. $S$-independence and $S_0$-independence. Let us denote by $S$ the family of all mappings $p \in M$ with $\text{ran } p \subset C(\text{dom } p)$ and by $S_0$ the family of all mappings $p \in M$ with $\text{ran } p \subset \text{dom } p$. 

The $S$-independence (in the sense of section 2) coincides with the independence 
"in sich", considered by Schmidt [6] and the $S_0$-independence — with the weak independence considered by Świerczkowski [7]. In view of the proposition of the preceding section we have

(i) $\text{Ind} \subseteq \text{Ind}(S) \subseteq \text{Ind}(S_0)$.

(ii) Families $\text{Ind}(S)$ and $\text{Ind}(S_0)$ are hereditary.

Moreover it is easy to check that

(iii) If $A$ is functionally complete, or, more generally, if every element of $A$ is an algebraic constant in $A$, then $\text{Ind}(S) = \text{Ind} = \{\emptyset\}$.

In fact, in view of (ii) it suffices to prove that no one-element set $\{c\}$ belongs to $\text{Ind}(S)$. Since, by hypothesis, $C(\{c\}) = A$, every mapping $c \rightarrow a \in A$ belongs to $S$. So, for one-element sets, the independence and the $S$-independence are equivalent and, as we know, no algebraic constant is independent.

Proposition (iii) is false for $S_0$-independence: see below 5 (ii).

4. A certain quasi-order and $G$-independence (1). We write $a \prec b$, whenever for any $f, g \in A^{(1)}$ if $f(a) = g(a)$, then $f(b) = g(b)$. More generally, $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$ whenever for any $f, g \in A^{(n)}$ if $f(a_1, \ldots, a_n)$

$= g(a_1, \ldots, a_n)$, then $f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n)$. The so defined relation is of course reflexive. Moreover, we obviously have

(i) If $c \prec a$, where $c$ is an algebraic constant, then $a = c$.

(ii) If $A$ is a group with the unity $e$, then $1^o a \prec e$ for every $a$, and 2$^o$ for any $a$ and $b$ of finite order we have $a \prec b$ iff the order of $a$ is divisible by the order of $b$.

A mapping $p : S \rightarrow A$, where $S \subseteq A$, is diminishing if $a \prec p(a)$ for every $a \in S$. It is totally diminishing if $(a_1, \ldots, a_n) \prec (p(a_1), \ldots, p(a_n))$ for every finite sequence $a_1, \ldots, a_n \in S$. Evidently this is another form of condition (a') of section 1, whence

(iii) A mapping $p$ of $S \subseteq A$ into $A$ can be extended to a homomorphism of $C(S)$ into $A$ iff $p$ is totally diminishing.

The class of all diminishing mappings will be denoted by $G$. The following proposition concerning the $G$-independence (in the sense of section 2) is an easy consequence of 2 (i) and 4 (iii):

(iv) Let $I$ be a subset of $A$. Then the following conditions are equivalent:

(g) $I$ is $G$-independent;

(1) This notion has been introduced (under the name of weak independence) by Grätzer (see [1] and [2]). Here I consider only individual algebras and I write $a \prec b$, while Grätzer considers equational classes and the relation $0(a) \prec 0(b)$. All propositions of this section are (explicitly or implicitly) contained in [1].
(g') every diminishing mapping of I into A is totally diminishing;

(g'') for any different elements \(a_1, \ldots, a_n\) of I, for any elements \(b_1, \ldots, b_n\) of I such that \(a_j \preceq b_j\) (for \(1, \ldots, n\)), and for every \(f, g \in A^{(n)}\), if

\[ f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n), \]

then

\[ f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n). \]

Proposition (iv) implies proposition

(v) The family \(\text{Ind}(G)\) is hereditary and any set \(I \subseteq A\) is \(G\)-independent iff every finite subset of I is \(G\)-independent.

Proposition (v) may also be deduced from propositions 2 (iv) and 2 (v).

We shall now prove proposition (see [1], p. 232)

(vi) If \(\mathcal{U} = (A; x+y, -x)\) is an Abelian group, then a set \(I \setminus \{0\} \subseteq A\) is \(G\)-independent iff I is linearly independent, i.e. if for every \(a_1, \ldots, a_n \in I\)

\[ \sum_{j=1}^{n} k_j a_j = 0 \]

implies \(k_j a_j = 0\) for \(j = 1, 2, \ldots, n\).

Let us suppose that \(I \setminus \{0\}\) is linearly independent and, moreover, that

\[ a_1, \ldots, a_n, b_1, \ldots, b_n \in I, \quad a_j \preceq b_j \text{ for } j = 1, \ldots, n, \]

and

\[ f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n). \]

Operations \(f\) and \(g\), as algebraic in \(\mathcal{U}\) are of the form

\[ f(x_1, \ldots, x_n) = \sum_j k_j x_j, \quad g(x_1, \ldots, x_n) = \sum_j l_j x_j. \]

It follows from (2) that

\[ \sum_j (k_j - l_j) a_j = 0, \]

whence, by the linear independence of \(I \setminus \{0\}\), we obtain \((k_j - l_j) a_j = 0\) then, by (1), \((k_j - l_j) b_j = 0\), and, consequently,

\[ f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n). \]

Thus the set I satisfies (g''), and by (iv), I is \(G\)-independent.

Let us suppose now that I is \(G\)-independent, and, moreover, that

\[ a_1, \ldots, a_n \in I \quad \text{and} \quad \sum_j k_j a_j = 0. \]
Putting
\[ f(x_1, \ldots, x_n) = \sum_{j} k_j x_j \quad \text{and} \quad g(x_1, \ldots, x_n) = 0 \]
we get
\[ f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n). \tag{3} \]

Let us fix \( j_0 \) such that \( 1 \leq j_0 \leq n \), and put \( b_{i_0} = a_{i_0} \) and \( b_j = 0 \) for \( j \neq j_0 \). Then \( a_j \preceq b_j \) for \( j = 1, 2, \ldots, n \) and, on account of (3), of the \( G \)-independence of \( I \) and of proposition (iv) (condition \( (g'') \)), we obtain
\[ k_{i_0} a_{i_0} = \sum_j k_j b_j = f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n) = 0. \]

5. Independence of one-point sets. As we know, for some families \( Q \) all one-point subsets of \( A \) are \( Q \)-independent. We can now give an obvious necessary and sufficient condition:

(i) \( \{a\} \in Ind(Q) \) iff every mapping \( p: \{a\} \to A \) belonging to \( Q \) is diminishing.

Of course, this condition is satisfied for every \( a \) by the family \( G \). It is also satisfied by the family \( S_0 \), since, if the mapping \( p: \{a\} \to A \) belongs to \( S_0 \), we have, by definition of \( S_0 \), \( p(a) = a \), whence \( p \) is diminishing. Hence

(ii) Every one-point set is \( G \)-independent ([1], p. 232) and \( S_0 \)-independent.

REFERENCES


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