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Existence of solutions of functional-differential inclusion in nonconvex case

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Abstract. In this paper we give conditions for existence of solutions of the functional-differential Cauchy problem in the form:

$$\begin{aligned} &\dot{x}(t) \in F(t, x(\cdot)) && \text{if } t \in [a, b], \\ &x(t) = \varphi(t) && \text{if } t \in [a-r, a], \end{aligned}$$

where $r \geqslant 0$, $x(\cdot)$ is a continuous function from interval $[a-r, b]$ into $\boldsymbol{R}^m$ and the map F defined on $[a, b]$ is $\mathscr{L} \otimes \mathscr{B}$-measurable, lower semicontinuous in x and has nonconvex values in $\boldsymbol{R}^m$.

1. Consider the following Cauchy problem:

$$\begin{aligned} &\dot{x}(t) \in F(t, x(\cdot)) && \text{if } t \in [a, b], \\ &x(t) = \varphi(t) && \text{if } t \in [a-r, a], \end{aligned} \tag{1}$$

where $r \geqslant 0$, $x(\cdot)$ belongs to the space $C([a-r, b], \boldsymbol{R}^m)$ of continuous functions from the closed interval $[a-r, b]$ into $\boldsymbol{R}^m$, φ is a fixed continuous function from $[a-r, a]$ into $\boldsymbol{R}^m$, and $F(t, x)$ is a multifunction from $[a, b] \times C([a-r, b], \boldsymbol{R}^m)$ into closed subsets of $\boldsymbol{R}^m$ not necessarily convex.

By a solution of (1) we mean a continuous function $x \in C([a-r, b], \boldsymbol{R}^m)$ such that $x(t) = \varphi(t)$ for each $t \in [a-r, a]$, x restricted to $[a, b]$ is absolutly continuous and $\dot{x}(t) \in F(t, x)$ almost everywhere in $[a, b]$.

Equation (1) is a generalization of a functional-differential equation and may be called a *functional-differential inclusion.*

A special case of (1) studied in literature is a functional-differential inclusion with deviating argument of the form

$$\begin{aligned} &\dot{x}(t) \in F(t, x_t) && \text{if } t \in [a, b], \\ &x(t) = \varphi(t) && \text{if } t \in [a-r, a], \end{aligned} \tag{2}$$

where x_t belongs to $C([-r, 0], \boldsymbol{R}^m)$ and is defined by $x_t(\theta) = x(t+\theta)$.

In this paper we give sufficient conditions for existence of solutions for problem (1). This more general setting does not require any new tools, quite opposite it simplifies the notation.

We prove the following:

EXISTENCE THEOREM. *Assume that the values of the map* $F(t, x)$ *are closed subsets of* $\mathbf{R}^m$ *for each* $(t, x) \in [a, b] \times C([a-r, b], \mathbf{R}^m)$ *and the following assumptions hold*:

(i) F *is* $\mathscr{L}([a, b]) \otimes \mathscr{B}(C([a-r, b], \mathbf{R}^m))$*-measurable* (*shortly*, $\mathscr{L} \otimes \mathscr{B}$ *measurable*), *where* $\mathscr{L}$ *is the Lebesgue* σ*-field of subsets of* $[a, b]$ *and* $\mathscr{B}$ *is the Borel* σ*-field of* $C([a-r, b], \mathbf{R}^m)$,

(ii) F *is lower semicontinuous* (*l.s.c.*) *in* x *for each fixed* t,

(iii) F *is integrably bounded*; *that is, there exists an integrable function* $p\colon [a, b] \to [0, +\infty]$ *such that*

$$\sup\{|z|\colon z \in F(t, x)\} \leqslant p(t) \quad \text{a.e. in } [a, b].$$

Under those assumptions problem (1) *admits a solution.*

This theorem extends similar results for differential inclusions of Fillippov [5], Kaczyński and Olech [7], Antosiewicz and Cellina [1], Bressan [2] and Łojasiewicz [8] (see also [4]).

One should notice that we do not assume convexity of F which makes the problem more difficult. The proof of this theorem follows the same idea as in Antosiewicz and Cellina [1], which consists in constructing a continuous selection $k(s)$ from a compact and convex subset S of $C([a-r, b], \mathbf{R}^m)$ into the Banach space $L^1([a, b], \mathbf{R}^m)$ such that $k(s)(t) \in F(t, s)$ a.e. in $[a, b]$. The existence of such selection is proved in author's paper [6]. This result is stated in Section 2. Section 3 contains the proof of the existence theorem above.

2. Selection theorem. Let S and Z be topological spaces. Denote by $\mathrm{cl}(Z)$ the space of closed subsets of Z and let $P\colon S \to \mathrm{cl}(Z)$ be a multifunction. The mapping P is lower semicontinuous (l.s.c.) if

$$P^- U = \{s \in S\colon P(s) \cap U \neq \emptyset\} \tag{3}$$

is open for each open subset U of Z.

If S and Z satisfy the first axiom of countability, then the l.s.c. can be expressed equivalently as:

(4) for each $s_0 \in S$ and $z_0 \in P(s_0)$ and any sequence $s_n \xrightarrow[n\to\infty]{} s_0$ there is $z_n \in P(s_n)$ such that $z_n \xrightarrow[n\to\infty]{} z_0$.

If additionally we assume that Z is a metric space with the metric d and values of P are compact, then (3) is equivalent to the condition:

(5) the map $(s, z) \to d(z, P(s))$ is upper semicontinuous.

Suppose that in S a σ-field Σ is given. Then the map P is Σ-measurable if $P^- U \in \Sigma$ for each open U.

For this and others properties of multifunctions we refer the reader to Castaing and Valadier [3] and Parthasarathy [10].

Consider now a compact space T with the σ-field $\mathfrak{M}$ of subsets of T generated by nonnegative regular Borel measure dt on T. By $L^1(T, \boldsymbol{R}^m)$ we denote the Banach space of integrable functions $u\colon T \to \boldsymbol{R}^m$ with the usual norm $\|u\| = \int_T |u(t)|\,dt$, where $|\cdot|$ stands for the norm in $\boldsymbol{R}^m$.

We say that $K \subset L^1(T, \boldsymbol{R}^m)$ is *decomposable* if, for each $u, v \in K$ and any $A \in \mathfrak{M}$, $u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K$, where χ_A is the characteristic function of A.

We say that the mapping $K\colon S \to \mathrm{cl}(L^1(T, \boldsymbol{R}^m))$ is *decomposable* if $K(s)$ is decomposable for each $s \in S$.

The following selection theorem is proved in [6]:

SELECTION THEOREM. *Assume that S is compact and $K\colon S \to \mathrm{cl}(L^1(T, \boldsymbol{R}^m))$ is decomposable and l.s.c. Then there exists a continuous selection $k\colon S \to L^1(T, \boldsymbol{R}^m)$ of K; that is, $k(s) \in K(s)$ for each $s \in S$.*

We will apply this theorem to the mapping K given by

$$K(s) = \{u \in L^1([a, b], \boldsymbol{R}^m)\colon u(t) \in F(t, s) \text{ a.e. in } [a, b]\} \tag{6}$$

for $s \in S$, where S is the compact space of continuous functions from $[a-r, b]$ into $\boldsymbol{R}^m$ such that $s(t) = \varphi(t)$ if $t \in [a-r, a]$, $s(t)$ is absolutely continuous on $[a, b]$ and $|\dot{s}(t)| \leqslant p(t)$. F and p are given in the existence theorem.

To apply the selection theorem to K defined by (6) we need to check that (i), (ii), (iii) imply that K is l.s.c., since K is obviously decomposable. We will prove that (4) holds. For this purpose fix $s_0 \in S$, $u_0 \in K(s_0)$ and $s_n \xrightarrow[n\to\infty]{} s_0$. Let $u_n \in K(s_n)$ be such that

$$|u_n(t) - u_0(t)| = d(u_0(t), F(t, s_n)).$$

The existence of such measurable u_n follows from (i). From (ii) and (5) we get that

$$\limsup_{n\to\infty} d(u_0(t), F(t, s_n)) \leqslant d(u_0(t), F(t, s_0)) = 0 \quad \text{a.e. in } [a, b],$$

which means that $u_n(t) \to u_0(t)$ a.e. in $[a, b]$.

Since from (iii), u_n are uniformly bounded by an integrable function, it follows that, $u_n \xrightarrow[n\to\infty]{} u_0$ in L^1-norm, which completes the proof that K is l.s.c.

3. Proof ot the Existence Theorem. Let S be defined as in the previous section. Clearly, S is a compact and convex subset of $C([a-r, b], \boldsymbol{R}^m)$. Consider the multifunction $K\colon S \to \mathrm{cl}(L^1([a, b], \boldsymbol{R}^m))$ defined by (6). We have proved that K is l.s.c. and decomposable. Therefore from the selection theorem there exists $k\colon S \to L^1([a, b], \boldsymbol{R}^m)$ which is continuous and such that

$$k(s)(t) \in F(t, s) \quad \text{a.e. in } [a, b]. \tag{7}$$

Define $l\colon S \to S$ by the formula

$$l(s)(t) = \begin{cases} \varphi(t) & \text{if } t \in [a-r, a], \\ \varphi(a) + \int_a^t k(s)(c)\,dc & \text{if } t \in [a, b]. \end{cases} \tag{8}$$

It is clear that $l(s)(t)$ is absolutely continuous on $[a, b]$ and $\left|\frac{d}{dt} l(s)(t)\right| \leqslant p(t)$ a.e. in $[a, b]$. Hence $l(s) \in S$. Continuity of k implies continuity of l. Therefore from the Schauder fixed-point theorem there exists $x \in S$ such that $l(x) = x$, and, by (7) and (8), $\dot{x}(t) \in F(t, x)$ a.e. in $[a, b]$ and $x(t) = \varphi(t)$ for $t \in [a-r, a]$. Hence x is a solution of (1) which completes the proof.

Remark 1. The existence of solution of problem (2) follows from the theorem we proved under the same assumption. It is enough to check that if $F(t, x_t)$ is l.s.c. in x_t, then the mapping $G(t, x) = F(t, x_t)$ for each $x \in C([a-r, b], R^m)$ is also l.s.c. in x. The $\mathscr{L} \otimes \mathscr{B}$-measurability of G follows from $\mathscr{L} \otimes \mathscr{B}$-measurability of F.

Remark 2. Notice that in the case where $F(t, x)$ is convex the mapping $K(s)$ is also convex and the existence of continuous selection $k(s)$ can be obtained from a general selection theorem of Michael [9].

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