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CERTAIN INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS

1. Introduction. Let $\varphi(t)$ be an infinitely divisible characteristic function (c.f.). Then $\{1-x\log\varphi(t)\}^{-a}$ is the c.f. of the power mixture on y of $\{\varphi(t)\}^{y}$, where y has a gamma distribution with c.f. $(1-xit)^{-a}$. Since the gamma distribution and $\varphi(t)$ are both infinitely divisible, $\{1-x\log\varphi(t)\}^{-a}$ is also infinitely divisible. Moreover, Steutel ([5], Theorem 3.5.2 and Definition 3.4.5) has established infinite divisibility for c.f.'s which are mixtures of the form

$$\int_0^\infty \{1-x\log\varphi(t)\}^{-a}dF(x)$$

when one of the following conditions holds:

(i) $0 < a \leqslant 1$,

(ii) $a = 2$ and $F(x)$ is the distribution function (d.f.) for a unimodal distribution,

(iii) $a = 2$ and $F(x)$ has at most four points of increase.

The purpose of this paper is to show that the transformation T mapping $\varphi(t)$ into

$$p\{1-x\log\varphi(t)\}^{-a}+q\{1-x\log\varphi(t)\}^{-a-1} \tag{1}$$

preserves infinite divisibility when $p, q > 0$, $p+q = 1$, $x > 0$, $a \geqslant 0$, and also to find conditions such that mixtures of the form

$$\int_0^\infty p\{1-x\log\varphi(t)\}^{-a}dF(x)+\int_0^\infty q\{1-x\log\varphi(t)\}^{-a-1}dF(x) \tag{2}$$

are also infinitely divisible.

Examples of c.f.'s having the forms (1) and (2) are then discussed.

Finally, certain further transformations are examined. The transformation

$$T_0 : \varphi(t) \to \{\varphi(t)-1\}/t\varphi'(0)$$

maps the c.f. with d.f. $F(x)$ into the c.f. with probability density function (p.d.f.) $\{1-F(x)\}/\mu$, i.e. into the renewal c.f. Lukacs [2] and Moran [3] give a number of other transformations which produce new c.f.'s from given c.f.'s. The present paper shows that under certain conditions these are self-transformations for (1).

2. The initial transformation. When $a = 1$ and $\varphi(t) = \exp(it)$, (1) becomes the superposition of the gamma c.f.'s with parameters 1 and 2, respectively. The p.d.f. is $(px+qy)\exp(-y/x)/x^2$, $0 \leqslant y < \infty$, and this is valid provided that $x > 0, p > 0, q > 0, p+q = 1$. Given these conditions, (1) remains a valid c.f. for all non-negative a.

THEOREM 1. *If p, $q > 0$, $p+q = 1$, $x > 0$, $a \geqslant 0$, and $\varphi(t)$ is an infinitely divisible c.f., then* (1) *is also an infinitely divisible c.f.*

Proof. Firstly, suppose that $\varphi(t) = \exp(it)$; then (1) takes the form

$$(1-pixt)/(1-ixt)^{a+1} = \varphi_1(t)\varphi_2(t),$$

where $\varphi_1(t)$ is the c.f. for the gamma distribution with parameter a, and

$$\varphi_2(t) = (1-pixt)/(1-ixt) = p+q/(1-ixt) \tag{3}$$
$$= \lambda/\{\lambda-\log\varphi_4(t)\},$$

where $\varphi_4(t) = \exp\big(\varphi_3(t)-1\big)$, $\varphi_3(t) = (1-bixt)/(1-pixt)$ and $b = p-\lambda+ +\lambda p$. By (3), $\varphi_2(t)$ is a valid c.f. Clearly, λ can be any value such that $0 < \lambda < p/q$, and so $\varphi_3(t)$ is also a valid c.f. Now $\varphi_4(t)$ is infinitely divisible because it is a Poisson mixture of $\varphi_3(t)$, and so $\varphi_2(t)$ is also infinitely divisible since it is a geometric mixture of $\varphi_4(t)$. Finally, $\varphi_1(t)\varphi_2(t)$ is infinitely divisible, since it is the product of two infinitely divisible c.f.'s.

Now, let $\varphi(t)$ be any infinitely divisible c.f. and let $G(y)$ be the d.f. corresponding to the c.f. $\varphi_1(t)\varphi_2(t)$. Then

$$\frac{1-px\log\varphi(t)}{\{1-x\log\varphi(t)\}^{a+1}} = \int_0^\infty \{\varphi(t)\}^y dG(y),$$

i.e. (1) is an infinitely divisible c.f.

3. Mixtures of the form (2).

THEOREM 2. *Mixtures of the form* (2) *are infinitely divisible when* $a = 0$.

Proof. When $a = 0$ and $\varphi(t) = \exp(it)$, (2) becomes

$$p[(1-ixt)^{-1}]_{x=0}^{\infty}+q\int_0^\infty (1-ixt)^{-1}dF(x), \tag{4}$$

which is infinitely divisible by Corollary 2.2.1 of Steutel [5]. If $\varphi(t)$ is any infinitely divisible c.f. and $G(y)$ is the d.f. with c.f. (4), then consideration of mixtures of the form

$$\int_0^\infty \{\varphi(t)\}^y dG(y)$$

completes the proof of the theorem.

THEOREM 3. *Mixtures of the form* (2) *are infinitely divisible when* $\alpha = 1$ *and* $F(x)$ *is such that* $\{pf(x) - qxf'(x)\}$ *changes its sign at most once, where* $f(x)$ *and* $f'(x)$ *are the continuous first and second derivatives of* $F(x)$, *and*

$$\lim_{x \downarrow 0} xf(x) = \lim_{x \to \infty} xf(x) = 0.$$

Proof. When $\alpha = 1$ and $\varphi(t) = \exp(it)$, (2) becomes, integrating by parts,

$$(5)\quad p\int_0^\infty (1-ixt)^{-1}f(x)\,dx + q[x(1-ixt)^{-1}f(x)]_0^\infty - q\int_0^\infty x(1-ixt)^{-1}f'(x)\,dx$$
$$= \int_0^\infty (1-ixt)^{-1}\{pf(x) - qxf'(x)\}\,dx.$$

By Corollary 2.2.1 of Steutel [5], this is infinitely divisible provided that $\{pf(x) - qxf'(x)\}$ changes its sign at most once. If $G(y)$ is now the d.f. corresponding to (5), then the infinite divisibility of (2) under these new conditions follows as before from consideration of

$$\int_0^\infty \{\varphi(t)\}^y dG(y).$$

The restriction for Theorem 3 that $\{pf(x) - qxf'(x)\}$ should change its sign at most once is not as severe as it at first appears. This condition holds for all sesquimodal distributions with non-negative support, and also for many of the widely-used mixing distributions such as the gamma, inverted-gamma, beta, inverted-beta-1 and inverted-beta-2 distributions.

4. Examples. Examples of distributions which are infinitely divisible by Theorem 1 can be obtained by taking $\varphi(t)$ to be the c.f. for the degenerate, normal, Poisson and Cauchy-type distributions, etc. For instance, using the Cauchy-type distribution (see [2], Theorem 4.5.3) we obtain the infinitely divisible c.f.

$$(1 + px|t|^c)/(1 + x|t|^c)^{\alpha+1}, \qquad 0 < c \leqslant 2.$$

Also the case $\alpha = 0$ for Theorem 1 is interesting in that it implies that any exponential mixture of an infinitely divisible c.f. remains infinitely divisible when zero-modified.

By Theorem 2, all mixtures of the above-mentioned distributions are infinitely divisible provided that $a = 0$. In particular, zero-modified mixtures of zero-modified exponential mixtures of infinitely divisible c.f.'s remain infinitely divisible.

By Theorem 3, all gamma, inverted-gamma, beta, inverted-beta-1 and inverted-beta-2 mixtures of c.f.'s of the form (1) are infinitely divisible provided that $\varphi(t)$ is infinitely divisible. Consider, for example, an inverted-gamma mixture of (1) with $\varphi(t) = \exp(it)$. The resultant distribution has the p.d.f.

$$c\beta^c\{(1+qc)y+p\beta\}/(y+\beta)^{c+2}$$

and the c.f.

$$pc\psi(1, 1-c; -\beta it)+qc(c+1)\psi(2, 1-c; -\beta it),$$

where $\psi(a_1, a_2; x)$ is the confluent hypergeometric function of second kind. (Note the relation to the F-distribution for $p = 0$ or 1, and see also [5], p. 39-40.)

Also, for $a = 0, 1$, an inverted-beta-2 mixture of (1), with $\varphi(t)$ being the c.f. of a Poisson distribution, yields

$$\int_0^\infty \frac{1-p\lambda x(e^{it}-1)}{\{1-\lambda x(e^{it}-1)\}^{a+1}} \frac{x^{c-1}(1+x)^{-c-d}}{B(c, d)}\, dx$$

$$= \frac{p\,{}_2F_1(a, c; a+c+d; \lambda e^{it})}{{}_2F_1(a, c; a+c+d; \lambda)} + \frac{q\,{}_2F_1(a+1, c; a+1+c+d; \lambda e^{it})}{{}_2F_1(a+1, c; a+1+c+d; \lambda)}.$$

For $\lambda = 1$, $a = 0$, this becomes the c.f. for a zero-modified Waring distribution, and for $\lambda = 1$, $a = 1$ we get the weighted sum of two Waring c.f.'s:

$$(1-cq)\frac{d}{c+d}\,{}_2F_1(1, c; c+d+1; e^{it})+\frac{cqd}{c+d+1}\,{}_2F_1(1, c+1; c+d+2; e^{it})$$

(using contiguity relations for the Gaussian hypergeometric function). For $a = 1$, $c = d+1$, $d = \frac{1}{2}$, this becomes the weighted sum of two "lost-games" c.f.'s (see [1]).

Finally, consider Steutel's conjecture (following his Theorem 2.2.2) that there seems to be no way to generate c.f.'s of the form

(6)

$$\prod_{k=1}^{n}\{\lambda_k/(\lambda_k-it)\}\prod_{j=1}^{m}\{(\mu_j-it)/\mu_j\},\ m \leqslant n,\ \sum_{k=1}^{l}\lambda_k \leqslant \sum_{j=1}^{l}\mu_j,\ l = 1, 2, \ldots, m,$$

other than by mixing exponential distributions. Lukacs' ([2], p. 323) operator

$$T_1 : \varphi(t) \to \{\varphi'(t) - \varphi'(0)\}/t\varphi''(0)$$

maps the c.f. for the exponential distribution, that is $(1-ixt)^{-1}$, into $(1-ixt/2)/(1-ixt)^2$, which is of the form (6) with $n = 2$, $m = 1$, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = 2$. This is an alternative derivation; for a physical interpretation see [2], p. 321-322.

5. The self-transformations. The renewal operator T_0 and Lukacs' ([2], p. 323) operator T_4 map the c.f. $\theta(t)$ into

$$\{\theta(t) - 1\}/t\theta'(0) \quad \text{and} \quad \{2\theta(t) - 1 - t\theta'(0)\}/t^2\theta''(0),$$

respectively. For $\theta(t) = (1-pixt)/(1-ixt)^2$, T_0 and T_4 yield

$$\{1 - ixt/(2-p)\}/(1-ixt)^2 \quad \text{and} \quad \{1-(2-p)\,ixt/(3-2p)\}/(1-ixt)^2,$$

i.e. both T_0 and T_4 give self-transformations for T (see (1)), when $a = 1$ and $\varphi(t) = \exp(it)$.

Now let

$$T_5 : \theta(t) \to \lambda\theta(t)/\{\lambda + 1 - \theta(t)\} \quad \text{and} \quad T_6 : \theta(t) \to \lambda/\{\lambda + 1 - \theta(t)\}.$$

Then T_5 is the operator for Moran's [3] transformation (8) with $b = 1$, whilst T_6 is the operator for Moran's [3] transformation (7), where the image is $P\{\theta(t)\}$ and $P(t)$ is the probability generating function for the geometric distribution (see [3], p. 275-276, and also [2], Theorem 12.2.3). For

$$\theta(t) = \{1 - px\log\varphi(t)\}/\{1 - x\log\varphi(t)\},$$

T_5 and T_6 map $\theta(t)$ into

$$\{1 - px\log\varphi(t)\}/\{1-(1+1/\lambda - p/\lambda)x\log\varphi(t)\}$$

and

$$\{1 - x\log\varphi(t)\}/\{1-(1+1/\lambda - p/\lambda)x\log\varphi(t)\},$$

respectively. Thus T_5 and T_6 both give self-transformations for T when $a = 0$.

References

[1] A. W. Kemp and C. D. Kemp, *On mixing processes and the lost-games distribution*, Zastos. Mat. 12 (1971), p. 167-173.

[2] E. Lukacs, *Characteristic functions*, Griffin, London 1970.

[3] P. A. P. Moran, *An introduction to probability theory*, Oxford Univ. Press, London 1968.
[4] F. W. Steutel, *A class of infinitely divisible mixtures*, Ann. Math. Statist. 39 (1968), p. 1153-1157.
[5] — *Preservation of infinite divisibility under mixing*, Mathematisch Centrum, Amsterdam 1970.

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O PEWNYCH NIESKOŃCZENIE PODZIELNYCH FUNKCJACH CHARAKTERYSTYCZNYCH

STRESZCZENIE

W notce pokazuje się, że pewne transformacje funkcji charakterystycznych zachowują własność nieskończonej podzielności. Transformacje te (i wymagane założenia) podane są w twierdzeniach 1-3.
