EIGENVALUES OF THE LAPLACIAN AND CURVATURE

BY

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Introduction. Let (M, g) be a closed, connected two-dimensional Riemannian manifold of genus zero with sectional curvature $\varkappa$, $\varkappa_0 := \min \varkappa$, $\varkappa_1 := \max \varkappa$. Then the first positive eigenvalue λ_1 of the Laplacian on functions fulfils the inequality

$$2\varkappa_0 \leqslant \lambda_1 \leqslant 2\varkappa_1,$$

and the equality on the left or the right holds iff (M, g) is isometrically diffeomorphic to a sphere (cf. [2], p. 179-180, and [9], also [16]; naturally the inequality on the left is of interest only if $\varkappa_0 > 0$).

Hersch's result (the right-hand inequality) suggests to investigate eigenvalues λ with $\lambda > 2\varkappa_1$. The following is our main result in the two--dimensional case:

THEOREM A. *Let (M, g) be a closed, connected Riemannian manifold, $\dim M = 2$, $\varkappa_0 > 0$. Then $\lambda > 2\varkappa_1$ implies $\lambda \geqslant 6\varkappa_0$, and $\lambda = 6\varkappa_0$ implies that the universal Riemannian covering $(\tilde{M}, \tilde{g})$ of (M, g) is isometrically diffeomorphic to a Euclidean sphere $S^2(\varkappa_0)$ with constant curvature $\varkappa_0$.*

As above, the result is of interest only if $6\varkappa_0 \geqslant 2\varkappa_1$, i.e. for pinched manifolds with pinching constant δ, $\delta \geqslant \frac{1}{3}$. So especially for manifolds "not too far from a Riemannian sphere" we get additional information on relations between curvature and the distribution of small eigenvalues.

One basic tool of our proof is an integral formula for eigenfunctions on two-dimensional manifolds which we prove in Lemma 2.1. The other tool are certain systems of differential equations for eigenfunctions on spheres (Obata [13], Tanno [20], Ferus [7], Gallot [8]; for Gallot's results cf. also Tanno [20]); they suggest to "compare" eigenfunctions corresponding to the eigenvalues λ_1 resp. λ_2 (where $\lambda_1 < \lambda_2$, as we do not regard their multiplicities) with corresponding eigenfunctions and eigenvalues on spheres (cf. (2.2)-(2.5)). To extend this method to higher eigenvalues λ_p ($p \geqslant 3$), one would need an analogue to (2.1) which corresponds to the

differential equations for higher eigenfunctions. Generalizing our method to higher dimensions in Section 3 we continue investigations on lower bounds for λ_2 on Einstein spaces (Tanno [19], Simon [16]); in Section 4 we improve local results from Lange-Simon [11]. In Section 5 we finally improve results from Simon [15] on minimal submanifolds of spheres. In the two-dimensional case we get the following generalization of a result of Lawson ([12], p. 195, Proposition 3):

THEOREM B. *Let (M, g) be a complete, connected Riemannian manifold,* $\dim M = 2$; *let $\tilde{x}$: $M \to S^N(1)$, $N > 2$, be an isometric minimal immersion. If $\frac{1}{3} \leqslant \varkappa \leqslant 1$, then either $\tilde{x}(M)$ is totally geodesic ($\varkappa \equiv 1$) or $\tilde{x}(M)$ is the Veronese surface in $S^4(1)$ ($\varkappa \equiv \frac{1}{3}$).*

Lawson additionally assumed M to be closed and $N = 4$.
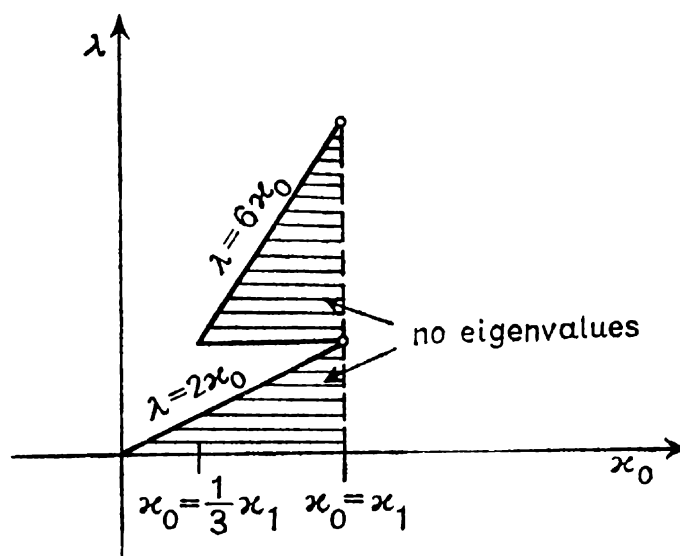
1. Preliminaries. Let (M, g) be a connected Riemannian manifold of class C^∞, $n = \dim M \geqslant 2$, denote by ∇ the corresponding covariant differentiation and by g_{ij} resp. g^{ij} the components of the metric tensor g resp. g^{-1} in local coordinates (u^i); denote by do the volume element on M and by $R^h{}_{ijk}$ resp. R_{ij} the components of the curvature tensor resp. the Ricci tensor (with the sign of [10], p. 201); let R denote the scalar curvature (such that $R = 1$ on the unit sphere). As usual raising and lowering of indices are defined.

Let f: $M \to R$ be a C^∞-function, let $f_{ij} := \nabla_j \nabla_i f$ denote the components of the Hessian Hess(f) and denote the Laplacian by $\Delta f := g^{ij} f_{ij}$.

1.1. LEMMA ([15], (7a-b)). *Let f: $M \to R$ be a C^∞-function. Then f fulfils the equation*

$$\frac{1}{2} \Delta(f_{ij} f^{ij}) = 2 \sum_{i<j} \varkappa_{ij}(\sigma_i - \sigma_j)^2 + f^{ij} \nabla_j \nabla_i (\Delta f) +$$

$$+ \nabla_k f_{ij} \nabla^k f^{ij} + f^{ij} f^k [2 \nabla_i R_{jk} - \nabla_k R_{ij}],$$

where $\sigma_1, \ldots, \sigma_n$ *are the eigenvalues of the Hessian,* $E_1, \ldots, E_n$ *are corresponding orthonormal eigenvectors and* $\varkappa_{ij}$ *is the sectional curvature of the plane* $\{E_i, E_j\}_{i \neq j}$.

1.2. Lemma. *Let* (M, g) *be closed (compact without boundary),* $\dim M \geqslant 2$. *Let* $f, h: M \to R$ *be* C^∞*-functions. Then*

$$\int f_{ij} h^{ij} do - \int \Delta f \Delta h \, do + \int R^{ij} f_i h_j \, do = 0.$$

This lemma generalizes the Bochner-Lichnerowicz formula (cf. [2], p. 131).

1.3. Remark. Let (M, g) be closed, connected, $\dim M \geqslant 2$. Then each eigenfunction f with eigenvalue λ fulfils

$$(n-1)\lambda \int \|\text{grad} f\|^2 \, do = n \int R^{ij} f_i f_j \, do + \int \sum_{i<j} (\sigma_i - \sigma_j)^2 \, do.$$

1.4. Lemma [13]. *Let* M *be a complete connected Riemannian manifold,* $\dim M = n \geqslant 2$. *There exist a nontrivial function* $f: M \to R$, $f \in C^\infty$, *and a real positive constant* c *which fulfil the system*

$$(1.4.1) \qquad n \cdot f_{ij} + c^2 f \cdot g_{ij} = 0$$

iff M *is isometrically diffeomorphic to a sphere* $S^n(c^2)$ *of sectional curvature* c^2.

1.5. Lemma ([20], [7], [8]). *Let* (M, g) *be simply connected and complete,* $\dim M \geqslant 2$. *There exist a constant* $c \in R$, $c \neq 0$, *and a nontrivial function* $f \in C^\infty(M)$ *such that*

$$(1.5.1) \qquad f_{ijk} + c^2 (f_j g_{ik} + f_i g_{jk} + 2 f_k g_{ij}) = 0$$

iff (M, g) *is isometrically diffeomorphic to an* n-*sphere* $S^n(c^2)$ *with sectional curvature* c^2.

2. Two-dimensional manifolds.

2.1. Lemma. *Let* (M, g) *be a closed, connected, two-dimensional Riemannian manifold. From* $\Delta f + \lambda f = 0$ *it follows that*

$$0 = -\frac{1}{2} \int (\lambda - 2\varkappa)(\sigma_1 - \sigma_2)^2 \, do + \int \|\nabla_k f_{ij}\|^2 \, do -$$

$$- \int \left\{ \varkappa \lambda + \frac{1}{2} (\lambda - 2\varkappa)^2 \right\} \|\text{grad} f\|^2 \, do,$$

where σ_1 *and* σ_2 *are defined in Lemma 1.1.*

Proof. To use formula 1.1 for $n = 2$, we make the following calculations:

(i) $2 f_{ij} f^{ij} = (\sigma_1 - \sigma_2)^2 + (\Delta f)^2.$

(ii) $n = 2$ implies $R_{ij} = \varkappa g_{ij}$; the Ricci identity and $\Delta f = -\lambda f$ give $\Delta(f_j) = (\varkappa - \lambda)f_j$; therefore

$$f^{ij}f^k \nabla_i R_{jk} = \{\nabla_i(R_{jk}f^k f^{ij})\} - \nabla_i(f^{ij})f^k R_{jk} - f^{ij}f_i^k R_{jk}$$

$$= \{\ldots\} + \varkappa(\lambda - \varkappa)\|\operatorname{grad}f\|^2 - \varkappa f_{ij}f^{ij}.$$

(iii) Analogously we have

$$f^{ij}f^k \nabla_k R_{ij} = \{\nabla_k(f^{ij}f^k R_{ij})\} - \varkappa((\Delta f)^2 - \lambda\|\operatorname{grad}f\|^2).$$

(iv) From (i), (ii) and (iii) it follows that

$$\int f^{ij}f^k[2\nabla_i R_{jk} - \nabla_k R_{ij}]do = -\int \varkappa(\sigma_1 - \sigma_2)^2 do + \int \varkappa(\lambda - 2\varkappa)\|\operatorname{grad}f\|^2 \, do.$$

(v) From (i) and Green's theorem we get

$$\int f^{ij}\nabla_j\nabla_i(\Delta f)do = -\lambda \int f_{ij}f^{ij}do = -\frac{\lambda}{2}\left[\int(\sigma_1 - \sigma_2)^2 do + \lambda \cdot \int \|\operatorname{grad}f\|^2 do\right].$$

The assertion follows from 1.1, (iv) and (v).

2.2. Remark. In the following we "compare" eigenfunctions on (M, g) and eigenfunctions on a sphere. Motivated by results of Obata [13] and Tanno [20] on the first resp. second eigenfunctions on spheres, we define for an eigenfunction f a $(3, 0)$-tensor $B(f)$ by

$$(2.2.1) \qquad B(f)_{ijk} := f_{ijk} + \frac{\lambda + 2\varkappa}{4}g_{ij}f_k + \frac{\lambda - 2\varkappa}{4}(g_{ik}f_j + g_{jk}f_i).$$

$B(f)$ vanishes identically on spheres if λ is the first or second eigenvalue and f a corresponding eigenfunction. Formula (2.2.1) implies

$$(2.2.2) \qquad \|B(f)\|^2 = \|f_{ijk}\|^2 - \frac{1}{2}\|\operatorname{grad}f\|^2\left\{\lambda^2 + \frac{1}{2}(\lambda - 2\varkappa)^2\right\},$$

and Lemma 2.1 gives

$$(2.2.3) \qquad 0 = \frac{1}{2}\int(2\varkappa - \lambda)(\sigma_1 - \sigma_2)^2 do + \frac{1}{4}\int \|\operatorname{grad}f\|^2(\lambda - 2\varkappa)(\lambda + 2\varkappa)do +$$

$$+ \int \|B(f)\|^2 do.$$

2.3. LEMMA. *If (M, g) is a closed connected two-dimensional Riemannian manifold, then each eigenfunction f fulfils*

$$\int(\sigma_1 - \sigma_2)^2 do = \int(\lambda - 2\varkappa)\|\operatorname{grad}f\|^2 do.$$

Proof. Apply 1.2, 2.1 (i) and Green's theorem.

2.4. LEMMA. *Let (M, g) and f be as in 2.3. Then*

$$0 \geqslant \int \|B(f)\|^2 do + \frac{1}{4}\int \|\operatorname{grad}f\|^2(\lambda - 2\varkappa)[4\varkappa_0 - (\lambda - 2\varkappa)]do.$$

Proof. (2.2.3) and Lemma 2.3.

2.5. THEOREM. *Let* (M, g) *be closed, connected,* $\dim M = 2$, $\varkappa \geqslant 0$. *Then* $\lambda > 2\varkappa_1$ *implies* $\lambda \geqslant 6\varkappa_0$, *and* $\lambda = 6\varkappa_0$ *implies that the universal covering* $(\tilde{M}, \tilde{g})$ *is isometrically diffeomorphic to a sphere* $S^2(\varkappa_0)$.

Proof. Let $\lambda > 2\varkappa_1$. Then $\lambda \geqslant 6\varkappa_0$ from 2.4. In case of equality $\lambda = 6\varkappa_0$ we get, from 2.4,

$$0 \geqslant \int \|B(f)\|^2 \, do + \frac{1}{2} \int \|\mathrm{grad} f\|^2 (\lambda - 2\varkappa)(\varkappa - \varkappa_0) \, do \geqslant 0,$$

which implies $B(f) \equiv 0$ and $(\varkappa - \varkappa_0)\|\mathrm{grad} f\|^2 \equiv 0$ on (M, g).

If $\|\mathrm{grad} f\| \neq 0$, then

(2.5.1) $$0 = B(f)_{ijk} = f_{ijk} + \varkappa_0(2g_{ij}f_k + g_{ik}f_j + g_{jk}f_i).$$

If $G := \{p \in M \mid \mathrm{grad} f|_p = 0\}$ is nowhere dense in M, then (2.5.1) and Tanno's result (1.5) imply the assertion. Assume there exists a nonempty open set $V \subset G$; then $\mathrm{grad} f = 0$ on V implies $df = 0$ and from this we get $0 = \Delta f = -\lambda f$, so $f \equiv 0$ on V, i.e. $V \subset N$, where N is the nodal set of f. But this contradicts known results on N (cf. [3] and [5]).

The last proof was shortened by a hint of S. Tanno.

3. Einstein spaces. Let (M, g) be a closed, connected Einstein space, $n = \dim M \geqslant 3$. Then each eigenvalue fulfils the inequality

$$\lambda \geqslant nR,$$

and $\lambda_1 = nR$ iff (M, g) is isometrically diffeomorphic to a sphere (Obata [13]), where R is the (constant) scalar curvature. Again we denote by $\varkappa_0$ the minimum of all sectional curvatures.

3.1. THEOREM. (a) *Let* (M, g) *be a closed, connected Einstein space,* $n \geqslant 3$. *Then there is no eigenvalue in the interval* $(nR, 2(n+2)\varkappa_0 - 2R)$, *i.e.* $\lambda > nR$ *implies*

$$\lambda \geqslant 2(n+2)\varkappa_0 - 2R.$$

(b) $\lambda = 2(n+2)\varkappa_0 - 2R$ *holds iff the universal covering space* $(\tilde{M}, \tilde{g})$ *is isometrically diffeomorphic to a sphere* $S^n(\varkappa_0)$, $\varkappa_0 = R$, *and* $\lambda = 2(n+1)\varkappa_0$ *is the second eigenvalue.*

3.2. Remarks. 1. Part 3.1 (a) is of interest only if $2(n+2)\varkappa_0 - 2R \geqslant nR$, i.e. $2\varkappa_0 \geqslant R$. Theorem 3.1 improves results of S. Tanno [19] and U. Simon [16]. Furthermore, as a corollary to 3.1, one can improve Theorem 1 in [14].

2. Theorem 3.1 together with a result of Cheng ([4], Theorem 2.4) implies a result which is closely related to a result of Berger ([1], Proposition 6.4).

Let (M, g) be a closed, connected Einstein space, $\dim M \geqslant 3$. From Cheng's theorem and 3.1 it follows that *if*

$$n(n+4)\pi^2 \cdot \varkappa_1 \leqslant (n+2)\varkappa_0 - 2R,$$

then $(\tilde{M}, \tilde{g})$ *is isometrically diffeomorphic to a sphere.*

3 (Berger). *If* $(n-2)/(n-1) < \varkappa < 1$, *then* (M, g) *is a space of constant curvature.*

The assumption on the pinching constant in Remark 2 works only for small n and generally is not as good as Berger's. But the discussion of equality in 3.1 (b) suggests that it must be possible to improve Cheng's result.

3.3. Proof of Theorem 3.1.

(a) Using the ideas from 2.2, we define

$$B(f)_{ijk} := f_{ijk} + \frac{1}{n+2}\,(\lambda+2R)g_{ij}f_k + \frac{1}{n+2}\,(\lambda - nR)(g_{ik}f_j + g_{kj}f_i),$$

which gives

(3.3.1) $\|B(f)_{ijk}\|^2 = \|f_{ijk}\|^2 - $

$$-\,\frac{1}{n+2}\,\|\mathrm{grad}f\|^2\,\{3\lambda^2 - 4\lambda R(n-1) + 2n(n-1)R^2\}.$$

We apply Lemma 1.1 to closed Einstein spaces $(R_{ij} = (n-1)Rg_{ij})$ and make the following calculations, using $\Delta f + \lambda f = 0$:

(A) 1.1 gives

$$0 = \int \sum_{i<j} 2\varkappa_{ij}(\sigma_i - \sigma_j)^2\,do - \lambda \int f_{ij}f^{ij}\,do + \int \nabla_k f_{ij}\nabla^k f^{ij}\,do;$$

(B) 1.2 gives

$$\int f_{ij}f^{ij}\,do = (\lambda - (n-1)R)\int \|\mathrm{grad}f\|^2\,do;$$

(C) 1.3 gives

$$\int \sum_{i<j}(\sigma_i - \sigma_j)^2\,do = (n-1)(\lambda - nR)\int \|\mathrm{grad}f\|^2\,do.$$

If (M, g) is not a sphere, then $\lambda > nR$, and (A)-(C) imply

(3.3.2) $0 = \int \sum_{i<j}(\sigma_i - \sigma_j)^2\left[2\varkappa_{ij} - \frac{\lambda(\lambda - (n-1)R)}{(n-1)(\lambda - nR)}\right]do + \int \|\nabla_k f_{ij}\|^2\,do.$

From (3.3.1) and (C) we infer for $\lambda > nR$ that

(3.3.3) $\int \|f_{ijk}\|^2\,do$

$$= \int \|B(f)_{ijk}\|^2\,do + \frac{1}{n+2}\,\{3\lambda^2 - 4\lambda R(n-1) + 2n(n-1)R^2\} \times$$

$$\times \frac{1}{(n-1)(\lambda - nR)}\int \sum_{i<j}(\sigma_i - \sigma_j)^2\,do$$

which, together with (3.3.2), gives

$$(3.3.4) \quad 0 = \int \sum_{i<j} (\sigma_i - \sigma_j)^2 \left[2\varkappa_{ij} - \frac{\lambda + 2R}{n+2} \right] do + \int \|B(f)_{ijk}\|^2 do.$$

(b) If $\lambda = 2[(n+2)\varkappa_0 - R]$, (3.3.4) gives

$$(3.3.5) \quad 0 = \int \|B(f)_{ijk}\|^2 do + \int \sum_{i<j} (\sigma_i - \sigma_j)^2 \cdot 2(\varkappa_{ij} - \varkappa_0) do$$

and, because of $\varkappa_{ij} - \varkappa_0 \geqslant 0$, we get

$$0 \equiv B(f)_{ijk} = f_{ijk} + 2\varkappa_0 g_{ij} f_k + (2\varkappa_0 - R)(g_{ik} f_j + g_{kj} f_i).$$

Application of Lemma 3.4 below implies $(\tilde{M}, \tilde{g})$ to be a sphere and $\varkappa_0 = R$.

3.4. LEMMA. *Let* (M, g) *be a connected Einstein manifold,* $\dim M \geqslant 3$.
(a) *If there exists* $f \in C^\infty(M)$ *such that* f *fulfils*

$$(3.4.1) \quad f_{ijk} + r g_{ij} f_k + s g_{ik} f_j + t g_{jk} f_i \equiv 0 \quad (\text{with } r, s, t \in R)$$

on M, *then* $s = t$ *and* f (*resp.* $g := f - c$ *for a constant* $c \in R$) *is an eigenfunction* $(\Delta f + \lambda f = 0)$ *and either*

$$(3.4.2a) \quad f_{jk} + \frac{\lambda}{n} f g_{jk} \equiv 0 \quad on \ M, \quad \lambda = nR,$$

or

$$(3.4.2b) \quad r = 2t \quad and \quad nr + 2t = 2(n-1)R + 2r = \lambda.$$

(b) *Let* (M, g) *be complete and simply connected and assume that* $r \in R$ *fulfils* $(n+2)r > 2R > 0$. *There exists* $f \in C^\infty(M)$ *which fulfils* (3.4.1) *iff* (M, g) *is isometrically diffeomorphic to a sphere and* f *is a first* (3.4.2a) *resp. a second* (3.4.2b) *eigenfunction*

Proof. (a) Because of the symmetry of (3.4.1) in (i, j) we get $t = s$. Formula (3.4.1) implies $(\Delta f)_k + (nr + 2t) f_k = 0$, so

$$(3.4.3) \quad \Delta(f - c) + \lambda(f - c) = 0, \quad \text{where } \lambda = nr + 2t.$$

As $g = f - c$ fulfils (3.4.1), without loss of generality we assume $c = 0$. From (3.4.1) we get

$$f_{jkil} = -r g_{jk} f_{il} - t(g_{ji} f_{kl} + g_{kl} f_{jl}),$$

so, on the one hand,

$$(3.4.4) \quad f_{jkil} - 2f_{ijkl} = r(2g_{ij} f_{kl} - g_{jk} f_{il}) + t(2g_{ik} f_{jl} + 2g_{jk} f_{il} - g_{ji} f_{kl} - g_{kl} f_{jl}).$$

The Ricci identities imply

(3.4.5) $f_{ijlk} - f_{iljk} = \nabla_k(f_s R^s_{ijl})$,

(3.4.6) $f_{ijkl} = f_{sj}R^s_{ikl} + f_{is}R^s_{jkl} + f_{ijlk} = f_{sj}R^s_{ikl} + f_{is}R^s_{jkl} + \nabla_k(f_s R^s_{ijl}) + f_{iljk}$,

so, on the other hand,

(3.4.7) $f_{jkil} - 2f_{ijkl} = [f_{jka} - f_{jikl}] - \{f_{ijkl}\}$

$= [\nabla_l(f_s R^s_{jki})] - \{f_{sj}R^s_{ikl} + f_{is}R^s_{jkl} + \nabla_k(f_s R^s_{ijl}) + f_{iljk}\}$.

As $R_{ij} = (n-1)Rg_{ij}$, $\Delta f = -\lambda f$, and $\nabla^t R_{sjki} = 0$ (from the second Bianchi identity for Einstein spaces), (3.4.4) and (3.4.7) imply (both after contraction with g^{il})

(3.4.8) $[\lambda - 2(R(n-1)+r)]f_{jk} + (2t-r)\lambda f g_{jk} = 0$.

Now either both coefficients vanish, which gives (3.4.2b), or (3.4.2a) holds and then from (3.4.3) and (3.4.8) we get $\lambda = (n+2)r - 2R$. Using this value for λ, covariant differentiation of (3.4.8) and comparison with the coefficients of (3.4.1) give $r = R$ resp. $\lambda = nR$.

(b) Apply Obata's [13] result in case (3.4.2a), resp. Tanno's [20] in case (3.4.2b).

4. Isometries with spheres. Following ideas of [11] we get the following analogue to Theorem 3.1:

4.1. THEOREM. *Let (M, g) be a connected Einstein space, $n = \dim M \geq 3$, which admits $m > 1$ linear independent eigenfunctions $f(a)$ $(a = 1, ..., m)$ corresponding to the same eigenvalue λ. Assume furthermore that*

(4.1.1) $\sum_a f(a)^2 = c$,

where c is a positive constant.

Then either

(4.1.2a) $\lambda = nR$ *and* $f(a)_{ij} + \frac{\lambda}{n}f(a)g_{ij} = 0$ *for* $a = 1, ..., m$,

or

(4.1.2b) $\lambda \geq 2[(n+2)\varkappa_0 - R]$.

Equality in (4.1.2b) holds iff

(4.1.2c) $f(a)_{ijk} + \varkappa_0\{2g_{ij}f(a)_k + g_{ik}f(a)_j + g_{jk}f(a)_i\} = 0$ *for* $a = 1, ..., m$.

Proof. Note that $\lambda > 0$ ([11], Lemma (2.7)) and $\lambda \geq nR$ ([11], Lemma (3.1)). Defining $B(a)_{ijk} := B(f(a))_{ijk}$ in analogy to (3.3.1), we get

$$\sum_a \|f(a)_{ijk}\|^2$$

$$= \sum_a \|B(a)_{ijk}\|^2 + \frac{1}{n+2}\sum_a \|\mathrm{grad}f(a)\|^2\{3\lambda^2 - 4(n-1)\lambda R + 2n(n-1)R^2\}.$$

This, together with [11], formulas (2.3), and Lemma 3.4, implies

(4.1.3) $-\dfrac{1}{2}(n-1)c\lambda\Delta R$

$$= \sum_a \sum_{i<j} 2\varkappa(a)_{ij}(\sigma(a)_i - \sigma(a)_j)^2 - (\lambda - (n-1)R)\lambda^2 c + \sum_a \|B(a)_{ijk}\|^2 +$$

$$+ \frac{1}{n+2}\lambda c\{3\lambda^2 - 4(n-1)\lambda R + 2n(n-1)R^2\};$$

here $\sigma(a)_1, \ldots, \sigma(a)_n$ are the eigenvalues of $f(a)_{ij}$ and $\varkappa(a)_{ij}$ is defined via corresponding pairs of eigenvectors (cf. Lemma 1.1). As $n \geqslant 3$, we have $R = $ const. Introducing now the notation $\varkappa_0 := \inf \varkappa$, from [11] (formula (3.4c)) and (4.1.3) we get

(4.1.4) $(\lambda - nR)(\lambda - 2[(n+2)\varkappa_0 - R]) \geqslant 0$.

So $\lambda > nR$ gives (4.1.2b). The case $\lambda = nR$ was discussed in [11] (Theorem 3.2), while the equality $\lambda = 2[(n+2)\varkappa_0 - R]$ implies $B(a)_{ijk} = 0$ from (4.1.3), and this together with 3.4 (a) gives (4.1.2c).

4.2. Lemma. *Let* (M, g) *be a connected Riemannian manifold,* $\dim M = 2$, *which admits* $m > 1$ *eigenfunctions* $f(a)$ $(a = 1, \ldots, m)$ *which fulfil* (4.1.1). *Then the curvature* $\varkappa$ *fulfils the following differential inequality:*

$$2\Delta\varkappa \leqslant (\lambda - 2\varkappa)(\lambda - 6\varkappa).$$

Equality holds iff $B(a)_{ijk} \equiv 0$ *for* $a = 1, \ldots, m$.

Proof. We use (4.1.3), where $n = 2$ and $R = \varkappa$, and

$$\sum_a (\sigma(a)_1 - \sigma(a)_2)^2 = \lambda c(\lambda - 2R)$$

(cf. [11], (3.4c)).

4.3. Theorem. *Let* (M, g) *be a complete, connected Riemannian manifold,* $\dim M = 2$, *with* m *linear independent eigenfunctions* $f(a)$ $(a = 1, \ldots, m)$ *corresponding to the eigenvalue* λ, *which fulfil* (4.1.1). *Then either* (M, g) *is isometrically diffeomorphic to a sphere* $S^2(\varkappa)$ *of curvature* $\varkappa$ *and*

$$\lambda = 2\varkappa_0 \quad \text{or} \quad \lambda \geqslant 6\varkappa_0.$$

$\lambda = 6\varkappa_0$ *holds iff the universal covering* $(\tilde{M}, \tilde{g})$ *is isometrically diffeomorphic to a sphere and* λ *is the second eigenvalue.*

The proof is similar to the proof of Theorem (5.3) in [11], but we apply Tanno's result (1.5) instead of Obata's result in [13].

It is obvious that the results of this paragraph improve results in [11] even if we did not formulate all the consequences of our results here (cf. especially [11], § 4).

5. Minimal submanifolds of spheres. Let $\tilde{x}$: $M \to S^{N-1}(1)$ be an isometric minimal immersion, $N-1 > n = \dim M$.

5.1. If j: $S^{N-1}(1) \to E^N$ is the canonical embedding into a Euclidean space such that the center of $S^{N-1}(1)$ is the origin of the canonical coordinate system of E^N, then the position vector x (with respect to this coordinate system) of the immersion $j\tilde{x}$ fulfils (cf. Takahashi [17])

$$(5.1.1) \qquad\qquad \Delta x + n x = 0.$$

5.2. Each coordinate function $x(a)$ is an eigenfunction corresponding to $\lambda = n$; furthermore $\langle x, x \rangle = 1$, where $\langle , \rangle$ denotes the inner product in E^N. Let $\tilde{S}$ resp. S denote the squares of the lengths of the second fundamental forms of the immersions $\tilde{x}$ resp. $j\tilde{x}$. We have

$$(5.2.1) \qquad\qquad S - n = \tilde{S} = n(n-1)(1-R).$$

5.3. Applying Lemma 1.1 to each coordinate function, we get (cf. [15])

$$(5.3.1) \qquad \frac{1}{2}\Delta\tilde{S} = \frac{1}{2}\Delta S = \sum_{a=1}^{N}\sum_{i<j}2\varkappa(a)_{ij}(\sigma(a)_i - \sigma(a)_j)^2 - nS + \langle x_{ijk}, x^{ijk}\rangle.$$

5.4. In analogy to (3.3.1) we define

$$(5.4.1) \qquad \bar{X}_{ijk} := x_{ijk} + \frac{1}{n+2}(n+2R)g_{ij}x_k +$$

$$+ \frac{n}{n+2}(1-R)(g_{ik}x_j + g_{kj}x_i)$$

which gives

$$(5.4.2) \qquad \langle \bar{X}_{ijk}, \bar{X}^{ijk}\rangle = \langle x_{ijk}, x^{ijk}\rangle - \frac{n^2}{n+2}[3n - 4R(n-1) + 2(n-1)R^2].$$

Furthermore (cf. [11], (3.4c))

$$(5.4.3) \qquad \sum_{a}\sum_{i<j}(\sigma(a)_i - \sigma(a)_j)^2 = n^2(n-1)(1-R),$$

$$(5.4.4) \qquad (1-R)nS = \frac{n-(n-1)R}{n-1}\sum_{a}\sum_{i<j}(\sigma(a)_i - \sigma(a)_j)^2.$$

5.5. Assume $\tilde{x}(M)$ not to be a sphere; (5.3.1) and (5.4.1)-(5.4.4) give

$$(5.5.1) \qquad \frac{1}{2}\Delta\tilde{S} = \sum_{a}\sum_{i<j}(\sigma(a)_i - \sigma(a)_j)^2\left\{2\varkappa(a)_{ij} - \frac{n+2R}{n+2}\right\} + \langle \bar{X}_{ijk}, \bar{X}^{ijk}\rangle,$$

which implies the following result:

5.6. Theorem. *Let* $\tilde{x}$: $M \to S^{N-1}(1)$ *be a minimal isometric immersion into the unit-sphere, where* (M, g) *is complete; assume*

$$(5.6.1) \qquad 2(n+2)\varkappa_0 \geqslant n+2R.$$

Then either $\tilde{x}(M)$ *is totally geodesic, i.e.* $\tilde{x}(M)$ *is a great n-sphere in* $S^{N-1}(1)$, *or* $\tilde{x}(M)$ *is an isometric immersion (but no embedding) of an n-sphere* $S^n(\varkappa_0)$, $\varkappa_0 = n/2(n+1)$.

5.7. Remark. (a) As $\tilde{x}(M) \subset S^{N-1}(1)$, we have $\varkappa \leqslant 1$ and $R \leqslant 1$. If $1 \geqslant \varkappa \geqslant \frac{1}{2}$, then (5.6.1) is fulfilled. So 5.6 improves Theorem I in [15]. The immersion of $S^n(\varkappa_0)$, $\varkappa_0 = n/2(n+1)$, is one of the examples given in [6].

5.8. Theorem. *Let* (M, g) *be closed and let* x: $M \to E^N$ *be an isometric immersion with parallel mean curvature vector* ξ *(therefore* $\langle \xi, \xi \rangle =$ const*).*

If $2n\{(n+2)\varkappa_0 - R\} \geqslant \langle \xi, \xi \rangle$, *then* $x(M)$ *is an n-sphere or* $\tilde{x}(M)$ *is an isometric minimal immersion (but no embedding) of an n-sphere* $S^n(\varkappa_0)$, $\varkappa_0 = \langle \xi, \xi \rangle/2n(n+1)$, *into* $S^{N-1}(\bar{\varkappa})$, $\bar{\varkappa} = \langle \xi, \xi \rangle/n^2$.

Proof. Cf. [15], Theorem II (in [15], without loss of generality, the constant $\langle \xi, \xi \rangle$ was chosen to be $\langle \xi, \xi \rangle = n^2$).

Especially for $n = 2$ we get from (5.5.1) Theorem B which was formulated in the introduction. Naturally there is an analogue to Theorem 5.8:

5.9. Theorem. *Let* (M, g) *be closed,* $\dim M = 2$, *and* x: $M \to E^N$ *be an isometric immersion with parallel mean curvature vector and* $12\varkappa_0 \geqslant \langle \xi, \xi \rangle$.

Then either $x(M)$ *is a great sphere* $S^2(\bar{\varkappa}) \subset S^4(\bar{\varkappa})$, $\bar{\varkappa} = \langle \xi, \xi \rangle/4$, *or* $x(M)$ *is a Veronese surface of constant curvature* $\hat{\varkappa} = \langle \xi, \xi \rangle/12$ *in* $S^4(\bar{\varkappa})$.

Addendum. As a result of discussion with K. Voss and P. Buser (Geometrietagung Oberwolfach 1978) we get the following corollary to Theorem A:

Theorem C. *Let* (M, g) *be closed, simply connected,* $\dim M = 2$, *and* $\varkappa_0 \geqslant \frac{1}{3}\varkappa_1 > 0$. *Then*

(a) *There are exactly three eigenvalues* $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ *(counted with their multiplicity) in the interval* $[2\varkappa_0, 2\varkappa_1]$, *and* $\lambda_4 \geqslant 6\varkappa_0 \geqslant 2\varkappa_1$.

(b) *If* $\lambda_1 = 2\varkappa_0$ *or* $\lambda_3 = 2\varkappa_1$, *then* (M, g) *is isometrically diffeomorphic to a sphere. If* $\lambda_4 = 6\varkappa_0$, *then* (M, g) *is isometrically diffeomorphic to* $S^2(\varkappa_0)$.

Proof. (a) Let (M, g) be given as above. For simplicity, assume that $\varkappa_0 \geqslant \frac{1}{3}$ and $\varkappa_1 = 1$. By the existence theorem of Weyl [1] one can realize the Riemannian manifold as an ovaloid in the Euclidean space

[1] Cf., e.g., L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Communications on Pure and Applied Mathematics 6 (1953), p. 337-394.

E^3, choosing $M = S^2(1)$ and considering the curvature $\varkappa$ as a given function of the outer normal ξ of $S^2(1)$. Define a one-parameter family $\big(S^2(1), g(t)\big)$ of ovaloids with metric $g(t)$ in the following way:

For each $t \in [0, 1]$ there exists an ovaloid with curvature $\varkappa(t)$, given as a function of the outer normal, where

$$\frac{1}{\varkappa(t)} = (1-t)\frac{1}{\varkappa} + t, \qquad t \in [0, 1].$$

As

$$\int \frac{1}{\varkappa}\, \xi d\omega = 0 \quad \text{and} \quad \int \xi d\omega = 0 \quad \text{on } S^2(1),$$

we get

$$\int \frac{1}{\varkappa(t)}\, \xi d\omega = 0 \quad \text{for each } t.$$

The metric $g(t)$ changes continuously with t and so do $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$. As $\lambda_1(0) = \lambda_2(0) = \lambda_3(0) = 2\varkappa_0 = 2\varkappa_1$ on the unit sphere, from Theorem A we get $2\varkappa_0(t) \leqslant \lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t) \leqslant 2\varkappa_1(t)$ for $t \in [0, 1]$, which especially holds for $t = 1$. Thus

$$\big(M, g(1)\big) = (M, g).$$

(b) Assume that an eigenvalue λ fulfils $\lambda = 2\varkappa_1$. Then 2.4 implies

$$0 \geqslant \int \|B(f)\|^2 d\omega + \int \|\mathrm{grad}f\|^2 (\varkappa_1 - \varkappa)(2\varkappa_0 + \varkappa - \varkappa_1)\, d\omega \geqslant 0.$$

As $G := \{p \in M \mid \mathrm{grad}f|_p = 0\}$ is nowhere dense in M (cf. 2.5), we get

$$(\varkappa_1 - \varkappa)(2\varkappa_0 + \varkappa - \varkappa_1) = 0 \quad \text{on } M,$$

which together with $\varkappa_1 \geqslant \varkappa \geqslant \tfrac{1}{2}\varkappa_1$ gives

$$(\varkappa_1 - \varkappa)(\varkappa - \varkappa_0) = 0 \quad \text{on } M.$$

But this is possible only if $\varkappa = \mathrm{const}$ on M. Therefore $\lambda_1 = 2\varkappa_0$, which implies the assertion.

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