ARTINIAN MODULES ARE COUNTABLY GENERATED

BY

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Let \( R \) be a commutative ring with identity. Bass [1] showed that any chain of submodules in a Noetherian \( R \)-module is countable. The purpose of this note is to prove the dual statement; namely, that any chain of submodules in an Artinian \( R \)-module is countable. Our general reference will be Lambek [5].

THEOREM. Let \( R \) be a commutative ring with identity. Then any Artinian \( R \)-module \( A \) is countably generated.

Proof. Let \( S_n \) be the \( R \)-submodule of \( A \) generated by all cyclic submodules of length less than or equal to \( n \). Then

\[
0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \quad \text{and} \quad \bigcup_{n=1}^{\infty} S_n = A.
\]

We show that \( \bigcup_{n=1}^{\infty} S_n = A \). Let \( x \in A \). Then \( Rx \) being a submodule of \( A \) is again Artinian. Now \( Rx \) and \( R/\text{ann}(x) \) are isomorphic as \( R \)-modules. Since \( R \) is commutative, \( \text{ann}(x) \) is an ideal of \( R \), and hence \( R/\text{ann}(x) \) is an Artinian ring. By Hopkins' Theorem [3], \( R/\text{ann}(x) \) is also Noetherian and hence of finite length. Thus \( Rx \) has finite length as an \( R \)-module. Thus \( x \in Rx \subseteq S_n \) for some \( n \). To complete the proof it suffices to show that each \( S_n \) is finitely generated. Now \( S_1 \) is just the socle of \( A \), that is, \( S_1 \) is the sum of all simple submodules of \( A \). Thus \( S_1 \) is isomorphic to a direct sum of simple modules. Since \( S_1 \) is Artinian (it is a submodule of \( A \)), \( S_1 \) is a finite direct sum of simple modules, and hence is finitely generated. Assume by induction that \( S_n \) is finitely generated. Then \( S_{n+1}/S_n \) is contained in the socle of the Artinian module \( A/S_n \). It follows that \( S_{n+1}/S_n \) is finitely generated. Since \( S_n \) and \( S_{n+1}/S_n \) are finitely generated, \( S_{n+1} \) is also finitely generated.

The following result is now immediate:

COROLLARY. Let \( R \) be a commutative ring with identity and let \( A \) be an Artinian \( R \)-module. Then any chain of submodules of \( A \) is countable.
Both our result and the result of Bass fail if \( R \) is allowed to be non-commutative. Fuchs [2], Example 3, constructs Artinian modules over a polynomial ring in non-commuting indeterminates which have any desired length. In this example there exist cyclic submodules not of finite length.

If \( R \) is a left Noetherian ring with identity, then \( R \) may have uncountable descending chains of two-sided ideals. For each ordinal \( \alpha \), Jategaonkar [4] constructs a local principal left-ideal domain \( R \) with \( J^\alpha \neq 0 \), where \( J \) is the Jacobson radical of \( R \).

REFERENCES


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