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ARTINIAN MODULES ARE COUNTABLY GENERATED

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Let R be a commutative ring with identity. Bass [1] showed that any chain of submodules in a Noetherian R-module is countable. The purpose of this note is to prove the dual statement; namely, that any chain of submodules in an Artinian R-module is countable. Our general reference will be Lambek [5].

THEOREM. *Let R be a commutative ring with identity. Then any Artinian R-module A is countably generated.*

Proof. Let S_n be the R-submodule of A generated by all cyclic submodules of length less than or equal to n. Then

$$0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \quad \text{and} \quad \bigcup_{n=1}^{\infty} S_n = A.$$

We show that $\bigcup_{n=1}^{\infty} S_n = A$. Let $x \in A$. Then Rx being a submodule of A is again Artinian. Now Rx and $R/\mathrm{ann}(x)$ are isomorphic as R-modules. Since R is commutative, $\mathrm{ann}(x)$ is an ideal of R, and hence $R/\mathrm{ann}(x)$ is an Artinian ring. By Hopkins' Theorem [3], $R/\mathrm{ann}(x)$ is also Noetherian and hence of finite length. Thus Rx has finite length as an R-module. Thus $x \in Rx \subseteq S_n$ for some n. To complete the proof it suffices to show that each S_n is finitely generated. Now S_1 is just the *socle* of A, that is, S_1 is the sum of all simple submodules of A. Thus S_1 is isomorphic to a direct sum of simple modules. Since S_1 is Artinian (it is a submodule of A), S_1 is a finite direct sum of simple modules, and hence is finitely generated. Assume by induction that S_n is finitely generated. Then S_{n+1}/S_n is contained in the socle of the Artinian module A/S_n. It follows that S_{n+1}/S_n is finitely generated. Since S_n and S_{n+1}/S_n are finitely generated, S_{n+1} is also finitely generated.

The following result is now immediate:

COROLLARY. *Let R be a commutative ring with identity and let A be an Artinian R-module. Then any chain of submodules of A is countable.*

Both our result and the result of Bass fail if R is allowed to be non-commutative. Fuchs [2], Example 3, constructs Artinian modules over a polynomial ring in non-commuting indeterminates which have any desired length. In this example there exist cyclic submodules not of finite length.

If R is a left Noetherian ring with identity, then R may have uncountable descending chains of two-sided ideals. For each ordinal α, Jategaonkar [4] constructs a local principal left-ideal domain R with $J^{\alpha} \neq 0$, where J is the Jacobson radical of R.

REFERENCES

[1] H. Bass, *Descending chains and the Krull ordinal of commutative Noetherian rings*, Journal of Pure and Applied Algebra (1971), p. 347-360.

[2] L. Fuchs, *Torsion preradicals and ascending Loewy series of modules*, Journal für die reine und angewandte Mathematik 239-240 (1969), p. 169-179.

[3] C. Hopkins, *Rings with minimal condition for left ideals*, Annals of Mathematics 40 (1939), p. 712-730.

[4] A. Jategaonkar, *A counter-example in ring theory and homological algebra*, Journal of Algebra 12 (1969), p. 418-440.

[5] J. Lambek, *Lectures on rings and modules*, Waltham, Mass., 1966.

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