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# Controllability and observability of control systems under uncertainty

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## 1. Introduction

The purpose of this paper is to review local controllability and/or observability of the system

(1.1) $$\text{for almost all} \quad t \in [0, T], \quad x'(t) \in F(t, x(t))$$

whose evolution is described by a differential inclusion.

The overall strategy consists in "linearizing" such a differential inclusion and deriving local results from the global controllability and/or observability of the linearized differential inclusion.

Results of this nature are useful to the extent where we know how to characterize controllability and/or observability of such a linearized differential inclusion: we shall provide necessary and sufficient conditions extending Kalman's celebrated rank condition and show that in this case, controllability and observability are dual concepts.

There is no longer any need to justify the use of differential inclusions, which provide a unifying framework for dealing with closed-loop control systems

$$x' = f(t, x, u), \quad u \in U(t, x)$$

or control systems defined in an implicit way

$$f(t, x, x', u) = 0, \quad u \in U(t, x)$$

or systems under uncertainty, where the set-valued map takes into account disturbances and/or perturbations (or even differential games).

**(a) Linearization through derivatives of set-valued maps.** Linearization of the differential inclusion requires naturally a differential calculus of set-valued maps, which will be presented in the fourth section.

The idea behind the construction of a differential calculus of set-valued maps is the simple idea of Fermat and is still the one to which all of us have been first acquainted during our teens. It starts with the concept of tangent to the graph of a function: the derivative is the slope of the tangent to the curve. We should say, now, that the tangent space to the graph of the curve is the graph of the differential. This is the statement that we take as a basis for adapting to the set-valued case the concept of derivative.

Consider a set-valued map $F$: $X \rightrightarrows Y$, which is characterized by its graph (the subset of pairs $(x, y)$ such that $y$ belongs to $F(x)$).

We need first an appropriate notion of tangent cone to a set in a Banach space at a given point, which coincides with the tangent space when the set is an embedded differentiable manifold and with the tangent cone of convex analysis when the set is convex. At the time, experience shows that four tangent cones seem to be useful:

(1) Bouligand's contingent cone, introduced in the 30's.

(2) Adjacent tangent cone, also known as the "intermediate cone".

(3) Clarke's tangent cone, introduced in 1975.

(4) Bouligand's paratingent cone, introduced in the 30's.

They correspond to different regularity requirements. The tangent cone of Clarke is always convex. There already exists a sufficiently detailed calculus of these cones (see [20], Ch. 4).

Once a concept of tangent cone is chosen, we can associate with it a notion of derivative of a set-valued map $F$ at a point $(x, y)$ of its graph:

> it is a set-valued map $F'(x, y)$ the graph of which is equal to the tangent cone to the graph of $F$ at the point $(x, y)$.

In this way, we associate with the contingent cone, the adjacent and the Clarke tangent cones the following concepts of derivatives:

(1) contingent derivative,

(2) adjacent derivative,

(3) circatangent derivative, corresponding to the continuous Fréchet derivative,

(4) paratingent derivative.

Derivatives of set-valued maps (and also of nonsmooth single-valued maps) are set-valued maps which are positively homogeneous. They are convex (in the sense that their graph is convex) when they depend in a "continuous" way on $(x, y)$. Such maps, whose graphs are closed convex cones, are the set-valued analogues of continuous linear operators, called *closed convex processes*.

They are presented in the second section, and one can say that almost all properties of continuous linear operators can be extended to closed convex processes (including Banach's closed graph and open mapping theorems and Banach–Steinhaus's theorem).

Therefore, the linearized differential inclusion of (1.1) around a given solution $z(\cdot)$ will have the form

$$\text{(1.2)} \qquad \text{for almost all } t \in [0, T], \quad w'(t) \in F'(t, z(t), z'(t))(w(t)).$$

Let $S$ denote the solution map (or the funnel) associating with any initial state $x_0$ the set of solutions to (1.1) starting at $x_0$.

Can such a linearized differential inclusion (1.2) be regarded as a *variational inclusion*, in the sense that the set of solutions $w(\cdot)$ of (1.2) starting at some $u$ is related to the derivative of the solution map at $(x_0, z(\cdot))$ in the direction $u$?

The answer is positive, and is the object of several *variational theorems* presented in the fifth section.

**(b) Local controllability.** Let $R(T, \xi) := \{x(T) \mid x \in S_T(\xi)\}$ be the reachable set at time $T$ and let $M \subset \boldsymbol{R}^n$, a closed subset, be the target. We shall say that the system is *locally controllable* around $M$ if

$$0 \in \operatorname{Int}(R(T, \xi) - M).$$

This means that there exists a neighborhood $U$ of 0 in $\boldsymbol{R}^n$ such that for all $u \in U$ there exists a solution $x(\cdot) \in S_T(\xi)$ such that $x(T) \in M + u$.

We shall say that the linearized system (1.2), where we take for derivative $F'$ the circatangent derivative, is controllable around $C_M(z(T))$ (the Clarke tangent cone to $M$ at $z(T)$) if

$$R^L(T, 0) - C_M(z(T)) = \boldsymbol{R}^n,$$

where $R^L(T, 0)$ denotes the reachable set of (1.2) from 0.

Under suitable assumptions, controllability of the linearized system implies local controllability of the original system.

We derive this result from a general inverse function theorem under constraints. It states that if the derivative $CF(x_0, y_0)$ of a set-valued map $F$ from a Banach space $X$ to a finite-dimensional space $Y$ is surjective, then $F$ is invertible around $y_0$ and its inverse enjoys some kind of Lipschitz property.

This result is a simple form of more powerful controllability results obtained by refinements of set-valued analysis (see [32]).

**(c) Local observability.** System (1.1) is observed through an observation map $H$, which is generally a set-valued map from the state space $X$ to some observation space $Y$, which associates with each solution to the differential inclusion (1.2) an observation $y(\cdot)$ satisfying

$$\text{(1.3)} \qquad \forall t \in [0, T], \quad y(t) \in H(x(t)).$$

Observability concepts deal with the possibility of recovering the initial state $x_0 = x(0)$ of the system knowing only the evolution of an observation

$t \in [0, T] \to y(t)$ during the interval $[0, T]$, and naturally, knowing the laws (1.1), (1.3). Once we get the initial state $x_0$, we may, by studying the differential inclusion, gather information about the solutions starting from $x_0$, using the many results provided by the theory of differential inclusions.

The set-valued character leads to two types of input-output (set-valued) maps:

Sharp Input-Output map which is the (usual) product

$$\forall x_0 \in X, \quad I_-(x_0) := (H \circ S)(x_0) := \bigcup_{x(\cdot) \in S(x_0)} H(x(\cdot)).$$

Hazy Input-Output map which is the *square product*

$$\forall x_0 \in X, \quad I_+(x_0) := (H \square S)(x_0) := \bigcap_{x(\cdot) \in S(x_0)} H(x(\cdot)).$$

The *sharp* Input-Output map tracks the evolution of *at least* a state starting at some initial state $x_0$ whereas the *hazy* Input-Output map tracks *all* such solutions.

Recovering the input $x_0$ from the outputs $I_-(x_0)$ or $I_+(x_0)$ means that the set-valued maps are "injective" in some sense.

We shall choose the following strategy for studying local observability:

(1) Provide a general principle of local injectivity of the set-valued maps $I_+$ and $I_-$, which derives these properties from the fact that the kernel of an adequate derivative of $I_+$ or $I_-$ is equal to 0.

(2) Supply chain rule formulas which allow to compute the derivatives of the usual product $I_-$ and the square product $I_+$ from the derivatives of the observation map $H$ and the solution map $S$.

(3) Use the various derivatives of the solution map $S$ in terms of the solution maps of the associated variational inclusions provided by the variational theorems.

**(d) Controllability and observability of convex processes.** For simplicity, consider now the case where $\xi$ is an equilibrium of a (time-independent) system, i.e. a solution to

$$0 \in F(\xi), \tag{1.4}$$

where $F$ is assumed to be smooth enough, so that its derivative $A := DF(\xi, 0)$ is a closed convex process.

So, local controllability around $\xi$ and observability of the system at $\xi$ can be derived from the controllability of the closed convex process

$$x'(t) \in A(x(t)), \quad x(0) = 0, \tag{1.5}$$

and the observability of this system through the linear operator $H'(\xi)$.

As continuous linear operators, closed convex processes can be transposed. Let $A$ be a convex process; we define its *transpose* $A^*$ by

$$p \in A^*(q) \Leftrightarrow \forall (x, y) \in \text{Graph } A, \quad \langle p, x\rangle \leqslant \langle q, y\rangle.$$

We introduce the adjoint differential inclusion

(1.6) $\qquad$ for almost all $\quad t \in [0, T], \quad -q'(t) \in A^*(q(t))$

the cones $Q_T$ and $Q$ defined by

(i) $\qquad Q_T := \{v \mid \exists q(\cdot)$, a solution to (1.6) satisfying $q(T) = v\}$,

(ii) $$Q := \bigcap_{T>0} Q_T.$$

We shall say that the adjoint system is "observable" if $Q = \{0\}$.

We denote by $R_T$ the *reachable set at time* $T$ defined by

$$R_T := \{x(T) \mid x(\cdot) \text{ is a solution to (1.5)}\}.$$

We also say that

$$R := \bigcup_{T>0} R_T \quad \text{is the } \textit{reachable set},$$

and that the differential inclusion (1.5) (or the convex process $A$) is *controllable* if the reachable set $R$ is equal to the whole space $\boldsymbol{R}^n$.

The duality method lies in the following statement:

(1.7) $R_T^+$ (the positive polar cone of $R_T$) is equal to $Q_T$ and $R^+ = Q$,

so that $A$ is controllable if and only if $A^*$ is observable. Actually, when the domain of the closed convex process $A$ is the whole space, we can provide eleven necessary and sufficient conditions for the controllability of the convex process $A$, which will be exposed in the third section.

The contents of this survey are the following.

We recall in the second section properties of closed convex processes, which we use for characterizing controllability and observability properties of linearized differential inclusions in the third section.

The fourth section is devoted to an exposition of tangent cones and derivatives of set-valued maps. We use these concepts to prove the variational theorems in the fifth section and abstract results on local injectivity and surjectivity in the sixth section.

The last two sections piece together the above results to prove the local controllability and local observability results which are the objectives of this paper.

## 2. Convex processes and their transposes

A set-valued map from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ is said to be a *convex process* if its graph is a convex cone. It is *closed* if its graph is closed. It is called *strict* if

$$\operatorname{Dom} A := \{x \in \boldsymbol{R}^n \mid A(x) \neq \varnothing\} \quad \text{is the whole space.}$$

Let $X$ be a Hilbert space and let $G \subset X$ be a subset. We denote by $G^+$ the (positive) *polar cone* of $G$, the closed convex cone defined by

$$G^+ := \{p \in X^* \mid \forall x \in G, \ \langle p, x\rangle \geqslant 0\}.$$

The separation theorem implies that the "bipolar" $G^{++}$ is the closed convex cone spanned by $G$. We shall use the following consequence of this fact.

LEMMA 2.1 (closed image Lemma). *Let $X$, $Y$ be two Hilbert spaces, let $\varphi$ be a continuous linear operator from $X$ to $Y$ and let $L$ be a closed convex cone of $Y$. Assume that*

$$\operatorname{Im} \varphi - L = Y \quad (\textit{surjectivity condition}).$$

*Then*

$$\varphi^{-1}(L)^+ = \varphi^*(L^+)$$

(see [14]).

We now recall some properties of convex processes from Rockafellar [69], [68], and [70] and Aubin–Frankowska [20], Chapter 2.

DEFINITION 2.2. Let $A$ be a convex process from $\boldsymbol{R}^n$ to itself. The *transpose* $A^*$ of $A$ is the set-valued map from $\boldsymbol{R}^n$ to itself given by

$$p \in A^*(q) \Leftrightarrow \forall (x, y) \in \operatorname{Graph}(A), \quad \langle p, x\rangle \leqslant \langle q, y\rangle.$$

In other words,

$$(q, p) \in \operatorname{Graph}(A^*) \Leftrightarrow (-p, q) \in (\operatorname{Graph} A)^+.$$

The transpose of $A^*$ is obviously a closed convex process and $A = A^{**}$ if and only if the convex process $A$ is closed. When $A$ is a linear operator, its transpose as a linear operator coincides with its transpose as a convex process.

If $A$ is a closed convex process, then

$$A(0) = (\operatorname{Dom} A^*)^+.$$

DEFINITION 2.3. Let $B$ denote the unit ball. When $A$ is a closed convex process, we define its *norm* by

$$\|A\| := \sup_{x \in B \cap \operatorname{Dom} A} \ \inf_{y \in A(x)} \|y\| \in [0, +\infty].$$

PROPOSITION 2.4. *Let $A$ be a strict closed convex process. Then*

(a) $\forall x, \ y \in \boldsymbol{R}^n$, $A(x) \subset A(y) + \|A\| \ \|x-y\| B$ *(i.e., $A$ is Lipschitzian with Lipschitz constant equal to $\|A\|$),*

(b) Dom $A^* = A(0)^+$ *and* $A^*$ *is upper semicontinuous with compact convex images, mapping the unit ball into the ball or radius* $\|A\|$,

(c) *the restriction of* $A^*$ *to the vector space* $\operatorname{Dom} A^* \cap (-\operatorname{Dom} A^*)$ *is single-valued and linear (and thus,* $A^*(0) = 0$).

(See [22].)

We observe that we always have

$$\sup_{p \in A^*(q_0)} \langle p, x_0 \rangle \leqslant \inf_{y \in A(x)} \langle q_0, y \rangle .$$

LEMMA 2.5. *Let* $A$ *be a closed convex process. For any* $x_0 \in$ Int Dom $A$, *and* $q_0 \in \operatorname{Dom} A^*$,

$$\sup_{p \in A^*(q_0)} \langle p, x_0 \rangle = \inf_{y \in A(x_0)} \langle q_0, y \rangle$$

(see [68]).

We now extend to the case of closed convex cones the concepts of invariant subspaces. When $K$ is a subspace and $F$ is a linear operator, we recall that $K$ is invariant by $F$ when $Fx \in K$ for all $x \in K$. When $A$ is a convex process, there are two ways of extending this notion: we shall say that $K$ is invariant by $A$ if, for any $x \in K$, $A(x) \subset K$ and that $K$ is a viability domain for $A$ if, for any $x \in K$, $A(x) \cap K \neq \emptyset$. We also need to extend these notions to the case when $K$ is a closed convex cone. We recall the

DEFINITION 2.6. If $K$ is a closed convex set and $x$ belongs to $K$, we say that

$$T_K(x) := \operatorname{cl}\left( \bigcup_{h>0} \frac{1}{h}(K - x) \right)$$

is the *tangent cone* to $K$ at $x$.

LEMMA 2.7. *When* $K$ *is a vector subspace, then, for all* $x \in K$, $T_K(x) = K$ *and when* $K$ *is a closed convex cone, then*

$$\forall x \in K, \qquad T_K(x) = \operatorname{cl}(K + \mathbf{R}x).$$

Now, we can introduce

DEFINITION 2.8. Let $K$ be a closed convex cone and $A$ a convex process. We say that $K$ is *invariant* by $A$ if

$$\forall x \in K, \qquad A(x) \subset T_K(x)$$

and that $K$ is a *viability domain* for $A$ if

$$\forall x \in K, \qquad A(x) \cap T_K(x) \neq \emptyset .$$

These are dual notions, as the following proposition shows.

PROPOSITION 2.9. *Let $A$ be a strict closed convex process and let $K$ be a closed convex cone containing $A(0)$. Then $K$ is invariant by $A$ if and only if $K^+$ is a viability domain for $A^*$.*

Proof. By Proposition 2.4(b) the condition $A(0) \subset K$ implies that $K^+ \subset A(0)^+ = \operatorname{Dom} A^*$. To say that $K$ is invariant by $A$ amounts to saying that

$$\forall x \in K, \quad \forall q \in T_K(x)^+, \quad \inf_{y \in A(x)} \langle q, y \rangle \geqslant 0. \tag{2.1}$$

Lemma 2.7 states $T_K(x) = \overline{\boldsymbol{R}x + K}$, $T_{K^+}(q) = \overline{\boldsymbol{R}q + K^+}$. Therefore,

$$q \in T_K(x)^+ \Leftrightarrow \langle q, x \rangle = 0, \quad q \in K^+ \Leftrightarrow x \in T_{K^+}(q)^+.$$

On the other hand, Lemma 2.5 implies that $\inf_{y \in A(x)} \langle q, y \rangle = \sup_{p \in A^*(q)} \langle p, x \rangle$. Therefore, condition (2.1) is equivalent to the condition

$$\forall q \in K^+, \quad \forall x \in T_{K^+}(q)^+, \quad \sup_{p \in A^*(q)} \langle p, x \rangle \geqslant 0. \tag{2.2}$$

By Proposition 2.4(b), for all $q \in K^+$, the set $A^*(q)$ is compact. The separation theorem implies that $A^*(q)$ has a nonempty intersection with $T_{K^+}(q)$ if and only if for all $x \in \boldsymbol{R}^n$, $\sup_{p \in A^*(q)} \langle p, x \rangle \geqslant \inf_{z \in T_{K^+}(q)} \langle z, x \rangle$. Since $T_{K^+}(q)$ is a cone, the latter inequality is equivalent to (2.2). This ends the proof. □

We introduce now the concepts of eigenvalues and eingenvectors of closed convex processes.

DEFINITION 2.10. We shall say that $\lambda \in \boldsymbol{R}$ is an *eigenvalue* of a convex process $A$ if $\operatorname{Im}(A - \lambda I) \neq \boldsymbol{R}^n$ and that $x \in \operatorname{Dom} A$ is an *eigenvector* of $A$ if $x \neq 0$ and if there exists $\lambda \in \boldsymbol{R}$ such that $\lambda x \in A(x)$.

We observe that half-lines spanned by eigenvectors of $A^*$ are viability domains for $A^*$.

LEMMA 2.11. *Let $A$ be a strict convex process. Then $A^*$ has an eigenvector if and only if $\operatorname{Im}(A - \lambda I) \neq \boldsymbol{R}^n$ for some $\lambda \in \boldsymbol{R}$.*

THEOREM 2.12. *Let $A$ be a strict closed convex process. If the largest viability domain $Q$ for $A^*$ is different from $\{0\}$ and contains no line, then $A^*$ has at least an eigenvector.*

EXAMPLE 2.13. Let $F$ be a linear operator from $\boldsymbol{R}^n$ to itself, let $L$ be a closed convex cone of controls and let $A$ be the strict closed convex process defined by $A(x) := Fx + L$.

A cone $K$ is invariant by $A$ if

$$\forall x \in K, \quad Fx + L \subset T_K(x)$$

and $\lambda$ is an eigenvalue of $A$ if

$$\operatorname{Im}(F-\lambda I)+L \neq \mathbf{R}^n.$$

The transpose $A^*$ of $A$ is defined by

$$A^*q = \begin{cases} F^*q & \text{if } q \in L^+, \\ \emptyset & \text{if } q \notin L^+. \end{cases}$$

A cone $P \subset L^+ = \operatorname{Dom} A^*$ is a viability domain for $A^*$ if and only if

$$\forall q \in P, \quad F^*q \in T_P(q).$$

An element $q \neq 0$ is an eigenvector of $A^*$ if and only if $q$ is an eigenvector of $F^*$ which belongs to the cone $L^+$.

Other examples of closed convex processes are provided by "circatangent derivatives" of set-valued maps (see Section 4).

Closed convex processes enjoy most of the properties of continuous linear operators, and in particular, the fundamental Banach

THEOREM 2.14 (Closed Graph Theorem). *A closed convex process A whose domain is the whole space is Lipschitz, in the sense that*

$$\forall x_1, x_2 \in X, \quad A(x_1) \subset A(x_2) + l\|x_1 - x_2\|B$$

whose open mapping formulation can be stated:

THEOREM 2.15 (Robinson–Ursescu's Open Mapping Theorem). *Assume that a closed convex process* $A: X \rightrightarrows Y$ *is surjective. Then there exists a constant* $l > 0$ *such that,*

$$\forall y \in Y, \quad \exists x \in A^{-1}(y) \quad \text{such that} \quad \|x\| \leqslant l\|y\|.$$

Banach–Steinhaus's uniform boundedness theorem can be extended to closed convex processes:

THEOREM 2.16 (Uniform Boundedness for Closed Convex Processes). *Let X and Y be reflexive Banach spaces and $A_h$ a family of closed convex processes from X to Y, "pointwise bounded" in the sense that*

$$\forall x \in X, \quad \exists y_h \in A_h(x) \quad \text{such that} \quad \sup_h \|y_h\| < +\infty.$$

*Then this family is "uniformly bounded" in the sense that*

$$\sup_h \|A_h\| < +\infty.$$

Hence we can speak of *bounded* families of closed convex processes, without specifying whether it is pointwise or uniform. We can deduce this very useful consequence:

THEOREM 2.17. *Let us consider a metric space $U$, reflexive Banach spaces $X$ and $Y$, and a set-valued map associating to each $u \in U$ a closed convex process $A(u)$: $X \rightrightarrows Y$. Let us assume that*

$$\textit{the family of closed convex processes } A(u) \textit{ is bounded.}$$

*Then the following conditions are equivalent:*

(i) *The set-valued map* $u \rightrightarrows \mathrm{Graph}\,(A(u))$ *is lower semicontinuous,*

(ii) *the set-valued map* $(u, x) \rightrightarrows A(u)(x)$ *is lower semicontinuous.*

(See [65], [23], [11].)

## 3. Controllability and observability of closed convex processes

We start this section by the duality theorem, which characterizes the polar cones of the reachable sets. Many results of this section as well as their proofs can be found in [22].

We denote by $W^{1,p}(0, T)$, $p \in [1, \infty]$, the Sobolev space of functions $x \in L^p(0, T; \boldsymbol{R}^n)$ such that $x'(\cdot)$ belongs to $L^p(0, T; \boldsymbol{R}^n)$.

Let us consider the Cauchy problem for the differential inclusion

$$(3.1)\quad \begin{aligned} &\text{(i)} \quad x'(t) \in A(x(t)) \quad \text{for almost all } t \in [0, T], \\ &\text{(ii)} \quad x(0) = 0. \end{aligned}$$

We recall that the reachable set $R_T$ is defined by

$$R_T := \{x(T) \mid x \in W^{1,1}(0, T) \text{ is a solution to } (3.1)\}.$$

We shall characterize its positive polar cone $R_T^+$. For that purpose, we associate with the differential inclusion (3.1) the adjoint inclusion

$$(3.2)\quad \begin{aligned} &\text{(i)} \quad -q'(t) \in A^*(q(t)) \quad \text{for almost all } t \in [0, T], \\ &\text{(ii)} \quad q(T) = \eta, \end{aligned}$$

and we denote by $Q_T \subset \mathrm{Dom}\,A^*$ the set of "final" values $\eta$ such that the differential inclusion (3.2) has a solution:

$$Q_T := \{\eta \mid \exists q \in W^{1,1}(0, T), \text{ a solution to } (3.2)\}.$$

THEOREM 3.1. *Let $A$ be a strict closed convex process. Then $R_T^+ = Q_T$.*

Proof. (a) We denote by $S$ the closed convex cone of solutions to the differential inclusion (3.1) in the Hilbert space

$$X := \{x \in W^{1,2}(0, T) \mid x(0) = 0\}.$$

Consider the continuous linear operator

$$\gamma_T\colon x(\cdot) \in X \to x(T) \in \boldsymbol{R}^n.$$

The transpose $\gamma_T^*$ maps $\boldsymbol{R}^n$ into the dual $X^*$ of $X$ and for all $\eta \in R_T^+$

$$\forall x \in S, \quad \langle \gamma_T^* \eta, x \rangle = \langle \eta, \gamma_T x \rangle \geqslant 0. \tag{3.3}$$

One can check that $S$ is dense in the $W^{1,1}(0, T)$-solutions to (3.1) in the metric of uniform convergence on $[0, T]$. This and (3.3) yield

$$R_T^+ = \{\eta \mid \gamma_T^* \eta \in S^+\}. \tag{3.4}$$

Let us set

(i) $Y := L^2(0, T; \boldsymbol{R}^n) \times L^2(0, T; \boldsymbol{R}^n)$,

(ii) $L := \{(x, y) \in Y \mid y(t) \in A(x(t)) \text{ a.e.}\}$,

(iii) $D$, the differential operator defined on $X$ by $Dx = x'$.

Then $S = (1 \times D)^{-1}(L)$. The closed image Lemma 2.1 applied to the continuous linear operator $\varphi = (1 \times D)$ states that

$$S^+ = (1 \times D)^*(L^+) \tag{3.5}$$

provided that the "surjectivity assumption"

$$\operatorname{Im}(1 \times D) - L = Y \tag{3.6}$$

is satisfied.

(b) It can be written

$$\forall (u, v) \in Y, \quad \exists x \in X \quad \text{such that} \quad x'(t) \in A(x(t) - u(t)) + v(t) \quad \text{a.e.}$$

Since the domain of $A$ is the whole space, $A$ is Lipschitzian.

The set-valued map $F(t, x) := A(x - u(t)) + v(t)$ is then measurable in $t$, Lipschitzian with respect to $x$, has closed images and satisfies the following estimate:

$$d(0, F(t, 0)) \leqslant \|A\| \|u(t)\| + \|v(t)\|.$$

The function $t \to \|A\| \; \|u(t)\| + \|v(t)\|$ being in $L^1(0, T)$, we can apply a Filippov Theorem [30] (see also [26]) which states the existence of a solution $x(\cdot)$ to the differential inclusion $x'(t) \in F(t, x(t))$, $x(0) = 0$, satisfying:

$$\|x'(t)\| \leqslant \|A\| \, e^{\|A\| T} \int_0^T d(0, F(t, 0)) dt + d(0, F(t, 0)).$$

Thus $x \in X$ and the surjectivity assumption (3.6) holds true.

(c) Therefore, by (3.4) and (3.5), we obtain the formula

$$R_T^+ = \{n \mid \gamma_T^* \eta \in (1 \times D)^*(L^+)\}. \tag{3.7}$$

Let $\eta \in Q_T$ and let $q$ be a solution to the adjoint inclusion (3.2). By Proposition 2.4(b), $q(\cdot) \in W^{1,\infty}(0, T)$ and for all $x \in S$

$$\langle \eta, x(T) \rangle = \langle (q', q), (x, x') \rangle_Y.$$

This is nonnegative by the definition of $A^*$. Thus $Q_T \subset R_T^+$. To prove the opposite, let $\eta$ belong to $R_T^+$. By (3.7), there exists $(p, q) \in L^+$ such that

$$\langle \eta, \gamma_T x\rangle = \langle p, x\rangle_{L^2} + \langle q, Dx\rangle_{L^2} \quad \forall x \in X. \tag{3.8}$$

By taking $x$ so that $x(T) = 0$, we deduce that $p = Dq$ in the sense of distributions. Since $p$ and $q$ belong to $L^2$, we infer that $q$ belongs to the Sobolev space $W^{1,2}(0, T)$. Thus $Dq = q'$. Integrating by parts in equation (3.8) and taking into account that $x(0) = 0$, we obtain

$$\langle \eta, \gamma_T x\rangle = \langle p - q', x\rangle_{L^2} + \langle q(T), x(T)\rangle = \langle q(T), x(T)\rangle.$$

The surjectivity of $\gamma_T$ implies that $\eta = q(T)$. Thus $q(\cdot)$ is a solution to (3.2) and then, $\eta$ belongs to $Q_T$. This completes the proof. □

We associate now with any $\eta \in \mathrm{Dom}\, A^*$ the "solution set" $S_T(\eta)$ of solutions to the adjoint differential inclusion (3.2) satisfying $q(T) = \eta$ and we denote by $Q_T$ the domain of the "solution map" $S_T$:

$$Q_T := \{\eta \in \mathrm{Dom}\, A^* \mid S_T(\eta) \neq \emptyset\}.$$

We observe that the sequence of the closed domains $Q_T$ decreases:

$$\text{if} \quad T_1 \geqslant T_2, \quad \text{then} \quad Q_{T_1} \subset Q_{T_2}.$$

We introduce the intersection $Q$ of these cones

$$Q := \bigcap_{T>0} Q_T.$$

Since the compact subsets $S^{n-1} \cap Q_T$ form a decreasing sequence, we observe that $Q \neq \{0\}$ if and only if all the cones $Q_T$ are different from 0. We shall say that $Q$ is the *largest viability domain*, thanks to the following theorem.

THEOREM 3.2. *Let $A$ be a strict closed convex process. Then the closed convex cone $Q$ is the largest closed convex cone which is a viability domain for $A^*$.*

Proof. It is not difficult to prove that $Q$ is a closed convex cone which contains any viability domain $P$. It remains to prove that $Q$ is a viability domain, i.e. that

$$\forall q \in Q, \quad A^*(q) \cap T_Q(q) \neq \emptyset.$$

Assume that $Q \neq \{0\}$. Thanks to the necessary condition of the viability theorem (see [46]), it is sufficient to prove that, for some $T > 0$,

$$\forall \eta \in Q, \quad \exists p(\cdot) \in S_T(\eta) \quad \text{which is viable on } Q.$$

Since $\eta$ belongs to $Q_{nT}$ for all $n \geqslant 2$, there exists a solution $p_n(\cdot) \in S_{nT}\{\eta\}$. By the very definition of $Q_t$, we know that $p(t) \in Q_t$ for all $t \leqslant nT$.

Therefore, the translated function $\hat{p}_n(\cdot)$ defined on $[0, T]$ by

$$\hat{p}_n(t) := p_n(t + (n-1)T)$$

belongs to $S_T(\eta)$ and satisfies for all $t \in [0, T]$, $k \leqslant n-1$,

$$\hat{p}_n(t) = p_n(t+(n-1)T) \in Q_{t+(n-1)T} \subset Q_{(n-1)T} \subset Q_{kT}.$$

But $S_T(\eta)$ is compact in $C(0, T; \boldsymbol{R}^n)$. Thus there exists a subsequence of $\hat{p}_n(\cdot)$ converging to some $\hat{p}(\cdot) \in S_T(\eta)$ uniformly on $[0, T]$. Since for all $t \in [0, T]$, $k \geqslant 1$, $\hat{p}(t) \subset Q_{kT}$, we infer that

$$\hat{p}(t) \subset \bigcap_{k \geqslant 1} Q_{kT} = Q. \quad \square$$

We translate now this result in terms of reachable sets $R_T$.

Since $0 \in A(0)$, the reachable cones $R(T)$ do form an increasing sequence. We define the reachable set of the inclusion (3.1) to be

$$R := \bigcup_{T>0} R(T).$$

It is a convex cone, which is equal to the whole space if and only if for some $T > 0$, $R(T) = \boldsymbol{R}^n$.

We say that the closure $\bar{R}$ of $R$ is the *smallest invariant cone* by $A$. This definition is motivated by the following consequences of both Theorem 3.1 and Theorem 3.2.

THEOREM 3.3. *Let $A$ be a strict closed convex process. Then the closed convex cone $\bar{R}$ is the smallest closed convex cone containing $A(0)$ and invariant by $A$.*

We consider now the largest subspace of $Q$:

$$Q \cap (-Q) \subset \operatorname{Dom} A^* \cap (-\operatorname{Dom} A^*).$$

PROPOSITION 3.4. *Let $A$ be a strict closed convex process. The subspace $Q \cap (-Q)$ is the largest subspace invariant by $A^*$ and its orthogonal space $R-R$ is invariant by $A$ in the sense that*:

$$\forall x \in R-R, \quad A(x) \subset R-R.$$

We consider now the cones $A(0)$, $A^2(0) := A(A(0)), \ldots, A^k(0) = A(A^{k-1}(0))$, etc. Since 0 belongs to $A(0)$, these convex cones form an increasing sequence. We introduce the cone

$$N := \operatorname{cl}\Big(\bigcup_{k \geqslant 1} A^k(0)\Big)$$

and the vector subspace

$$M \text{ spanned by } N.$$

THEOREM 3.5. *Let $A$ be a strict closed convex process. Then*

(a) $A(N) \subset N$,

(b) $\bar{R} \subset N \subset M \subset R - R$,

(c) $Q \cap (-Q) \subset \bigcap_{k \geqslant 1} A^k(0)^{\perp} \subset \bigcap_{k \geqslant 1} A^k(0)^{+} \subset Q$.

Remark. When the reachable set $R$ is a vector space, the subsets $R$, $N$, $M$ and $R-R$ coincide. This happens when, for instance, $A$ is symmetric (in the sense that $A(-x) = -A(x)$), i.e., when the graph of $A$ is a vector subspace.

By duality Theorem 3.1, the following dual version of this theorem holds true.

THEOREM 3.6. *Let $A$ be a strict closed convex process. Assume that the reachable set $R$ is different from $\boldsymbol{R}^n$ and spans the whole space. Then $A$ has at least one eigenvalue.*

We shall deduce from the preceding results several characterizations of the controllability of closed convex processes.

DEFINITION 3.7. We shall say that (3.1) is *controllable at time $T$* (respectively, *controllable*) if $R_T = \boldsymbol{R}^n$ (respectively, $R = \boldsymbol{R}^n$). We shall say that the adjoint inclusion (3.2) is *observable at time $T$* (respectively, *observable*) if $Q_T = \{0\}$ (respectively, $Q = \{0\}$).

We also observe the following property.

LEMMA 3.8. *Let $A$ be a strict closed convex process. The three following properties are equivalent.*

$$\text{(3.9)}\quad \begin{aligned} &\text{(a)} \quad \exists m \geqslant 1 \quad \textit{such that} \quad A^m(0) - A^m(0) = \boldsymbol{R}^n,\\ &\text{(b)} \quad \exists m \geqslant 1 \quad \textit{such that} \quad A^m(0)^{\perp} = \{0\},\\ &\text{(c)} \quad \exists m \geqslant 1 \quad \textit{such that} \quad \operatorname{Int} A^m(0) \neq \emptyset. \end{aligned}$$

It is convenient to introduce the

RANK CONDITION. We say that a convex process $A$ satisfies the *rank condition* if one of the equivalent properties (3.9) holds true.

LEMMA 3.10. *Consider the strict closed convex process $A(x) = Fx + L$, where $F \in \boldsymbol{R}^{n \times n}$ is a matrix and $L$ is a vector subspace of $\boldsymbol{R}^n$. Then $A$ satisfies the rank condition if and only if $A^n(0) - A^n(0) = \boldsymbol{R}^n$.*

We begin by stating characteristic properties of observability of the adjoint system (3.2) and then, use the duality results to infer the equivalent characteristic properties of controllability of system (3.1).

THEOREM 3.11. *Let $A$ be a strict closed convex process. The following properties are equivalent.*

(a*) *The adjoint inclusion* (3.2) *is observable.*

(b*) *The adjoint inclusion* (3.2) *is observable at time $T > 0$ for some $T$.*

(c*) *$\{0\}$ is the largest closed convex cone which is a viability domain for $A^*$.*

(d*) *$A^*$ has neither proper invariant subspace nor eigenvectors.*

(e*) *The rank condition is satisfied and $A^*$ has no eigenvectors.*

THEOREM 3.12. *Let $A$ be a strict closed convex process. The equivalent properties* (a*), (b*), (c*), (d*) *and* (e*) *of Theorem* 3.11 *are equivalent to the following properties.*

(a) *Differential inclusion* (3.1) *is controllable.*

(b) *Differential inclusion* (3.1) *is controllable at some time $T > 0$.*

(c) *$\boldsymbol{R}^n$ is the smallest closed convex cone containing $A(0)$ which is invariant by $A$.*

(d) *$A$ has neither proper invariant subspace nor eigenvalues.*

(e) *The rank condition is satisfied and $A$ has no eigenvalues.*

(f) *For some $m \geqslant 1$, $A^m(0) = (-A)^m(0) = \boldsymbol{R}^n$.*

In the case where the set-valued map $A$ is defined by $A(x) := Fx + L$, we derive known results due to Kalman when $L$ is a vector space of control and to Brammer, Korobov, Saperstone and Yorke [25], [53], [71] when $L$ is an arbitrary set of controls containing 0.

## 4. Tangent cones and derivatives of set-valued maps

We devote this section to the definitions of some (and may be, too many) of the tangent cones which have been used in applications, in particular, for defining derivatives of set-valued maps. Unfortunately, for arbitrary subsets, we are forced to introduce and study several concepts of tangent cones, which correspond to different regularity requirements.

But the idea is the same: implement one of the possible mathematical descriptions of the concept of tangency, without requiring a priori a vector space of tangent vectors, as in differential geometry.

DEFINITION 4.1 (Tangent cones). Let $K \subset X$ be a subset of a Banach space $X$ and let $x \in \bar{K}$ belong to the closure of $K$. We denote by

$$S_K(x) := \bigcup_{h>0} \frac{K-x}{h}$$

the cone spanned by $K - x$.

We introduce the following four tangent cones:

(1) The *contingent cone* $T_K(x)$, defined by

$$T_K(x) := \{v \mid \liminf_{h \to 0+} d_K(x+hv)/h = 0\}$$

(from the Latin *contingere*, to touch on all sides, introduced by G. Bouligand in 1931).

(2) The *adjacent cone* $T_K^b(x)$, defined by

$$T_K^b(x) := \{v \mid \lim_{h \to 0+} d_K(x+hv)/h = 0\}$$

(from the Latin *adiacere*, to lie near, recently used under the name *intermediate cone* by Frankowska [32] and the name of *derivable cone* by Rockafellar).

(3) The *Clarke tangent cone* $C_K(x)$, defined by

$$C_K(x) := \{v \mid \lim_{\substack{h \to 0+ \\ K \ni x' \to x}} d_K(x'+hv)/h = 0\}$$

(from the Canadian Clarke [26]; we shall use the adjective *circatangent* to mention properties derived from this tangent cone, for instance, circatangent derivatives).

(4) If $L \subset K$ is a subset of $K$, the *paratingent cone* $P_K^L(x)$ to $K$ relative to $L$ at $x \in L$, defined by

$$P_K^L(x) := \{v \mid \limsup_{\substack{h \to 0+ \\ L \ni x' \to x}} d_K(x'+hv)/h = 0\}$$

(introduced by Bouligand in 1931).

We see at once that these tangent cones are closed, that these tangent cones to $K$ and the closure $\bar{K}$ of $K$ do coincide, that

$$C_K(x) \subset T_K^b(x) \subset T_K(x) \subset S_K(x),$$

and that

$$\text{if} \quad x \in \operatorname{Int}(K), \quad \text{then} \quad C_K(x) = X.$$

The Clarke tangent cone $C_K(x)$ is a closed convex cone satisfying the following properties:

(i) $C_K(x) + T_K(x) \subset T_K(x)$ and

(ii) $C_K(x) + T_K^b(x) \subset T_K^b(x)$.

DEFINITION 4.2. We shall say that a subset $K \subset X$ is *sleek at* $x \in K$ if the set-valued map

$$K \ni x' \rightrightarrows T_K(x') \quad \text{is lower semicontinuous at } x$$

and *sleek* if and only if it is sleek at every point $x$ of $K$.

We shall say that $K$ is *derivable at* $x \in K$ if and only if $T_K^b(x) = T_K(x)$ and *derivable* if and only if it is derivable at every $x \in K$.

The following property is very useful:

THEOREM 4.3 (Tangent Cones of Sleek Subsets). *Let $K$ be a closed subset of a Banach space. If $K$ is sleek at $x \in K$, then the contingent and Clarke tangent cones do coincide, and consequently, are convex* (see [20]).

EXAMPLE (Tangent Cones to Convex Sets). Let us assume that $K$ is convex. Then the contingent cone $T_K(x)$ to $K$ at $x$ is convex and

$$C_K(x) = T_K^b(x) = T_K(x) = \overline{S_K(x)}.$$

Furthermore any closed convex subset is sleek.

The same is true for the embedded smooth manifolds (see [14]).

Remark. We are led to introduce this ménagerie of tangent cones because each of them corresponds to a classical regularity requirement. We shall see that the contingent cone is related to Gâteaux derivative, the adjacent cone to the Fréchet derivative and the Clarke tangent cone to the continuous Fréchet derivative.

The contingent cone plays a crucial role to characterize the subsets $K \subset \boldsymbol{R}^n$ which enjoy the viability property: for every $x_0 \in K$, there exists a solution to the differential inclusion $x' \in F(x)$ which is viable in the sense that $x(t) \in K$ for all $t \geqslant 0$.

When $F$ is upper semi-continuous with closed convex images and linear growth, Haddad's viability theorem (see [46]), an extension of the 1943 Nagumo theorem, states that $K$ enjoys the viability property if and only if

$$\forall x \in K, \qquad F(x) \cap T_K(x) \neq \emptyset .$$

Adjacent tangent cones play an important role in Lebesgue and Sobolev spaces.

The charm of the Clarke tangent cone (and thus, of sleek subsets) is the convexity, which allows to state dual formulations and statements by polarity and transposition. But the price to pay in terms of loss of information for playing with duality just to be able to conserve some familiar dual formulation is indeed too high in many situations. This is one of the reasons why we shall not use normal cones and generalized gradients here.

We now derive from each concept of tangent cone to a subset an associated concept of *graphical derivative* of a set-valued map $F$ from a topological vector space $X$ to another $Y$.

The idea is very simple, and goes back to the prehistory of the differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of the tangent to the graph of a function.

The tangent space to the graph of a function $f$ at a point $(x, y)$ of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \to f'(x)u$.

It is possible to implement this idea for any set-valued map $F$ since we have introduced (unfortunately, several) ways to implement the concept of tangency for any subset of a topological vector space. Therefore, in the framework of a given problem, we can choose the adequate concept of tangent cone, and thus, regard this tangent cone to the graph of the set-valued map $F$ at some point $(x, y)$ of its graph as the graph of the associated "graphical" derivative of $F$ at this point $(x, y)$.

Since the tangent cones are at least ... cones, all these derivatives are at least *positively homogeneous* set-valued maps (also called *processes*). This is what remains of the familiar, but luxurious, requirement of linearity.

However, they are closed convex processes, i.e., set-valued analogues of continuous linear operators, when the tangent cones happen to be closed and convex (this is the case when we use the Clarke tangent cone).

Hence we start with some definitions and notations.

DEFINITION 4.4. Let $F\colon X \rightrightarrows Y$ be a set-valued map from a Banach vector space $X$ to another $Y$.

We introduce the four following graphical derivatives

(1) the *contingent derivative* $DF(x, y)$, defined by

$$\mathrm{Graph}(DF(x, y)) := T_{\mathrm{Graph}\,(F)}(x, y),$$

(2) the *adjacent derivative* $D^bF(x, y)$, defined by

$$\mathrm{Graph}(D^bF(x, y)) := T^b_{\mathrm{Graph}\,(F)}(x, y),$$

(3) the *circatangent derivative* $CF(x, y)$, defined by

$$\mathrm{Graph}(CF(x, y)) := C_{\mathrm{Graph}\,(F)}(x, y),$$

(4) the *paratingent derivative* $PF(x, y)$, defined by

$$\mathrm{Graph}(PF(x, y)) := P_{\mathrm{Graph}\,(F)}(x, y).$$

We shall say that $F$ is *sleek at* $(x, y) \in \mathrm{Graph}\,(F)$ if and only if

$$(x', y') \rightrightarrows \mathrm{Graph}(DF(x', y')) \quad \text{is lower semicontinuous at } (x, y)$$

and it is *sleek* if it is sleek at every point of its graph.

We shall say that $F$ is *derivable at* $(x, y) \in \mathrm{Graph}(F)$ if and only if the contingent and adjacent derivatives coincide:

$$DF(x, y) := D^bF(x, y)$$

and that it is *derivable* if it is derivable at every point of its graph.

But what about Newton and Leibniz, who introduced the derivatives as limits to differential quotients? Our first duty is to characterize the various "graphical definitions" as adequate limits of differential quotients. Unfortunately, the formulas become very often quite ugly, and nobody in a right frame of mind would have invented them from scratch if they were not derived from the graphical approach.

But all these limits are pointwise limits, which classify all these generalized derivatives in a class different from the class of distributional derivatives introduced by L. Schwartz and S. Sobolev in the fifties, for solving partial differential equations. (Their objective was to keep the linearity of the differential operators, by allowing convergence of the differential quotients in weaker and weaker topologies, the price to be paid being that derivatives may no longer be functions, but distributions.)

For instance, the *contingent derivative* $DF(x, y)$ of $F$ at $(x, y)$ is the set-valued map from $X$ to $Y$ defined by

$$v \in DF(x, y)(u) \Leftrightarrow \liminf_{\substack{h \to 0+ \\ u' \to u}} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0$$

and the *paratingent derivative* $PF(x, y)$ of $F$ at $(x, y)$ is the set-valued map from $X$ to $Y$ defined by

$$v \in PF(x, y)(u) \Leftrightarrow \liminf_{\substack{h \to 0+ \\ (x',y') \to_F (x,y) \\ u' \to u}} d\left(v, \frac{F(x'+hu')-y'}{h}\right) = 0,$$

where $\underset{F}{\to}$ denotes the convergence in Graph$(F)$.

When $F$ is lipschitzian around $x \in \mathrm{Int}(\mathrm{Dom}(F))$, the above formulas become

$$\text{(i)} \qquad v \in DF(x, y)(u) \Leftrightarrow \liminf_{h \to 0+} d\left(v, \frac{F(x+hu)-y}{h}\right) = 0,$$

$$\text{(ii)} \qquad v \in PF(x, y)(u) \Leftrightarrow \liminf_{\substack{h \to 0+ \\ (x',y') \to_F (x,y)}} d\left(v, \frac{F(x'+hu)-y'}{h}\right) = 0.$$

Moreover, if $k$ denotes the Lipschitz constant of $F$ at $x$, then for every $y \in F(x)$ the derivative $DF(x, y)$ has nonempty images and is $k$-lipschitzian.

Despite the fact that both adjacent and circatangent derivatives can be defined as limits of difference quotients for any set-valued map $F$, the formulas are simpler when we deal with lipschitzian set-valued maps. Since we use them only in this context in this paper, we provide their formulas in this limited case.

Assume that $F$ is lipschitzian around an element $x \in \mathrm{Int}(\mathrm{Dom}(F))$, then the *adjacent derivative* $D^bF(x, y)$ and the *circatangent derivative* $CF(x, y)$ are the set-valued maps from $X$ to $Y$ respectively defined by

$$v \in D^bF(x, y)(u) \Leftrightarrow \lim_{h \to 0+} \left(v, \frac{F(x+hu)-y}{h}\right) = 0$$

and

$$v \in CF(x, y)(u) \Leftrightarrow \lim_{\substack{h \to 0+ \\ (x',y') \to_F (x,y)}} d\left(v, \frac{F(x'+hu)-y'}{h}\right) = 0.$$

Several remarks are in order. First, all these derivatives are positively homogeneous and their graphs are closed.

We observe the obvious inclusions

$$CF(x, y)(u) \subset D^bF(x, y)(u) \subset DF(x, y)(u) \subset PF(x, y)(u)$$

and that the definitions of contingent and adjacent derivatives on the one hand, the paratingent and circatangent derivatives, on the other one, are symmetric. When $F := f$ is single-valued, we set

$$Df(x) := Df(x, f(x)), \quad D^b f(x) := D^b f(x, f(x)), \quad Cf(x) := Cf(x, f(x)).$$

We see easily that

$$\begin{aligned} Df(x)(u) &= f'(x)u && \text{if } f \text{ is Lipschitz and Gâteaux differentiable,} \\ D^b f(x)(u) &= f'(x)u && \text{if } f \text{ is Fréchet differentiable,} \\ Cf(x)(u) &= f'(x)u && \text{if } f \text{ is continuously differentiable.} \end{aligned}$$

This allows also to define and use derivatives of restrictions $F := f|_K$ of single-valued maps $f$ to subsets $K \subset X$, which are defined by

$$f|_K(x) := \begin{cases} f(x) & \text{if } x \in K, \\ \emptyset & \text{if } x \notin K. \end{cases}$$

If $f$ is continuously differentiable around a point $x \in K$, then the *derivative of the restriction is the restriction of the derivative to the corresponding tangent cone.*

The most familiar instance of set-valued maps is the inverse of a noninjective single-valued map. The *derivative of the inverse of a set-valued map F is the inverse of the derivative*:

$$D(F)^{-1}(y, x) = DF(x, y)^{-1}, \quad D^b(F)^{-1}(y, x) = D^b F(x, y)^{-1},$$
$$C(F)^{-1}(y, x) = CF(x, y)^{-1}$$

and enjoy a now well investigated calculus.

*The circatangent derivatives are closed convex processes,* because their graph are closed convex cones, i.e., they are set-valued analogues of the continuous linear operators.

Remark (Kernel of the Derivative). The kernels of the various derivatives characterize the associated tangent cones to the inverse image.

PROPOSITION 4.5. *Let $F$: $X \rightrightarrows Y$ be a set-valued map and let $(x, y)$ belong to its graph. Then*

$$\text{(i)} \qquad T_{F^{-1}(y)}(x) \subset \ker DF(x, y) := DF(x, y)^{-1}(0),$$

$$\text{(ii)} \qquad T^b_{F^{-1}(y)}(x) \subset \ker D^b F(x, y).$$

*If $F^{-1}$ is pseudo-lipschitzian around $(y, x)$, in the sense that there exists $l > 0$ such that for any $(\hat{x}, \hat{y}) \in \mathrm{Graph}(F)$ in a neighborhood of $(x, y)$, $d(\hat{x}, F^{-1}(y)) \leqslant l \|y - \hat{y}\|$ we have*

$$\text{(i)} \qquad \ker DF(x, y) = T_{F^{-1}(y)}(x),$$

$$\text{(ii)} \qquad \ker D^b F(x, y) = T^b_{F^{-1}(y)}(x),$$

$$\text{(iii)} \qquad \ker CF(x, y) \subset C_{F^{-1}(y)}(x).$$

We now provide chain rule formulas for computing the composition product of a set-valued map $G: X \rightrightarrows Y$ and a set-valued map $H: Y \rightrightarrows Z$.

One can conceive two dual ways for defining composition products of set-valued maps (which coincide when $G$ is single-valued):

DEFINITION 4.6. Let $X$, $Y$, $Z$ be Banach spaces and let $G: X \rightrightarrows Y$, $H: Y \rightrightarrows Z$ be set-valued maps:

(1) The usual composition product (called simply the *product*) $H \circ G: X \rightrightarrows Z$ of $H$ and $G$ at $x$ is defined by

$$(H \circ G)(x) := \bigcup_{y \in G(x)} H(y).$$

(2) The *square product* $H \square G: X \rightrightarrows Z$ of $H$ and $G$ at $x$ is defined by

$$(H \square G)(x) := \bigcap_{y \in G(x)} H(y).$$

Let us recall that there are two manners to define the inverse image by a set-valued map $G$ of a subset $M$:

$$\text{(a)} \qquad G^{-}(M) := \{x \mid G(x) \cap M \neq \emptyset\},$$

$$\text{(b)} \qquad G^{+}(M) := \{x \mid G(x) \subset M\}.$$

We deduce the following formulas

$$\text{(i)} \qquad \operatorname{Graph}(F \circ G) = (G \times 1)^{-} \operatorname{Graph}(H) = (1 \times H) \operatorname{Graph}(G),$$

$$\text{(ii)} \qquad \operatorname{Graph}(F \square G) = (G \times 1)^{+} \operatorname{Graph}(H),$$

as well as the formulas which state that the inverse of a product is the product of the inverses (in reverse order):

$$\text{(i)} \qquad (H \circ G)^{-1}(y) = G^{-}(H^{-1}(y)),$$

$$\text{(ii)} \qquad (H \square G)^{-1}(y) = G^{+}(H^{-1}(y)).$$

We begin by the simple result:

THEOREM 4.7. *Let us consider a set-valued map $G: X \rightrightarrows Y$ and a set-valued map $H: Y \rightrightarrows Z$.*

*Let us assume that $H$ is lipschitzian around $y$ where $y$ belongs to $G(x)$. Then, for any $z \in H(y)$, we have*

$$D^{b}H(y, z) \circ DG(x, y) \subset D(H \circ G)(x, z).$$

*Let us assume that $G$ is lipschitzian around $x$. Then, for all $y \in G(x)$ and $z \in (H \square G)(x)$, we have*

$$D(H \square G)(x, z) \subset DH(y, z) \square D^{b}G(x, y).$$

*In particular, if $G := g$ is single-valued and lipschitzian around $x$, we obtain*

$$D(Hg)(x, z)(u) \subset DH(g(x), z)(g'(x)u)$$

*and equality holds true when $H$ is lipschitzian around $g(x)$.*

We state now a more powerful result which can be derived from the inverse function theorem of the next section.

THEOREM 4.8. *Let us consider a set-valued map $G: X \rightrightarrows Y$ and a set-valued map $H: Y \rightrightarrows Z$.*

*We suppose that*

$$\operatorname{Im}(CG(x_0, y_0)) - \operatorname{Dom}(CH(y_0, z_0)) = Y.$$

*If the dimension of $Y$ is finite, then*

(i) $$D^b H(y_0, z_0) \circ DG(x_0, y_0) \subset D(H \circ G)(x_0, z_0),$$

(ii) $$D^b H(y_0, z_0) \circ D^b G(x_0, y_0) = D^b (H \circ G)(x_0, z_0),$$

(iii) $$CH(y_0, z_0) \circ CG(x_0, y_0) \subset C(H \circ G)(x_0, z_0).$$

The next proposition provides chain rule formulas for square products.

PROPOSITION 4.9. *Let us consider a set-valued map $G$ from a Banach space $X$ to a Banach space $Y$ and a single-valued map $H$ from $Y$ to a Banach space $Z$. Assume that $G$ is lipschitzian around $x^*$. If $H$ is differentiable around some $y^* \in G(x^*)$, then*

(1) *the contingent derivative of $H \square G$ is contained in the square product of the derivative of $H$ and the adjacent derivative of $G$: for all $u \in \operatorname{Dom}(D^b G(x^*, y^*))$ we have*

$$D(H \square G)(x^*, H(y^*))(u) \subset H'(y^*) \square D^b G(x^*, y^*)(u);$$

(2) *if $H$ is continuously differentiable around $y^*$ then the paratingent derivative of $H \square G$ is contained in the square product of the derivative of $H$ and the circatangent derivative of $G$: $\forall u \in \operatorname{Dom}(CG(x^*, y^*))$ we have*

$$P(H \square G)(x^*, H(y^*))(u) \subset H'(y^*) \square CG(x^*, y^*)(u).$$

We can extend this theorem to the case where $H$ is set-valued. For that purpose, we have to define the lop-sided paratingent derivatives $P_x F(x, y)$ and $P_y F(x, y)$ in the following way:

$$\operatorname{Graph}(P_1 F(x, y)) := P^{\operatorname{Dom}(F)}_{\operatorname{Graph}(F)}(x, y) \,\&\, \operatorname{Graph}(P_2 F(x, y)) := P^{\operatorname{Im}(F)}_{\operatorname{Graph}(F)}(x, y).$$

THEOREM 4.10. *Assume that $G$ is lipschitzian around $x$. Then*

(1) *if $Y$ is a finite dimensional vector-space and $G(x)$ is bounded, then*

$$D(H \circ G)(x, z) \subset \bigcup_{y \in G(x)} P_1 H(y, z) \circ P_2 G(x, y),$$

*and*

(2) $$P(H \square G)(x, z) \subset \bigcap_{y \in G(x)} PH(y, z) \square CG(x, y).$$

Proofs of the above results can be found in [19] and [20].

## 5. Variational inclusions

We now provide estimates of the contingent, adjacent and circatangent derivatives of the solution map $S$ associated to the differential inclusion

$$x'(t) \in F(t, x(t)). \tag{5.1}$$

We shall express these estimates in terms of the solution maps of adequate linearizations of differential inclusion (5.1) of the form

$$w'(t) \in F'(t, x(t), x'(t))(w(t))$$

where for almost all $t$, $F'(t, x, y)(u)$ denotes one of the (contingent, adjacent or circatangent) derivatives of the set-valued map $F(t, \cdot, \cdot)$ at a point $(x, y)$ of its graph (in this section the set-valued map $F$ is regarded as a family of set-valued maps $x \rightrightarrows F(t, x)$ and *the derivatives are taken with respect to the state variable only*).

These linearized differential inclusions can be called the *variational inclusions*, since they extend (in various ways) the classical variational equations of ordinary differential equations.

Let $\bar{x}$ be a solution of the differential inclusion (5.1). We assume that $F$ satisfies the following assumptions:

$$\text{(5.2)} \quad \begin{cases} \text{(i)} & \forall x \in X, \text{ the set-valued map } F(\cdot, x) \text{ is measurable,} \\ \text{(ii)} & \forall t \in [0, T],\ \forall x \in X,\ F(t, x) \text{ is a closed set,} \\ \text{(iii)} & \exists \beta > 0,\ k(\cdot) \in L^1(0, T) \text{ such that for almost all } t \in [0, T] \text{ the} \\ & \text{map } F(t, \cdot) \text{ is } k(t)\text{-Lipschitz on } \bar{x}(t) + \beta B. \end{cases}$$

Consider the *adjacent variational inclusion*, which is the "linearized" inclusion along the trajectory $\bar{x}$

$$\text{(5.3)} \quad \begin{cases} w'(t) \in D^b F(t, \bar{x}(t), \bar{x}'(t))(w(t)) & \text{a.e. in } [0, T], \\ w(0) = u, \end{cases}$$

where $u \in X$. In Theorems 5.1, 5.2 below we consider the solution map $S$ as the set-valued map from $\boldsymbol{R}^n$ to the Sobolev space $W^{1,1}(0, T; \boldsymbol{R}^n)$. We provide first a short proof of a result from [32].

THEOREM 5.1 (Adjacent variational inclusion). *If assumptions* (5.2) *hold true then for all $u \in X$, every solution $w \in W^{1,1}(0, T; X)$ to the linearized inclusion* (5.3) *satisfies $w \in D^b S(\bar{x}(0), \bar{x})(u)$. In other words,*

$$\{w(\cdot) \mid\ w'(t) \in D^b F(t, \bar{x}(t), (t))(w(t)),\ w(0) = u\} \subset D^b S(\bar{x}(0), \bar{x})(u).$$

Proof. Filippov's theorem (see for example [13], Theorem 2.4.1, p. 120) implies that the map $u \to S(u)$ is pseudo-lipschitzian on a neighborhood of $(\bar{x}(0), \bar{x}(\cdot))$. Let $h_n > 0$, $n = 1, 2, \dots$ be a sequence converging to 0. Then, by the very definition of the adjacent derivative, for almost all $t \in [0, T]$,

$$\lim_{n \to \infty} d\left(w'(t), \frac{F(t, \bar{x}(t) + h_n w(t)) - \bar{x}'(t)}{h_n}\right) = 0. \tag{5.4}$$

Moreover, since $\bar{x}'(t) \in F(t, \bar{x}(t))$ a.e. in $[0, T]$, by (5.2), for all sufficiently large $n$ and almost all $t \in [0, T]$

$$d(\bar{x}'(t) + h_n w'(t), F(t, \bar{x}(t) + h_n w(t))) \leqslant h_n(\|w'(t)\| + k(t)\|w(t)\|).$$

This, (5.4) and the Lebesgue dominated convergence theorem yield

$$\int_0^T d(\bar{x}'(t) + h_n w'(t), F(t, \bar{x}(t) + h_n w(t)))dt = o(h_n), \tag{5.5}$$

where $\lim_{n \to \infty} o(h_n)/h_n = 0$. By the Filippov Theorem (see for example [13], Theorem 2.4.1, p. 120) and by (5.5) there exist $M \geqslant 0$ and solutions $y_n \in S(\bar{x}(0) + h_n u)$ satisfying

$$\|y_n' - \bar{x}' - h_n w\|_{L^1(0,T;X)} \leqslant Mo(h_n).$$

Since $(y_n(0) - \bar{x}(0))/h_n = u = w(0)$, this implies that

$$\lim_{n \to \infty} \frac{y_n - \bar{x}}{h_n} = w \quad \text{in } C(0, T; x); \quad \lim_{n \to \infty} \frac{y' - \bar{x}'}{h_n} = w' \quad \text{in } L^1(0, T; X).$$

Hence

$$\lim_{n \to \infty} d\left(w, \frac{S(\bar{x}(0) + h_n u) - \bar{x}}{h_n}\right) = 0.$$

Since $u$ and $w$ are arbitrary, the proof is complete.

Consider next the *circatangent variational inclusion*, which is the linearization involving circatangent derivatives:

$$\begin{aligned} &w'(t) \in CF(t, \bar{x}(t), \bar{x}'(t))(w(t)) \quad \text{a.e. in } [0, T], \\ &w(0) = u, \end{aligned} \tag{5.5}$$

where $u \in X$.

THEOREM 5.2 (Circatangent variational inclusion). *Assume that conditions (5.2) hold true. Then for all $u \in X$, every solution $w \in W^{1,1}(0, T; X)$ to the linearized inclusion (5.5) satisfies $w \in CS(\bar{x}(0), \bar{x})(u)$. In other words,*

$$\{w(\cdot) \mid w'(t) \in CF(t, \bar{x}(t), \bar{x}'(t))(w(t)), \ w(0) = u\} \subset CS(\bar{x}(0), \bar{x})(u).$$

Proof. By Filippov's theorem, the map $u \to S(u)$ is pseudo-lipschitzian on a neighborhood of $(\bar{x}(0), \bar{x}(\cdot))$. Consider a sequence $x_n$ of trajectories of (5.1) converging to $\bar{x}$ in $W^{1,1}(0, T; X)$ and let $h_n \to 0+$. Then there exists a subsequence $x_j = x_{n_j}$ such that

$$\lim_{j\to\infty} x_j'(t) = \bar{x}'(t) \quad \text{a.e. in } [0, T]. \tag{5.6}$$

Set $\lambda_j = h_{n_j}$. Then, by definition of circatangent derivative and by (5.6), for almost all $t \in [0, T]$

$$\lim_{j\to\infty} d\left(w'(t), \frac{F(t, x_j(t)+\lambda_j w(t)) - x_j'(t)}{\lambda_j}\right) = 0. \tag{5.7}$$

Moreover, using the fact that $x_j'(t) \in F(t, x_j(t))$ a.e. in $[0, T]$, we obtain that for almost all $t \in [0, T]$

$$d\big(x_j'(t)+\lambda_j w'(t), F(t, x_j(t)+\lambda_j w(t))\big) \leqslant \lambda_j(\|w'(t)\| + k(t)\|w(t)\|).$$

This, (5.7) and the Lebesgue dominated convergence theorem yield

$$\int_0^T d\big(x_j'(t)+\lambda_j w'(t), F(t, x_j(t)+\lambda_j w(t))\big)dt = o(\lambda_j), \tag{5.8}$$

where $\lim\limits_{j\to\infty} o(\lambda_j)/\lambda_j = 0$. By the Filippov Theorem and (5.8), there exist $M \geqslant 0$ and solutions $y_j \in S(x_j(0)+\lambda_j u)$ satisfying

$$\|y_j' - x_j' - \lambda_j w'\|_{L^1(0,T;X)} \leqslant M o(h_j).$$

Since $(y_j(0)-x_j(0))/\lambda_j = u = w(0)$, this implies that

$$\lim_{j\to\infty} \frac{y_j - x_j}{h_{n_j}} = w \quad \text{in } C(0, T; X); \qquad \lim_{j\to\infty} \frac{y_j' - x_j'}{h_{n_j}} = w' \quad \text{in } L^1(0, T; X).$$

Hence

$$\lim_{j\to\infty} d\left(w, \frac{S(x_j(0)+h_{n_j}u) - x_j}{h_{n_j}}\right) = 0. \tag{5.9}$$

Therefore we have proved that for every sequence of solutions $x_n$ to (5.1) converging to $\bar{x}$ and every sequence $h_n \to 0+$, there exists a subsequence $x_j = x_{n_j}$ which satisfies (5.9). This yields that for every sequence of solutions $x_n$ converging to $\bar{x}$ and $h_n \to 0+$

$$\lim_{n\to\infty} d\left(w, \frac{S(x_n(0)+h_n u) - x_n}{h_n}\right) = 0.$$

Since $u$ and $w$ are arbitrary, the proof is complete.

We consider now the contingent variational inclusion

$$(5.10)\qquad \begin{aligned} &w'(t)\in\overline{co}DF(t,\bar{x}(t),\bar{x}'(t))(w(t)) \quad \text{a.e. in } [0,T],\\ &w(0)=u. \end{aligned}$$

THEOREM 5.3 (Contingent variational inclusion). *Let us consider the solution map $S$ as a set-valued map from $\mathbf{R}^n$ to $W^{1,\infty}(0,T;\mathbf{R}^n)$ supplied with the weak-* topology and let $\bar{x}(\cdot)$ be a solution of the differential inclusion* (5.10) *starting at $x_0$. Then the contingent derivative $DS(x_0,\bar{x}(\cdot))$ of the solution map is contained in the solution map of the contingent variational inclusion* (5.10), *in the sense that*

$$(5.11)\quad DS(x_0,\bar{x}(\cdot))(u)\subset\{w(\cdot)\mid w'(t)\in\overline{co}DF(t,\bar{x}(t),\bar{x}'(t))(w(t)),\ w(0)=u\}.$$

Proof. Fix a direction $u\in\mathbf{R}^n$ and let $w(\cdot)$ belong to $DS(x_0,\bar{x}(\cdot))(u)$. By the definition of the contingent derivative, there exist sequences of elements $h_n\to 0+$, $u_n\to u$ and $w_n(\cdot)\to w(\cdot)$ in the weak-* topology of $W^{1,\infty}(0,T;\mathbf{R}^n)$ and $c>0$ satisfying

$$(5.12)\qquad \begin{aligned} &\text{(i)} && \|w_n'(t)\|\leqslant c \quad \text{a.e. in } [0,T],\\ &\text{(ii)} && \bar{x}'(t)+h_nw_n'(t)\in F(t,\bar{x}(t)+h_nw_n(t)) \quad \text{a.e. in } [0,T],\\ &\text{(iii)} && w_n(0)=u_n. \end{aligned}$$

Hence

$$(5.13)\qquad \begin{aligned} &\text{(i)} && w_n(\cdot) \text{ converges pointwise to } w(\cdot),\\ &\text{(ii)} && w_n'(\cdot) \text{ converges weakly in } L^1(0,T;\mathbf{R}^n) \text{ to } w'(\cdot). \end{aligned}$$

By Mazur's Theorem and (5.13) (ii), a sequence of convex combinations

$$v_m(t):=\sum_{p=m}^{\infty}a_m^pw_p'(t)$$

converges strongly to $w'(\cdot)$ in $L^1(0,T;X)$. Therefore a subsequence (again denoted) $v_m(\cdot)$ converges to $w'(\cdot)$ almost everywhere. By (5.12) (i) and (ii) for all $p$ and almost all $t\in[0,T]$

$$w_p'(t)\in\left(\frac{1}{h_p}F(t,\bar{x}(t)+h_pw_p(t))-\bar{x}'(t)\right)\cap cB.$$

Let $t\in[0,T]$ be a point where $v_m(t)$ converges to $w'(t)$ and $x'(t)\in F(t,x(t))$. Fix an integer $n\geqslant 1$ and $\varepsilon>0$. By (5.13) (i), there exists $m$ such that $h_p\leqslant 1/n$ and $\|w_p(t)-w(t)\|\leqslant 1/n$ for all $p\geqslant m$.

Then, by setting

$$\Phi(y,h):=\frac{1}{h}\left(F(t,\bar{x}(t)+hy)-\bar{x}'(t)\right)\cap cB$$

we obtain that

$$v_m(t) \in K_n := co\left( \bigcup_{\substack{h \in [0, 1/n] \\ y \in w(t) + 1/nB}} \Phi(y, h) \right)$$

and therefore, by letting $m$ go to $\infty$, that

$$w'(t) \in \overline{co}\left( \bigcup_{\substack{h \in [0, \alpha] \\ y \in w(t) + 1/nB}} \Phi(y, h) \right).$$

Since this is true for any $n$, we deduce that $w'(t)$ belongs to the convex upper limit:

$$w'(t) \in \bigcap_{n \geq 1} \overline{co}\left( \bigcup_{\substack{h \in [0, 1/n] \\ y \in w(t) + 1/nB}} \Phi(y, h) \right).$$

Since the subsets $\Phi(y, h)$ are contained in the ball of radius $c$, we infer that $w'(t)$ belongs to the closed convex hull of the Kuratowski upper limit:

$$w'(t) \in \overline{co} \bigcap_{\varepsilon > 0, n \geq 1} \left( \bigcup_{\substack{h \in [0, 1/n] \\ y \in w(t) + 1/nB}} \Phi(y, h) + \varepsilon B \right).$$

We observe now that

$$\bigcap_{\substack{\varepsilon > 0 \\ n \geq 1}} \left( \bigcup_{\substack{h \in [0, 1/n] \\ y \in w(t) + 1/nB}} \Phi(y, h) + \varepsilon B \right) \subset DF(t, \bar{x}(t), \bar{x}'(t))(w(t))$$

to conclude that $w(\cdot)$ is a solution to the differential inclusion

$$\begin{aligned} &w'(t) \in \overline{co} DF(t, \bar{x}(t), \bar{x}'(t))(w(t)) \quad \text{a.e. in } [0, T], \\ &w(0) = u. \end{aligned}$$

Since $w \in DS(x_0, \bar{x}(\cdot))(u)$ is arbitrary, we proved (5.11).

## 6. Local injectivity and surjectivity of set-valued maps

Let $\mathscr{F}$ be a set-valued map from a Banach space $X$ to a Banach space $Y$. We study its local invertibility (injectivity and surjectivity) at a point $(x^*, y^*)$ of its graph.

We shall derive local injectivity of a set-valued map $\mathscr{F}: X \rightrightarrows Y$ from a general principle based on the differential calculus of set-valued maps.

For that purpose, we use its contingent and paratingent derivatives $D\mathscr{F}(x^*, y^*)$ and $P\mathscr{F}(x^*, y^*)$, which are closed processes from $X$ to $Y$.

Since $0 \in D\mathscr{F}(x^*, y^*)(0)$, we observe that the "linearized system" $D\mathscr{F}(x^*, y^*)$ enjoys the inverse univocity, which in particular implies that the inverse image

$D\mathscr{F}(x^*, y^*)^{-1}(0)$ contains only one element, i.e., that its kernel, which is naturally defined by

$$\operatorname{Ker} D\mathscr{F}(x^*, y^*) := D\mathscr{F}(x^*, y^*)^{-1}(0),$$

is reduced to zero.

THEOREM 6.1. *Let $\mathscr{F}$ be a set-valued map from a finite dimensional vector space $X$ to a Banach space $Y$ and $(x^*, y^*)$ belong to its graph.*

(1) *If the kernel of the contingent derivatives $D\mathscr{F}(x^*, y^*)$ of $\mathscr{F}$ at $(x^*, y^*)$ is equal to $\{0\}$, then there exists a neigborhood $N(x^*)$ such that*

$$\{x \text{ such that } y^* \in \mathscr{F}(x)\} \cap N(x^*) = \{x^*\}.$$

(2) *Let us assume that there exists $\gamma > 0$ such that $\mathscr{F}(x^* + \gamma B)$ is relatively compact and that $\mathscr{F}$ has a closed graph. If for all $y \in \mathscr{F}(x^*)$ the kernels of the paratingent derivatives $P\mathscr{F}(x^*, y)$ of $\mathscr{F}$ at $(x^*, y)$ are equal to $\{0\}$, then $\mathscr{F}$ is locally injective around $x^*$.*

Proof. We provide only the proof of the second statement. The proof of the first one can be found in [18].

Assume that $\mathscr{F}$ is not locally injective. Then there exists a sequence of elements $x_n^1$, $x_n^2 \in N(x^*)$, $x_n^1 \neq x_n^2$, converging to $x^*$ and $y_n$ satisfying

$$\forall n \geqslant 0, \quad y_n \in \mathscr{F}(x_n^1) \cap \mathscr{F}(x_n^2).$$

Let us set $h_n := \|x_n^1 - x_n^2\|$ which converges to 0, and $u_n := (x_n^1 - x_n^2)/h_n$.

The elements $u_n$ do belong to the unit sphere, which is compact. Hence a subsequence (again denoted) $u_n$ does converge to some $u$ different from 0. Then for all large $n$

$$y_n \in \mathscr{F}(x_n^1) \cap \mathscr{F}(x_n^2) := \mathscr{F}(x_n^2 + h_n u_n) \cap \mathscr{F}(x_n^2) \subset \mathscr{F}(x^* + \gamma B)$$

so that we deduce that a subsequence (again denoted) $y_n$ converges to some $y \in \mathscr{F}(x^*)$ (because $\operatorname{Graph}(\mathscr{F})$ is closed). Since the above equation implies that

$$\forall n \geqslant 0, \quad y_n + h_n 0 \in \mathscr{F}(x_n^2 + h_n u_n),$$

we deduce that

$$0 \in \mathrm{P}\mathscr{F}(x^*, y)(u).$$

Hence we have proved the existence of a nonzero element of the kernel of $P\mathscr{F}(x^*, y)$ which is a contradiction. □

Generalizations of this result can be found in [20].

For local surjectivity, we shall obtain furthermore some regularity property of $\mathscr{F}^{-1}$ around $y^* \in \mathscr{F}(x^*)$. We need for that purpose the following

DEFINITION 6.2. A set-valued map $G$ from $Y$ to $Z$ is *pseudo-Lipschitz* around $(y^*, z^*) \in \operatorname{Graph}(G)$ if there exist neighborhoods $V$ of $y^*$ and $W$ of $z^*$ and a constant $l$ such that

(i) $\forall y \in V$, $G(y) \neq \emptyset$,

(ii) $\forall y_1, y_2 \in V$, $G(y_1) \cap W \subset G(y_2) + l\|y_1 - y_2\| B$.

THEOREM 6.3. *Let $\mathscr{F}$ be a set-valued map from a Banach space $X$ to a finite dimensional space $Y$ and let $(x^*, y^*)$ belong to the graph of $\mathscr{F}$. If the circatangent derivative $C\mathscr{F}(x^*, y^*)$ is surjective, then $\mathscr{F}^{-1}$ is pseudo-Lipschitz around $(y^*, x^*) \in \mathrm{Graph}(\mathscr{F}^{-1})$.*

See [18] for the proof of the above result.

As a corollary we get the following inverse function theorem for single-valued maps under constraints.

COROLLARY 6.4. *Let $X$ be a Banach space, $Y$ a finite dimensional space, $K \subset X$ a closed subset of $X$ and let $x_0$ belong to $K$. Let $A$ be a differentiable map from a neighborhood of $K$ to $Y$. We assume that $A'$ is continuous at $x_0$ and that*

$$A'(x_0) C_K(x_0) = Y.$$

*Then $A(x_0)$ belongs to the interior of $A(K)$ and there exist constants $\varrho$ and $l$ such that, for all $y_1$, $y_2 \in A(x_0) + \varrho B$ and any solution $x_1 \in K$ to the equation $A(x_1) = y_1$ satisfying $\|x_0 - x_1\| \leqslant l\varrho$, there exists a solution $x_2 \in K$ to the equation $A(x_2) = y_2$ satisfying $\|x_1 - x_2\| \leqslant l \|y_1 - y_2\|$.*

For further extensions of inverse function theorems for maps from a complete metric space to a Banach space and higher order results, see [37], [38], [42], [43], [45].

## 7. Local observability of differential inclusions

Let us consider a set-valued input-output system of the following form built through a differential inclusion

$$\text{(7.1)} \qquad \text{for almost all} \quad t \in [0, T], \quad x'(t) \in F(t, x(t))$$

whose dynamics are described by a set-valued map $F$ from $[0, T] \times X$ to $X$, where $X$ is a finite dimensional vector-space (the *state space*) and $0 < T \leqslant \infty$. It governs the (uncertain) evolution of the state $x(\cdot)$ of the system. The *inputs* are the *initial states* $x_0$ and the *outputs* are the *observations* $y(\cdot) \in H(x(\cdot))$ of the evolution of the state of the system through a single-valued (or set-valued) map $H$ from $X$ to an *observation space* $Y$.

Let $S := S_F$ from $X$ to $C(0, T; X)$ denote the solution map associating with every initial state $x_0 \in X$ the (possibly empty) set $S(x_0)$ of solutions to the differential inclusion (7.1) starting at $x_0$ at the initial time $t = 0$.

In other words, we have introduced an Input-Output system where the

(1) *inputs*, are the *initial states* $x_0$,

(2) *outputs*, are the *observations* $y(\cdot) \in H(x(\cdot))$ of the evolution of the state of the system through $H$

$$\begin{array}{ccccc}
\textit{Inputs} & \overset{S}{\rightrightarrows} & \textit{States} & \overset{H}{\rightrightarrows} & \textit{Outputs} \\
\downarrow & & \downarrow & & \downarrow \\
X \ni x_0 & \rightrightarrows & x(\cdot) \in S(x_0) & \rightrightarrows & y(\cdot) \in H(x(\cdot)) \\
\uparrow & & \uparrow & & \uparrow \\
\text{Initial States} & & \left\{ \begin{array}{l} x'(t) \in F(t, x(t)) \\ x(0) = x_0 \end{array} \right\} & & \text{Observations}
\end{array}$$

It remains to define an Input-Output map. But, because of the set-valued character (the presence of uncertainty), one can conceive two dual ways for defining composition products of the set-valued maps $S$ from $X$ to the space $C(0, T; X)$ and $H$ from $C(0, T; X)$ to $C(0, T; Y)$. So, for systems under uncertainty, we have to deal with *two Input-Output maps* from $X$ to $C(0, T; Y)$: the

*Sharp Input-Output map*, which is the (usual) product

$$\forall x_0 \in X, \quad I_-(x_0) := (H \circ S)(x_0) := \bigcup_{x(\cdot) \in S(x_0)} H(x(\cdot)).$$

*Hazy Input-Output map*, which is the *square product*

$$\forall x_0 \in X, \quad I_+(x_0) := (H \square S)(x_0) := \bigcap_{x(\cdot) \in S(x_0)} H(x(\cdot)).$$

The *sharp* Input-Output map tracks *at least* the evolution of a state starting at some initial state $x_0$ whereas the *hazy* Input-Output map tracks *all* such solutions.

Opinions may differ about which would be the "right" Input-Output map, just because they depend upon the context in which a given problem is stated. So, we shall study observability properties of *both* the sharp and hazy Input-Output maps.

Observe that when the observation map is single-valued, the use of a nontrivial hazy Input-Output map requires that all solutions $x(\cdot) \in S(x_0)$ yield the same observation $y(\cdot) = H(x(\cdot))$. Hence we have to study when this possibility occurs, by projecting the differential inclusion (7.1) onto a differential equation which "tracks" all the solutions to the differential inclusion.

We shall tackle this issue by "projecting" the differential inclusion given in the state space $X$ onto a differential inclusion in the observation space $Y$ in such a way that solutions to the projected differential inclusion are observations of solutions of the original differential inclusion.

We project the differential inclusion (7.1) to a differential inclusion (or a differential equation) on the observation space $Y$ described by a set-valued map $G$ (or a single-valued map $g$)

$$(7.2) \qquad y'(t) \in G(t, y(t)) \quad (\text{or } y'(t) = g(t, y(t))), \quad y(0) = y_0$$

which allows to track partially or completely solutions $x(\cdot)$ to the differential inclusion (7.2) in the following sense:

$$(7.3) \quad \begin{cases} \text{(a)} & \forall (x_0, y_0) \in \mathrm{Graph}(H), \text{ there exist solutions } x(\cdot) \text{ and } y(\cdot) \text{ to (7.1) and (7.2) such that } \forall t \in [0, T],\ y(t) \in H(x(t)), \\ \text{(b)} & \forall (x_0, y_0) \in \mathrm{Graph}(H), \text{ all solutions } x(\cdot) \text{ and } y(\cdot) \text{ to (7.1) and (7.2) satisfy } \forall t \in [0, T],\ y(t) \in H(x(t)). \end{cases}$$

The second property means that the differential inclusion (7.2) is so to speak "blind" to the solutions to the differential inclusion (7.1). When it is satisfied, we see that for all $x_0 \in H^{-1}(y_0)$, all the solutions to the differential inclusion (7.1) do satisfy

$$\forall t \in [0, T], \qquad y(t) \in H(x(t)).$$

In the next Proposition we denote by $DH(x, y)$ the contingent derivative of $H$ at $(x, y)$.

PROPOSITION 7.1. *Let us consider a closed set-valued map $H$ from $X$ to $Y$.*

(1) *Let us assume that $F$ and $G$ are nontrivial upper semicontinuous set-valued maps with nonempty compact convex images and with linear growth. We posit the assumption*

$$(7.4) \qquad \forall (x, y) \in \mathrm{Graph}(H), \qquad G(t, y) \cap (DH(x, y) \circ F)(t, x) \neq \emptyset.$$

*Then property* (7.3) (a) *holds true.*

(2) *Let us assume that $F \times G$ is lipschitzian on a neighborhood of the graph of $H$ and has a linear growth. We posit the assumption*

$$(7.5) \qquad \forall (x, y) \in \mathrm{Graph}(H), \qquad G(t, y) \subset (DH(x, y) \square F)(t, x).$$

*Then property* (7.3) (b) *is satisfied.*

(See [19] for the proof.)

In particular, we have obtained a sufficient condition for the hazy Input-Output set-valued map $I_+$ to be nontrivial.

First, it will be convenient to introduce the following definition.

DEFINITION 7.2. Let us consider $F$: $[0, T] \times X \rightrightarrows X$ and $H$: $[0, T] \times X \rightrightarrows Y$. We shall say that a set-valued map $G$: $[0, T] \times Y \rightrightarrows Y$ is a *lipschitzian square projection* of a set-valued map $F$: $[0, T] \times X \rightrightarrows X$ by $H$ if and only if

(i) $F \times G$ is lipschitzian around $[0, T] \times \mathrm{Graph}(H)$

(ii) $\forall (x, y) \in \mathrm{Graph}(H)$, $G(t, y) \subset (DH(x, y) \square F)(t, x)$.

Therefore, for being able to use nontrivial hazy Input-Output maps, we shall use the following consequence of Proposition 7.1.

PROPOSITION 7.3. *Let us assume that* $F\colon [0, T] \times X \rightrightarrows X$ *and* $H\colon X \rightrightarrows Y$ *are given. If there exists a lipschitzian square projection of* $F$ *by* $H$, *then the hazy Input-Output map* $I_{+} := H \square S$ *has nonempty values for any initial value* $y_0 \in H(x_0)$.

We observe that when the set-valued maps $F$ and $G$ are time-independent, Proposition 7.1 can be reformulated in terms of commutativity of schemes for square products.

Let us denote by $\Phi$ the solution map associating to any $y_0$ a solution to the differential inclusion (equation) (7.2) starting at $y_0$ (when $G$ is single-valued such solution is unique).

Then we can deduce that property (7.3) b) is equivalent to

$$\forall y_0 \in \operatorname{Im}(H), \qquad \Phi(y_0) \subset ((H \square S) \square H^{-1})(y_0).$$

Condition (7.5) becomes: for all $y \in \operatorname{Im}(H)$,

$$G(y) \subset \bigcap_{x \in H^{-1}(y)} \bigcap_{v \in F(x)} DH(x, y)(v) := (DH(x, y) \square F) \square H^{-1}(y).$$

In other words, the second part of Proposition 7.1 implies that if the scheme

$$\begin{array}{ccccc} & X & \overset{F}{\rightrightarrows} & X & \\ H^{-1} & \uparrow & & \downarrow & DH(x, y) \\ & Y & \overset{G}{\rightrightarrows} & Y & \end{array}$$

is "commutative for the square products", then the derived scheme

$$\begin{array}{ccc} X & \overset{S}{\rightrightarrows} & C(0, T; X) \\ {\scriptstyle H}\downarrow\uparrow{\scriptstyle H^{-1}} & & \downarrow{\scriptstyle H} \\ Y & \overset{\Phi}{\rightrightarrows} & C(0, T; Y) \end{array}$$

is also commutative for the square products.

With these definitions at hand, we are able to adapt some of the observability concepts to the set-valued case.

DEFINITION 7.4. Assume that the sharp and hazy Input-Output maps are defined on nonempty open subsets. Let $y^* \in H(S(x_0))$ be an observation associated with an initial state $x_0$.

We shall say that the system is *sharply observable at* (respectively *locally sharply observable at*) $x_0$ if and only if the sharp Input-Output map $I_{-}$ enjoys the global inverse univocity (respectively *local*).

*Hazily observable* and locally hazily observable systems are defined in the same way, when the sharp Input-Output map is replaced by the hazy Input-Output map $I_+$.

The system is said to be *hazily* (*locally*) *observable around* if the hazy Input-Output map $I_+$ is (locally) injective.

Remark 7.5. Several obvious remarks are in order. We observe that the system is sharply locally observable at $x_0$ if and only if there exists a neighborhood $N(x_0)$ of $x_0$ such that

$$\text{if} \quad x(\cdot)\in S(N(x_0)) \quad \text{is such that} \quad y^*(\cdot)\in H(x(\cdot)), \quad \text{then} \quad x(0)=x_0,$$

i.e., sharp observability means that an observation $y^*(\cdot)$ characterizes the input $x_0$.

The system is hazily locally observable at $x_0$ if and only if there exists a neighborhood $N(x_0)$ of $x_0$ such that, for all $x_1\in N(x_0)$,

$$\text{if} \quad \forall x(\cdot)\in S(x_1), \quad y^*(\cdot)\in H(x(\cdot)), \quad \text{then} \quad x_1=x_0.$$

It is also clear that sharp local (respectively global) observability implies hazy local (respectively global) observability.

We mention that if we consider two systems $\mathscr{F}_1$ and $\mathscr{F}_2$ such that

$$\forall x\in X, \quad \mathscr{F}_1(x)\subset\mathscr{F}_2(x)$$

then

(1) If $\mathscr{F}_2$ is sharply locally (respectively globally) observable, so is $\mathscr{F}_1$.

(2) If $\mathscr{F}_1$ is hazily locally (respectively globally) observable, so is $\mathscr{F}_2$.

We piece together in this section the general principle on local inverse univocity and local injectivity (Theorem 6.1), the chain rule formulas and the estimates of the derivatives of the solution map in terms of solution maps of the variational equations (Theorems 5.1, 5.2 and 5.3) to prove the statements on local hazy and sharp observability we have announced.

We assume from now on that $H$ is differentiable and $F$ has a linear growth. We impose also some regularity assumptions on the set-valued map $F$.

In the next theorem it is assumed that $F$ is derivable in the sense that its contingent and adjacent derivatives do coincide.

THEOREM 7.6. *Let us assume that $F$ is derivable, satisfies assumptions* (5.2), *that it has a lipschitzian square projection $G$ by $H$. Let $\bar{x}(\cdot)\in S(x_0)$. If the contingent variational inclusion*

$$\text{(7.6)} \qquad \textit{for almost all} \quad t\in[0,T], \quad w'(t)\in DF(t,\bar{x}(t),\bar{x}'(t))(w(t))$$

*is globally hazily observable through $H'(\bar{x}(\cdot))$ at* 0, *then system* (7.1) *is locally hazily observable through $H$ at $x_0$.*

In the next theorem we assume that $F$ is sleek, so that its contingent and circatangent derivatives do coincide.

THEOREM 7.7. *Let us assume that $F$ is sleek, has convex images, satisfies assumptions* (5.2), *and that it has a lipschitzian square projection $G$ by $H$. If for all $\bar{x}(\cdot)\in S(x_0)$ the contingent variational inclusion* (7.6) *is globally hazily observable through $H'(\bar{x}(\cdot))$ at* 0, *then the system* (7.1) *is hazily observable through $H$ around $x_0$.*

We consider now the sharp Input-Output map.

THEOREM 7.8. *Let us assume that the graphs of the set-valued maps $F(t,\cdot)$: $X \rightrightarrows X$ are closed and convex. Let $H$ be a linear operator from $X$ to another finite dimensional vector-space $Y$. Let $\bar{x}(\cdot)$ be a solution to the differential inclusion* (7.1). *If the contingent variational inclusion* (7.6) *is globally sharply observable through $H$ around* 0, *then the system* (7.1) *is globally sharply observable through $H$ around $x_0$.*

Whenever we know that the chain rule holds true, we can state the following proposition, a consequence of the general principle (Theorem 6.1) and of Theorem 5.3 on the estimate of the contingent derivative of the solution map.

PROPOSITION 7.9. *Let us assume that the solution map of the differential inclusion* (7.1) *and the differentiable observation map $H$ do satisfy the chain rule*

$$DI_{-}(x_0, y_0)(u) = \big(H'(\bar{x})\circ S(x_0, \bar{x}(\cdot))\big)(u).$$

*If the contingent variational inclusion*

$$\text{for almost all} \quad t\in[0, T], \quad w'(t)\in \overline{co}DF(t, \bar{x}(t), \bar{x}'(t))(w(t))$$

*is globally sharply observable through $H'(\bar{x}(\cdot))$ around* 0, *then the system* (7.1) *is locally sharply observable through $H$ around $x_0$.*

However, we can bypass the chain rule formula and attempt to obtain directly other criteria of local sharp observability in the nonconvex case.

THEOREM 7.10. *Assume that $F$ has closed convex images, is continuous, Lipschitz in the second variable with a constant independent of $t$ and that the growth of $F$ is linear with respect to the state. Let $H$ be a twice continuously differentiable function from $X$ to another finite dimensional vector-space $Y$. Consider an observation $y^*\in I_{-}(x_0)$ and assume that for every solution $\bar{x}(\cdot)$ to the differential inclusion* (7.1) *satisfying $y^*(\cdot) = H(\bar{x}(\cdot))$ and for all $t\in[0, T]$ we have*

$$\operatorname{Ker} H'(\bar{x}(t)) \subset \big(F(t, \bar{x}(t)) - F(t, \bar{x}(t))\big)^{\perp}.$$

*If for all $\bar{x}$ as above the contingent variational inclusion*

$$\text{for almost all} \quad t\in[0, T], \quad w'(t)\in \overline{co}DF(t, \bar{x}(t), \bar{x}'(t))(w(t))$$

*is globally sharply observable through $H'(\bar{x}(t))$ around* 0, *then the system* (7.1) *is locally sharply observable through $H$ at $(x_0, y^*)$.*

EXAMPLE. OBSERVABILITY AROUND AN EQUILIBRIUM. Let us consider the case of a time-independent system $(F, H)$: this means that the set-valued map $F\colon X \rightrightarrows X$ and the observation map $H\colon X \rightrightarrows Y$ do not depend upon the time.

We shall observe this system around an equilibrium $\bar{x}$ of $F$, i.e., a solution to the equation

$$0 \in F(\bar{x}).$$

For simplicity, we shall assume that the set-valued map $F$ is *sleek* at the equilibrium. Hence all the derivatives of $F$ at $(\bar{x}, 0)$ do coincide with the contingent derivative $DF(\bar{x}, 0)$, which is a closed convex process from $X$ to itself.

The theorems on local observability reduce the local observability around the equilibrium $\bar{x}$ to the study of the observability properties of the variational inclusion

$$w'(t) \in DF(\bar{x}, 0)(w(t)) \tag{7.7}$$

through the observation map $H'(\bar{x})$ around the solution 0 of this variational inclusion.

We mention below a characterization of sharp observability of the variational inclusion in terms of "viability domains" of the restriction of the derivative $DF(\bar{x}, 0)$ to the kernel of $H'(\bar{x})$.

PROPOSITION 7.11. *Let us assume that $F$ is sleek at its equilibrium $\bar{x}$ and that $H$ is differentiable at $\bar{x}$. Then the variational inclusion* (7.7) *is sharply observable at* 0 *if and only if the largest closed viability domain of the restriction to* $\ker H'(\bar{x})$ *of the contingent derivative $DF(\bar{x}, 0)$ is equal to zero.*

On the other hand the variational inclusion is hazily observable if and only if the largest closed invariance domain of the restriction to $\ker H'(\bar{x})$ of the derivative $DF(\bar{x}, 0)$ is equal to zero.

Therefore we derive from the duality results of the first section that the sharp observability of the variational inclusion at 0 is equivalent to the controllability of the adjoint system

$$-p'(t) \in DF(\bar{x}, 0)^*(p(t)) + H'(\bar{x})^* u(t), \qquad u(t) \in Y^*.$$

PROPOSITION 7.12. *We posit the assumptions of Proposition* 7.11, *we assume that $DF(\bar{x}, 0)(0) = 0$ and we suppose that*

$$\ker H'(\bar{x}) + \mathrm{Dom}(DF(\bar{x}, 0)) = X.$$

*Then the concepts of sharp and hazy observability of the variational inclusion coincide and are equivalent to the controllability of the adjoint system.*

## 8. Applications to local controllability

Let us consider a bounded set-valued map $F$ from a closed subset $K \subset \boldsymbol{R}^n$ to $\boldsymbol{R}^n$ with closed graph and convex values, satisfying

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset.$$

By Haddad's Theorem, we know that for all $\xi \in K$, the subset $S_T(\xi)$ of viable solutions (a trajectory $t \to x(t)$ is viable if, for all $t \in [0, T]$, $x(t) \in K$) to the differential inclusion

$$x'(t) \in F(x(t)), \quad x(0) = \xi \tag{8.1}$$

is non-empty and closed in $C(0, T; \boldsymbol{R}^n)$ for all $\xi \in K$.

Let $R(T, \xi) := \{x(T) | \ x \in S_T(\xi)\}$ be the reachable set and let $M \subset \boldsymbol{R}^n$ the target, be a closed subset. We shall say that the system is locally controllable around $M$ if

$$0 \in \operatorname{Int}(R(T, \xi) - M).$$

This means that there exists a neighborhood $U$ of 0 in $\boldsymbol{R}^n$ such that for all $u \in U$, there exists a solution $x(\cdot) \in S_T(\xi)$ such that $x(T) \in M + u$. We denote by $\mathscr{K} \subset S_T(\xi)$ the subset of solutions $x \in S_T(\xi)$ such that $x(T) \in M$.

Let $z(\cdot) \in \mathscr{K}$ be such a solution. We linearize the differential inclusion (8.1) around $z(\cdot)$ using the circatangent derivative:

$$w'(t) \in CF(z(t), z'(t))(w(t)), \quad w(0) = 0, \tag{8.2}$$

and we denote by $R^L(T, 0)$ its reachable set from zero at time $T$.

When $\xi$ is an equilibrium and $z(\cdot) \equiv \xi$, the differential inclusion (8.2) becomes

$$w'(t) \in CF(\xi, 0)(w(t)), \quad w(0) = 0,$$

where $CF(\xi, 0)$ is a closed convex process. Its controllability then can be derived from Theorems 3.11 and 3.12.

THEOREM 8.1. *We posit the assumptions of Theorem* 5.2. *If*

$$R^L(T, 0) - C_M(z(T)) = \boldsymbol{R}^n \tag{8.3}$$

(i.e. if the linearized system is controllable around the Clarke tangent cone to $M$ at $z(T)$), *then the original system is locally controllable around $M$ and there exists a neighborhood $U$ of $z$ and a constant $l > 0$ such that, for any solution $x \in S_T(\xi)$ in $U$,*

$$d(x(\cdot), \mathscr{K}) \leqslant l d_M(x(T)).$$

Proof. We apply Theorem 6.3 to the continuous linear map $A$ from $C(0, T; \boldsymbol{R}^n) \times \boldsymbol{R}^n$ to $\boldsymbol{R}^n$ defined by $A(x, y) := x(T) - y$, to the subset $S_T(\xi) \times M$, at $(z, z(T)) \in S_T(\xi) \times M$. We observe that $A(z, z(T)) = 0$ and that condition (8.3) can be written

$$AC_{S_{T(\xi)}}(z) - C_M(z(T)) = \mathbf{R}^n.$$

Hence 0 belongs to the interior of $A(S_T(\xi) \times M) = R(T, \xi) - M$ and there exist constants $r > 0$ and $l > 0$ such that $u \to A^{-1}(u) \cap (S_T(\xi) \times M)$ is pseudo-Lipschitz around $(0, z, z(T))$. Let us consider now a ball $U$ of center $z$ and radius $r$. Let us take a solution $x \in S_T(\xi) \cap U$ to the inclusion (8.1) so that $d_M(x(T)) \leqslant \|x(T) - z(T)\| \leqslant r$. Let $y$ belong to $\pi_M(z(T))$. Then $\|A(x, y)\| = d_M(x(T))$ and we deduce from the fact that $u \to A^{-1}(u) \cap (S_T(\xi) \times M)$ is pseudo-Lipschitz that there exists $\tilde{x}$ such that $A(\tilde{x}, \tilde{x}(T)) = 0$ (i.e., an element $\tilde{x} \in \mathscr{K}$) such that

$$d(x, \mathscr{K}) \leqslant \|x - \tilde{x}\| \leqslant l\|0 - A(x, y)\| = l d_M(x(T)).$$

Remark. When $M = \{\xi\}$, the considered notion of controllability around $\xi$ coincides with the one often used in the literature. In this case a strong result was proved in [32] under the assumption that $\xi$ is an equilibrium a "larger" linearization was considered namely

$$w'(t) \in CF(\xi, 0)w(t) + T_{coF(\xi)}(0), \qquad w(0) = 0. \tag{8.4}$$

Observe that the map $x \to \overline{CF(\xi, 0)x + T_{coF(\xi)}(0)}$ is a convex process with closed images. If, moreover, $\operatorname{Dom} CF(\xi, 0) = \mathbf{R}^n$ then it is also closed. Hence we may apply results from Section 3 to study controllability of (8.4).

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