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# *THE TIME TRANSPORTATION PROBLEM*

**1. Introduction.** Given: a system $(T, M)$ where $T = \{t_{ij}\}$ is a $m\times n$ matrix with $t_{ij} \geqslant 0$ and $M = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ is a system of $m+n$ positive numbers $a_i$ and $b_j$ such that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Consider an $m\times n$ matrix $X = \{x_{ij}\}$ where $x_{ij} \geqslant 0$ and satisfy the conditions

$$(1)\qquad \begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \qquad i = 1, \ldots, m, \\ \sum_{i=1}^{m} x_{ij} &= b_j, \qquad j = 1, \ldots, n. \end{aligned}$$

By $\theta_X$ we denote the set of all $(i, j)$ for which $x_{ij} > 0$. The problem is to find a matrix $X$ which satisfies (1) and minimizes

$$(2)\qquad t_X = \max_{(i,j)\in\theta_X} t_{ij}.$$

Every matrix $X = \{x_{ij}\}$ where $x_{ij}$ are non negative and satisfy (1) we call a *solution*. By an *optimal solution* we mean each solution minimizing (2). One can give the following interpretation to this problem. There are $n$ suppliers which offer some product in amounts $a_1, \ldots, a_m$ for $n$ consumers who need that product in amounts $b_1, \ldots, b_n$. It is assumed that the supply and demand totals are equal.

By $t_{ij}$ we denote the amount of time necessary to deliver any amount of the product from the $i$th supplier to the $j$th consumer. By $X = \{x_{ij}\}$ we mean a transportation program where $x_{ij}$ denotes the amount of the product to be sent from the $i$th supplier to the $j$th consumer.

Then $t_X$ is the time (operation time) necessary to perform the whole transportation program.

The problem, which will be called the *Time Transportation Problem* (TTP), is to find a transportation program whose operation time is minimal.

The TTP was posed and solved in 1959 by A. S. Barsow ([1]). The method of solution is based on the simplex method. E. P. Nesterov ([3]) solved this problem by an adaptation of Kantorovitch's linear programming method. In [3] there is also given a method by I. W. Romanowski based on the reduction of TTP to a classical transportation problem with a cost matrix changing in the course of the iterative solving procedure. W. Grabowski ([2]) solved the TTP by transforming the problem into a single classical transportation problem.

This paper presents a method of solving the TTP based on the theory of graphs as developed in [4].

**2. Definitions and theorems.** Let $\Phi$ be the set of all points $(i, j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Any subset $\Omega$ of $\Phi$ we call a *set of nodes*. Two nodes $(i_1, j_1)$, $(i_2, j_2)$ are said to *lie on one line* if $i_1 = i_2$ or $j_1 = j_2$. Two nodes of $\Omega$ are *neighbouring* if they lie on one line and between them there is no node of $\Omega$ on the same line. Let $p$ and $q$ be two neighbouring nodes. By a *link* $pq$ we mean a straight line segment of end-points $p$ and $q$. We assume that $pq = qp$.

A graph $G_\Omega$ is called a *set of nodes* $\Omega$ and a *set of all possible links in* $\Omega$. Graph $G_{\Omega'}$ is a *subgraph* of $G_\Omega$ if $\Omega' \subset \Omega$. By a *route* $p_1$—$p_k$ we mean a sequence of different links $p_1p_2, p_2p_3, \ldots, p_{k-1}p_k$ where every two consecutive links are perpendicular and at most two nodes of the route are on one line. By a *cycle* we mean either a route $p_1$—$p_k$ where $p_1 = p_k$ or a graph $G_\Gamma$ where $\Gamma$ is the set of all nodes in the route $p_1$—$p_k$. We say that $G_\Omega$ *contains a cycle* if there exists a subgraph of $G_\Omega$ which is a cycle.

$G_\Omega$ is said to be *connected* if to any two nodes of $\Omega$ there exists a subgraph (of $G_\Omega$) whose all links form a route $p$—$q$. We then say that $G_\Omega$ contains a route $p$—$q$.

Let $B$ be a subset of $\Phi$ consisting of $m+n-1$ nodes. $B$ is called a *basis* if $G_B$ contains no cycle.

It is known ([4]) that $G_B$ is a connected graph. It is also known ([4]) that to each basis $B$ there exists exactly one matrix $Y = \{y_{ij}\}$ whose elements satisfy (1) and also the conditions

$$y_{ij} = 0 \quad \text{for all } (i, j) \notin B.$$

If in addition all $y_{ij}$ are $\geqslant 0$, then $Y$ is called a *basic solution* and $B$ — a *feasible basis*. Such a solution we denote by $X(B) = \{x_{ij}^B\}$.

Let $(k, l) \in B$. Consider $G_{B-(k,l)}$. It is easy to see that this graph consists of two connected subgraphs, say $G_{\Omega_1}$ and $G_{\Omega_2}$, where $\Omega_1$ and $\Omega_2$ are two disjoint sets (one of these sets may be empty). By $\Omega_1$ we mean either an empty set if $(k, l)$ is the only node of $B$ in column $l$ or that set which contains a node in column $l$.

By $I_1$ we denote the set of rows, and by $J_1$ the set of columns, of the $m \times n$ rectangular table in which lie the nodes of $\Omega_1$. In a similar way we define the set of rows $I_2$ and the set of columns $J_2$ which are determined by $\Omega_2$.

Let $I$ be the set of all rows and $J$ the set of all columns of an $m \times n$ table. Further let $\bar{I}_1 = I - I_1$, $\bar{J}_2 = J - J_2$.

We now introduce the following definition of a set $\Psi$ $(\Psi \subset \Phi)$

$$\Psi = \bar{I}_1 \times \bar{J}_2 - (k, l). \tag{3}$$

Given a basis $B$ and a node $(k, l) \in B$, the set $\Psi$ is uniquely determined. To examine the properties of $\Psi$ take an arbitrary element $(i, j)$ of $\Psi$. Then ([4]) $G_{B+(i,j)}$ contains exactly one cycle, say $G_\Gamma$, and the following theorem holds:

THEOREM 1. *The node $(k, l)$ belongs to $\Gamma$ and both routes $(i, j)$—$(k, l)$ of $G_\Gamma$ consist of an even number of nodes.*

Proof. First a remark. From the definition of $\Omega_1$ and $\Omega_2$ it follows that horizontal (vertical) links in $G_B$ leading to $(k, l)$ connect this node with the nodes of $\Omega_2$ $(\Omega_1)$.

To prove that $(k, l) \in \Gamma$ it is sufficient to show that $G_{B+(i,j)}$ contains two different routes $(i, j)$—$(k, l)$ with no common nodes except $(i, j)$ and $(k, l)$. Then both routes form a cycle. Now consider $G_{\Omega_1+(k,l)+(i,j)}$. This graph is connected and contains no cycle. Therefore ([4]) this graph contains exactly one subgraph, say $G_{\theta_1}$, whose all nodes form a route $(i, j)$—$(k, l)$ of one of two kinds (Fig. 1).

In the first case $\Omega_1$ is empty and $(i, j)$ consists of one segment. Consider now the second case. The first and the last segments in $(i, j)$—$(k, l)$ are vertical. This follows from: 1) the remark given at the beginning of the proof, 2) the fact that in column $j$ there is at least one node of $\Omega_1$ (see the definition of $\Psi$) and in row $i$ there is no node of $\Omega_1$. It is easy to prove (by induction) that the number of nodes in this route is even.

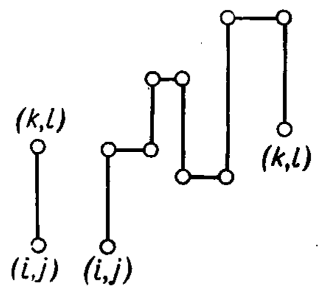


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Fig. 1

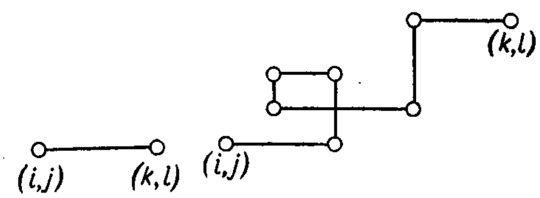


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Fig. 2

Now consider $G_{\Omega_2+(k,l)+(i,j)}$. This graph is also connected and without cycles and therefore ([4]) contains exactly one subgraph, say $G_{\theta_2}$, whose all nodes form a route $(i, j)$—$(k, l)$ of one of two kinds (Fig. 2).

In the first case $\Omega_2$ is empty. So, as before, one can show that $(k, l)$—$(i, j)$ consists of an even number of nodes (here the first and last segments are horizontal; $(i, j)$ is the only node of $\Omega_2+(i, j)+(k, l)$ in column $j$, in row $i$ there is at least one node of $\Omega_2+(k, l)$).

Since $(i, j)$ and $(k, l)$ are the only nodes which belong to both routes $(i, j)$—$(k, l)$, $G_{\theta_1+\theta_2}$ is a cycle. Since $G_{B+(i,j)}$ contains only one cycle $G_\Gamma$, we have $\Gamma = \theta_1+\theta_2$. Thus the theorem has been proved.

Let us introduce, instead of (3), a set $\Psi'$ defined as follows:

$$\Psi' = I_1 \times J_2. \tag{3'}$$

Then the following theorem is true:

THEOREM 1'. *The node $(k, l)$ belongs to the cycle contained in $G_{B+(k,l)}$ and both routes $(i, j)$—$(k, l)$ of this cycle consist of an odd number of nodes.*

The proof of Theorem 1' is quite similar to the proof of Theorem 1.

Remark. Let $(i, j)$ be an arbitrary element of the set $\Phi-(B+\Psi+ +\Psi')$. $G_{B+(i,j)}$ contains, as was said before, exactly one cycle $G_\Gamma$. One can prove that $(k, l) \notin \Gamma$.

Let $X(B)$ be a basic solution and such that $x_{kl}^B > 0$ (then $(k, l) \in B$) and let $\Pi$ be any set disjoint with $B$.

By a *$\Pi$ solution* we mean each solution of TTP which satisfies additional conditions

$$x_{ij} = 0 \quad \text{for all } (i, j) \in \Pi.$$

We can state the following

THEOREM 2. *If $\Psi \subset \Pi$ then there exists no $\Pi$ solution $X = \{x_{ij}\}$ whose element $x_{kl} = 0$* (¹).

Proof. First the following remark. It is easy to show that

$$x_{kl}^B = \begin{cases} \sum_{j \in J_1} b_j - \sum_{i \in I_1} a_i & \text{if } \Omega_1 \text{ is not empty;} \\ b_l & \text{if } \Omega_1 \text{ is empty.} \end{cases}$$

If $\Omega_1$ is empty, then $x_{kl}$ is equal to $b_l$ ($b_l > 0$) for all cells in column $l$ except $(k, l)$ belonging to $\Psi$, which is a subset of $\Pi$. Let us turn to the case where $\Omega_1$ is not empty. Then $x_{kl}^B = \sum_{j \in J_1} b_j - \sum_{i \in I_1} a_i > 0$. Assume now that there exists a solution $X = \{x_{ij}\}$ where $x_{kl} = 0$. Consider all columns of $J_1$. Since $\Psi \subset \Pi$ and $x_{kl} = 0$, all positive $x_{ij}$, $j \in J_1$, can appear only in rows belonging to $I_1$ (see Fig. 3.)

Remark. All black nodes belong to $\Omega_1$, all white nodes belong to $\Omega_2$ and $\Psi$ is the set of all crossed cells. Here $(k, l) = (2, 4)$.

(¹) One can prove even a stronger theorem: *If $\Psi \subset \Pi$, then there exists no $\Pi$ solution $\{x_{ij}\}$ whose element $x_{kl}$ is $< x_{kl}^B$.*

So
$$\sum_{j\in J_1} b_j = \sum_{i\in I_1}\sum_{j\in J_1} x_{ij} \leqslant \sum_{i\in I_1}\sum_{j\in J} x_{ij} = \sum_{i\in I_1} a_i$$
which contradicts the assumption that
$$x_{kl}^B = \sum_{j\in J_1} b_j - \sum_{i\in I_1} a_i > 0 .$$
This contradiction implies that the theorem is true.

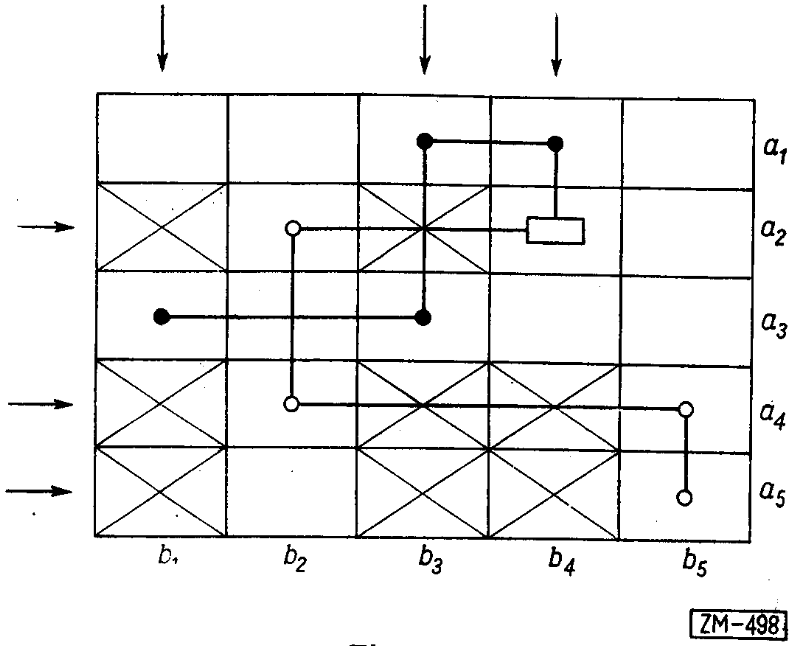



Fig. 3

**3. Method of solving the TTP.** The method of solving the TTP is the following.

1. Find an initial basic solution $X(B_1)$ by any of the known methods (for example by the minimum row method).

2. Find $t_{X(B_1)} = \max_{(i,j)\in\theta X(B_1)} t_{ij} = t_{kl}$. Define $\Pi_1$ as follows:
$$\Pi_1 = \{(i,j)\,|\,(i,j)\notin B_1,\ t_{ij} \geqslant t_{kl}\}$$
and consider from now on $\Pi_1$ solutions only.

3. Find $\Psi$. There are two cases:

(a) $\Psi \subset \Pi_1$. Then $X(B_1)$ is the optimal (and basic) solution of TTP. This follows from Theorem 2.

(b) $\Psi\overline{\Pi}_1 \neq 0$ (²). Then proceed to

4. Find $\min_{(i,j)\in\Psi\overline{\Pi}_1} t_{ij} = p_{pq}$. Graph $G_{B_1+(p,q)}$ contains exactly one cycle, say $G_\Gamma$. Divide $\Gamma$ into two subsets $\Gamma_1$ and $\Gamma_2$ assigning neighbouring

(²) $\overline{\Pi}_1 = \Phi - \Pi_1$.

nodes of $\Gamma$ to different sets and assigning $(p, q)$ to $\Gamma_1$. Then $(k, l)$ belongs to $\Gamma_2$ (this follows from Theorem 1).

5. Find $\min_{(i,j)\in\Gamma_2} x_{ij}^{B_1} = \bar{x} = x_{rs}^{B_1}$, determine a new basis $B_2 = B_1 + (p, q) - (r, s)$ and a new basic solution $X(B_2) = \{x_{ij}^{B_2}\}$ defined by the formulas

$$x_{ij}^{B_2} = \begin{cases} x_{ij}^{B_1} + \bar{x} & \text{if} \quad (i, j) \in \Gamma_1, \\ x_{ij}^{B_1} - \bar{x} & \text{if} \quad (i, j) \in \Gamma_2, \\ x_{ij}^{B_1} & \text{if} \quad (i, j) \notin \Gamma. \end{cases}$$

6. Repeat steps 2-5 for $B_2$ with the restriction to $\Pi_2$ solutions only, where

$$\Pi_2 = \{(i, j) \,|\, (i, j) \notin B_2,\ t_{ij} \geqslant t_{X(B_2)}\}$$

and continue this iteration until in the sequence

$$X(B_1),\ X(B_2),\ \ldots,\ X(B_s) \tag{4}$$

either we obtain an optimal solution $X(B_s)$ (which will be established in step 3) or $X(B_s) = X(B_r)$ for an $r < s$.

In the latter case proceed to step

7. Perform a perturbation, i.e. solve by using steps 1-6 the TTP for a system $(T, \bar{M})$ where $\bar{M} = (\bar{a}_1, \ldots, \bar{a}_m;\ \bar{b}_1, \ldots, \bar{b}_n)$ and

$$\bar{a}_i = a_i + \varepsilon, \quad i = 1, \ldots, m;$$

$$\bar{b}_j = \begin{cases} b_j, & j = 1, \ldots, n-1, \\ b_j + m\varepsilon, & j = n, \end{cases}$$

where $\varepsilon$ is a positive number chosen in such a way that for all possible sets $I^*, J^*$ — which are real subsets of $I$ and $J$ respectively — the following relation is satisfied

$$\sum_{i\in I^*} \bar{a}_i \neq \sum_{j\in J^*} \bar{b}_j \tag{$*$}$$

(it is known ([4]) that we can always choose such a number $\varepsilon_0$ that $(*)$ is satisfied for all $\varepsilon$ from the interval $\langle 0, \varepsilon_0)$. Setting zero instead of $\varepsilon$ in the optimal solution of this problem we get the optimal (and basic) solution of the original problem.

Let $X(B_t)$ be the optimal solution of TTP. Take any $\Pi_t$ solution $X$ of TTP. Then $X$ is also an optimal solution of this problem, because

$$t_{X(B_t)} = t_X = t_{kl} \quad (\text{so } x_{kl} > 0 \text{ and } x_{kl}^{B_t} > 0).$$

The following theorem is true.

THEOREM 3. *If* TTP *is solved by the method given in steps* 1-7 *then the number of iterations leading from* $X(B_1)$ *to the optimal basic solution is finite.*

Proof. The number of all bases is finite because it is less than $\binom{mn}{m+n-1}$. To each basis there exists at most one basic solution, and so the set of basic solutions is finite. This implies that there exists at least one basic solution minimizing (2). The solution procedure of TTP orders the basic solutions $X(B_s)$ (which form the sequence (4)) in such a way that

$$t_{X(B_s)} \geqslant t_{X(B_{s+1})}, \quad s = 1, 2, \ldots, \tag{5}$$

where also

$$\Pi_s \subset \Pi_{s+1}, \quad s = 1, 2, \ldots \tag{6}$$

Suppose that the TTP has been solved without using step 7. Then (4) consists of different elements and therefore is a finite sequence, q. e. d.

Suppose now that we have solved the problem by perturbation getting a sequence

$$\bar{X}(B_1),\ \bar{X}(B_2),\ \ldots,\ \bar{X}(B_u),\ \ldots,\ \bar{X}(B_v), \ldots \tag{4'}$$

where $\bar{X}(B_s)$ are basic solutions of TTP for the system $(T, \bar{M})$. Here (5) and (6) are also satisfied.

All we have to do is to prove that (4') consists of different basic solutions. Suppose to the contrary that in (4') appear two identical solutions, say $\bar{X}(B_u)$ and $\bar{X}(B_v)$, where $v > u$. Then we must have: $B_u = B_v$.

It is known ([4]) that for each basic solution $\bar{X}(B_s)$ of the perturbed problem we have

$$x_{ij}^{B_s} > 0 \quad \text{for all } (i, j) \in B_s. \tag{7}$$

Let $t_{X(B_u)}$ be equal to $t_{kl}$. Then $(k, l)$ belongs not only to $B_u$ but also to $B_v$ (otherwise $B_u$ would not be identical with $B_v$). Consider $x_{kl}^{B_s}$ for $s = u, \ldots, v$. From step 5 and also from (7) it follows that the value of $x_{kl}^{B_s}$ decreases as $s$ increases (if $x_{kl}^{B_{s+1}} = x_{kl}^{B_s}$ then $\bar{x}$ (see step 5) is equal to zero, which contradicts (7)), which is inconsistent with the assumption that $x_{kl}^{B_u} = x_{kl}^{B_v}$. So (4') consists of different solutions and therefore is a finite sequence. This completes the proof.

**4. Example.** Let us consider a $4 \times 5$ TTP

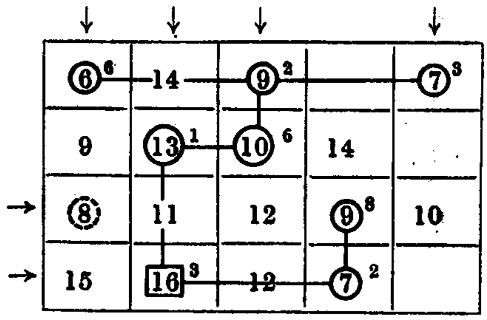
| | | | | | |
|---|---|---|---|---|---|
| 6 | 14 | 9 | 18 | 7 | 11 |
| 9 | 13 | 10 | 14 | 16 | 7 |
| 8 | 11 | 12 | 9 | 10 | 8 |
| 15 | 16 | 12 | 7 | 19 | 5 |
| 6 | 4 | 8 | 10 | 3 | |

The numbers $a_i$ and $b_j$ are on the right and below the matrix $T = \{t_{ij}\}$ respectively.

First using the minimum row method we find the initial solution $X(B_1)$

| | | | | |
|---|---|---|---|---|
| 6 | | 2 | | 3 |
| | 1 | 6 | | |
| | | | 8 | |
| | 3 | | 2 | |

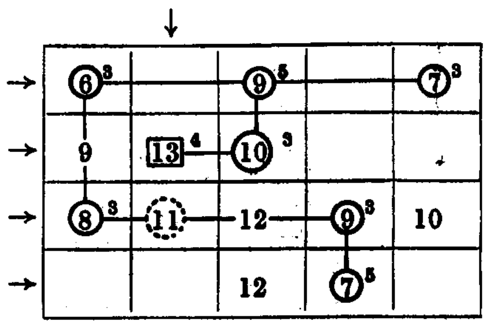
Here $t_{X(B_1)} = \max\limits_{(i,j)\epsilon\theta_{X}(B_1)} t_{ij} = t_{42} = 16$. So $\Pi_1 = [(1,4), (2,5), (4,5)]$. We restrict ourselves to $\Pi_1$ solutions and consider the graph $G_{B_1}$



Determine $\Omega_1$. This set occupies rows 1 and 2 and columns 1, 2, 3 and 5. Consider $\Psi$ (i.e. the set of cells, except (4,2), which are on the intersection of the row and the column arrows). Find $\min\limits_{(i,j)\epsilon\Psi\bar{\Pi}_1} t_{ij} = t_{31} = 8$. Graph $G_{B_1+(3,1)}$ contains exactly one cycle $G_\Gamma$ where $\Gamma = [(3,1), (1,1), (1,3), (2,3), (2,2), (4,2), (4,4), (3,4)]$. Divide $\Gamma$ into $\Gamma_1$ and $\Gamma_2$ and assign (3,1) to $\Gamma_1$. We find

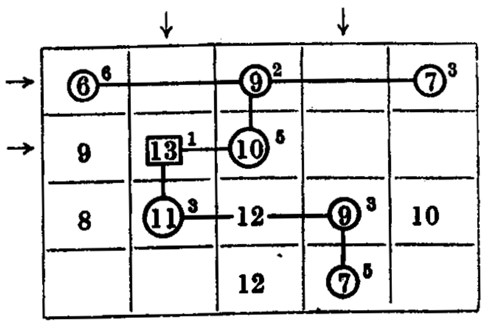
$$\min_{(i,j)\epsilon\Gamma_2} x_{ij}^{B_1} = x_{42}^{B_1} = 3.$$

So $B_2 = B_1 + (3,1) - (4,2)$ and $X(B_2)$ is as follows:



Here $t_{X(B_2)} = 13 = t_{22}$ and $\Pi_2 = \Pi_1 + [(4,1), (1,2), (4,2), (2,4)]$.

Now $\Omega_1$ is empty. We find $\Psi$ (the whole second column except (2,2)). The set $\Psi\overline{\Pi}_2$ contains only one element (3,2). We repeat the procedure from step 4 and obtain $X(B_3)$




Here $\Pi_3 = \Pi_2$. Determine the corresponding set $\Psi$. Since the set $\Psi\overline{\Pi}_3$ is empty, $X(B_3)$ is a basic optimal solution of TTP. Note that each $\Pi_3$ solution is an optimal solution. See two examples given below:

| | | | | |
|---|---|---|---|---|
| 1 | | 7 | | 3 |
| 5 | 1 | | | |
| | 3 | | 3 | |
| | | | 5 | |

| | | | | |
|---|---|---|---|---|
| 3 | | 5 | | 3 |
| 3 | 1 | 2 | | |
| | 3 | | 3 | |
| | | | 5 | |

Here the second optimal solution is not a basic one.

### References

[1] А. С. Барсов, *Что такое линейное программирование*, Москва 1959, pp. 90-101.

[2] W. Grabowski, *Problem of transportation in minimum time*, Bull. Acad. Polon. Sci. 12 (1964), pp. 107-108.

[3] Е. П. Нестеров, *Транспортные задачи линейного программирования*, Москва 1962, pp. 72-80.

[4] W. Szwarc, *Zagadnienie transportowe*, Zastosow. Matem. 6 (1962), pp. 149-187.

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## *ZAGADNIENIE TRANSPORTOWE Z KRYTERIUM CZASU*

### STRESZCZENIE

Mamy $m$ dostawców, którzy oferują pewien określony towar w ilościach $a_1, \ldots, a_m$ i $n$ odbiorców, których zapotrzebowania na ten towar wynoszą $b_1, \ldots, b_n$. Zakładamy, że $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Znane są liczby $t_{ij}$ oznaczające czas potrzebny do dostarczenia towaru od $i$-tego dostawcy do $j$-tego odbiorcy. Przez $x_{ij}$ oznaczmy ilość towaru jaki $i$-ty dostawca dostarcza $j$-temu odbiorcy. Liczby $x_{ij}$ utworzą macierz prostokątną $X = \{x_{ij}\}$, którą można nazwać planem transportowym. Wprowadźmy oznaczenia

$$(2) \qquad \theta_X = \{(i,j) | x_{ij} > 0\}, \quad t_X = \max_{(i,j)\in\theta_X} t_{ij}.$$

Liczbę $t_X$ nazwiemy czasem wykonania planu transportowego (jest to czas najdłużej trwającej dostawy).

Zagadnienie transportowe z kryterium czasu (TTP) polega na znalezieniu planu transportowego o najkrótszym czasie wykonania. Problem ten da się zapisać następująco. Znaleźć macierz $X$ o nieujemnych elementach $x_{ij}$, spełniających warunki

$$(1) \qquad \begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \quad i = 1, \ldots, m, \\ \sum_{i=1}^{m} x_{ij} &= b_j, \quad j = 1, \ldots, n, \end{aligned}$$

dla której $t_X$ osiąga wartość najmniejszą. Liczby $a_i$, $b_j$ są dane i dodatnie, przy czym $\sum_i a_i = \sum_j b_j$.

W pracy korzysta się z pojęć, które wprowadzone zostały w [4]. Nie będę więc tych pojęć powtórnie definiował odsyłając czytelnika do pracy [4].

Rozpatrzmy rozwiązanie podstawowe $X(B) = \{x_{ij}^B\}$ i niech $(k,l)\in B$. Graf $G_{B-(k,l)}$ składa się z dwóch spójnych grafów $G_{\Omega_1}$ i $G_{\Omega_2}$, gdzie $\Omega_1$ i $\Omega_2$ są rozłączne. Oznaczmy przez $I_1, J_1$ i $I_2, J_2$ zbiór wierszy i kolumn macierzy prostokątnej $m\times n$, na których znajdują się elementy odpowiednio zbiorów $\Omega_1$ i $\Omega_2$.

Przez $\Omega_1$ oznaczymy albo zbiór pusty, jeśli $(k,l)$ jest jedynym elementem bazy $B$ w kolumnie $l$ albo ten ze zbiorów, który zawiera element w $l$-tej kolumnie. Niech $I = (1, \ldots, m)$, $J = (1, \ldots, n)$.

Wprowadzamy następujące oznaczenie:

$$\Psi = (I - I_1)\times(J - J_2) - (k,l).$$

Metoda rozwiązania postawionego zagadnienia opiera się na dwóch twierdzeniach.

Rozpatrzmy dowolny element $(i,j)\in\Psi$. Graf $G_{B+(i,j)}$ zawiera dokładnie jeden cykl $G_\Gamma$. Zachodzi następujące

TWIERDZENIE 1. *Węzeł $(k,l)\in\Gamma$ i obie drogi $(i,j)-(k,l)$ cyklu $G_\Gamma$ zawierają parzystą ilość węzłów.*

Dane jest rozwiązanie podstawowe $X(B)$, takie że $x_{kl}^B > 0$.

Niech $\Pi$ będzie dowolnym zbiorem rozłącznym z bazą $B$. Prawdziwe jest

TWIERDZENIE 2. *Jeśli $\Psi \subset \Pi$, to nie istnieje macierz $X = \{x_{ij}\}$ spełniająca* (1), *i dla której $x_{ij} = 0$ dla wszystkich $(i, j) \in \Pi$ a także $x_{kl} = 0$.*

W pracy podana jest metoda rozwiązania, która prowadzi do optymalnego rozwiązania po skończonej ilości kroków (patrz dowód twierdzenia 3). Przytoczony tu został także przykład liczbowy.

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## *ТРАНСПОРТНАЯ ЗАДАЧА С КРИТЕРИЕМ ВРЕМЕНИ*

### РЕЗЮМЕ

Имеется $m$ поставщиков предлагающих определенный товар в количествах $a_1, \ldots, a_m$ и $n$ потребителей потребования которых на этот товар равны $b_1, \ldots, b_n$. Принимаем что $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Известны числа $t_{ij}$ обозначающие время необходимое для поставки товара от $i$-того поставщика $j$-тому потребителю. Через $x_{ij}$ обозначим количество товара доставленного $i$-тым поставщиком для $j$-того потребителя.

Числа $x_{ij}$ составляют прямоугольную матрицу $X = \{x_{ij}\}$ которую можно назвать транспортным планом.

Введем обозначения

$$\theta_X = \{(i, j) | x_{ij} > 0\}, \qquad t_X = \max_{(i,j) \in \theta_X} t_{ij}. \tag{2}$$

Число $t_X$ обозначает время выполнения транспортного плана (это время найболее длительной поставки).

Транспортная задача с критерием времени состоит в определении транспортного плана с найболее коротким временем выполнения. Проблему эту можно записать следующим образом. Найти матрицу $X$ с неотрицательными элементами $x_{ij}$ исполняющими

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \quad i = 1, \ldots, m, \\ \sum_{i=1}^{m} x_{ij} &= b_j, \quad j = 1, \ldots, n \end{aligned} \tag{1}$$

для которой $t_X$ минимальное, числа $a_i$ и $b_j$ известны и положительные причем $\sum_i a_i = \sum_j b_j$.

В работе используются понятия которые были введены в [4]. Не буду здесь этих понятий повторно определять отсылая читателя к работе [4].

Рассмотрим базисное решение $X(B) = \{x_{ij}^B\}$ и пусть $(k, l) \in B$. Граф $G_{B-(k,l)}$ состоит из двух соединенных графов $G_{\Omega_1}$ и $G_{\Omega_2}$, где $\Omega_1$ и $\Omega_2$ не имеют общих элементов. Обозначим через $I_1, J_1$ и $I_2, J_2$ множество строк и колонок прямоугольной матрицы $m \times n$ в которых находятся элементы соответственно множеств $\Omega_1$ и $\Omega_2$.

Через $\Omega_1$ обозначим или пустое множество, если $(k, l)$ единственный элемент базы $B$ в колонке $l$ или же множество которое имеет элемент в $l$-той колонке.

Пусть $I = (1, \ldots, m)$, $J = (1, \ldots, n)$.

Введем следующее обозначение

$$\Psi = (I - I_1) \times (J - J_2) - (k, l).$$

Метод решения поставленной проблемы опирается на двух теоремах.

Рассмотрим любой элемент $(i, j) \in \Psi$. Граф $G_{B+(i,j)}$ имеет точно один цикл, скажем $G_\Gamma$. Справедлива следующая.

Теорема 1. *Узел $(k, l)$ принадлежит к $\Gamma$ и оба пути $(i, j)-(k, l)$ цикла $G_\Gamma$ состоят из четного количества узлов.*

Имеется базисное решение $X(B)$ в котором $x_{kl}^B > 0$.

Пусть $\Pi$ любое множество не имеющее с $B$ общих элементов. Справедлива

Теорема 2. *Если $\Psi \subset \Pi$ то не существует матрица $X = \{x_{ij}\}$ исполняющая* (1), *для которой $x_{ij} = 0$ для всех $(i, j) \in \Pi$ и где $x_{kl} = 0$.*

В работе приводится метод решения, который ведет к оптимальному решению после конечного количества шагов (смотри доказательство теоремы 3). Приводится также числовой пример.