COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 2

A NOTE ON HOMOGENEOUS LATTICES

BY

JOEL BERMAN (CHICAGO, ILLINOIS)

In [2] Dwinger defines a lattice L to be *homogeneous* if, for any $a, b \in L$, there exists an automorphism f of L such that $f(a) = b$ and such that if $a < b$, then $x < f(x)$ for all $x \in L$, and if a and b are incomparable, then x and $f(x)$ are incomparable for each $x \in L$. He shows that every lattice-ordered group is homogeneous as a lattice. He then asks if there exists a homogeneous lattice which does not admit a lattice-ordered group structure (P 887). In [1] a slightly more general definition of a homogeneous lattice is studied and there is given an example of a chain (essentially, the long line) which does not admit a lattice-ordered group structure, is not homogeneous in the sense of [2], but is homogeneous in the sense of [1]. To complete this strict chain of inclusions and to answer Ph. Dwinger's question we exhibit in this note an example of a homogeneous lattice which does not admit a lattice-ordered group structure.

Let R denote the chain of real numbers. Form the lattice

$$P = \prod_{i \in N} R_i, \quad \text{where } N = \{1, 2, 3, \ldots\}.$$

Define $\overline{n} \in P$ by $(\overline{n})_i = n$ for all $i \in N$. Let

$$L = \{s \in P \mid s < \overline{n} \text{ for some } n \in N\}.$$

Clearly, L is a sublattice of P. Moreover, L is homogeneous. For let $a, b \in L$. If $i \in N$ is such that $a_i < b_i$, we define $f_i\colon R \to R$ by

$$f_i(r) = \begin{cases} r + b_i - a_i & \text{for } r \leqslant a_i, \\ \dfrac{r - a_i}{b_i - a_i} + b_i & \text{for } a_i \leqslant r \leqslant b_i, \\ r + 1 & \text{for } b_i \leqslant r. \end{cases}$$

If $i \in N$ is such that $a_i > b_i$, we define $f_i \colon R \to R$ by

$$f_i(r) = \begin{cases} r + b_i - a_i & \text{for } r \leqslant a_i, \\ (r - a_i)(a_i - b_i) + b_i & \text{for } a_i \leqslant r \leqslant a_i + 1, \\ r - 1 & \text{for } a_i + 1 \leqslant r. \end{cases}$$

Finally, if $a_i = b_i$, then we put $f_i(r) = r$ for all $r \in R$. In all three cases, $f_i(a_i) = b_i$ and f_i is a lattice automorphism of R. In the case $a_i < b_i$ we have $r < f_i(r)$ for all $r \in R$. Similarly, if $a_i > b_i$, then $r > f_i(r)$ for all $r \in R$. Define $f \colon L \to P$ by $(f(s))_i = f_i(s_i)$. If $s \in L$ and $s < \bar{k}$ and $b < \bar{n}$, then $(f(s))_i < 1 + \max\{n, k\}$ for all i. Hence $f(L) \subseteq L$. Since each f_i is a lattice automorphism, it is easily seen that f is a lattice automorphism of L and $f(a) = b$. If $a < b$, then $a_i \leqslant b_i$ for all i and $a_j < b_j$ for some j. Hence $r \leqslant f_i(r)$ and $r < f_j(r)$ for any $r \in R$, and $s < f(s)$ for all $s \in L$. Similarly, if a and b are not comparable, then s and $f(s)$ are not comparable for all $s \in L$. Thus L is homogeneous.

If L admitted a lattice-ordered group structure, then L would have some dual lattice automorphism a. Define $s \in P$ by $s_n = (a(\bar{n}))_n - 1$ for $n \in N$. Since $a(\bar{n}) \leqslant a(\bar{1})$, we have $s \in L$. Moreover, $s_n < (a(\bar{n}))_n$ for $n \in N$. Take $t \in L$ with $a(t) = s$ and $t < \bar{m}$. So $s = a(t) > a(\bar{m})$ and, therefore, $s_m \geqslant (a(\bar{m}))_m$, a contradiction.

REFERENCES

[1] J. Berman, *Homogeneous lattices and lattice-ordered groups*, Colloquium Mathematicum 32 (1974), p. 13-24.

[2] Ph. Dwinger, *Subdirect products of chains*, ibidem 29 (1974), p. 201-207.

UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE
CHICAGO, ILLINOIS

Reçu par la Rédaction le 29. 3. 1974