Connections between symmetry or asymmetry of the equations $f_1(x_1+ \ldots +x_n) = \sum\limits_{(i_1,\ldots,i_n)} f(x_{i_1}) \ldots f_n(x_{i_n})$ and their solutions

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It is easy to verify that equations of the form

$$(1) \qquad f_1(x_1 + \ldots +x_n) = \sum_{(i_1, \ldots, i_n) \epsilon P_k} f_1(x_{i_1}) \ldots f_n(x_{i_n})$$

(where P_k is a set of k permutations of the numbers $1, \ldots, n$) are satisfied by the functions

$$(2) \quad f_1(x) = A_1 e^{ax}, \quad \ldots, \quad f_{n-1}(x) = A_{n-1} e^{ax}, \quad f_n(x) = \frac{1}{kA_2 \ldots A_{n-1}} e^{ax},$$

where $A_1, \ldots, A_{n-1}, a$ $(A_1 \neq 0, \ldots, A_{n-1} \neq 0)$ are arbitrary constants.

All the symmetrical and some asymmetrical equations (1) (see Definitions 1 and 3, page 258) are satisfied also by the functions

$$(3) \quad \begin{aligned} f_1(x) &= A_1 x e^{ax}, \quad f_2(x) = A_2 e^{ax}, \quad \ldots, \quad f_{n-1}(x) = A_{n-1} e^{ax}, \\ f_n(x) &= \frac{1}{k_{11} A_2 \ldots A_{n-1}} e^{ax}, \end{aligned}$$

where $A_1, \ldots, A_{n-1}, a$ $(A_1 \neq 0, \ldots, A_{n-1} \neq 0)$ are arbitrary constants, and k_{11} is the number of components of the right side of equation (1) with $f_1(x_1)$.

We look for non-trivial (cf. Definition 6) solutions of equations (1) such that at least one function $f_i(x)$ has a point of continuity. It will be shown that functions (2) are the only solutions with this property for a certain class of asymmetrical equations (1). We shall also show a larger class of asymmetrical equations (1) such that for every non-trivial solution we have $f_1(x) \neq 0, \ldots, f_n(x) \neq 0$ everywhere. It will be proved that all their non-trivial solutions with the first derivatives must have form (2). Moreover, we shall prove that functions (2) and (3) are the only non-trivial solutions with first derivatives for all the other asymmetrical equations (1).

Symmetrical equations (1) have many solutions different from (2) and (3). Their form depends on k and n and it seems impossible to give a general formula. We do not solve them in this paper.

The following problem remains open: Can an asymmetrical equation (1) be satisfied by functions with a point of continuity and without a first derivative different from (2) and (3)?

Notation.

$$f_0 = f(0), \quad f_0' = f'(0), \quad f_{ia} = f_i(a),$$

$$F_{ia} = f_{1a} \cdots f_{i-1,a} f_{i+1,a} \cdots f_{na}, \quad F_i = F_{i0},$$

$$\varphi_{i\xi} = \varphi_i(\xi), \quad \varphi_{i\xi}' = \varphi_i'(\xi),$$

$$\Phi_{i\xi} = \varphi_{1\xi}' \cdots \varphi_{i-1,\xi}' \varphi_{i+1,\xi}' \cdots \varphi_{n\xi}',$$

$$\widetilde{\Phi}_{i\xi} = \widetilde{\varphi}_{1\xi}' \cdots \widetilde{\varphi}_{i-1,\xi}' \widetilde{\varphi}_{i+1,\xi}' \cdots \widetilde{\varphi}_{n\xi}',$$

$$\overset{\xi}{\Phi}_{i\xi} = \overset{\xi}{\varphi}_{1\xi}' \cdots \overset{\xi}{\varphi}_{i-1,\xi}' \overset{\xi}{\varphi}_{i+1,\xi}' \cdots \overset{\xi}{\varphi}_{n\xi}',$$

k_{ij} — number of components of the right side of (1) with $f_j(x_i)$,

$$\sum_{i=1}^{n} k_{ij} = \sum_{j=1}^{n} k_{ij} = k \quad (i, j = 1, \ldots, n),$$

$$\bar{k}_{ij} = \begin{cases} k_{i1} - k & \text{for} \quad j = 1, \\ k_{ij} & \text{for} \quad j = 2, \ldots, n \end{cases} \quad (i = 1, \ldots, n),$$

$r[a_{ij}]$ — rank of the matrix $[a_{ij}]$,

$|a_{ij}|$ — determinant of the matrix $[a_{ij}]$.

Definitions.

DEFINITION 1. We say that equation (1) is *symmetrical* if all the numbers k_{ij} are equal.

DEFINITION 2. We say that equation (1) is *symmetrical by* f_r if all the numbers k_{ir} are equal.

DEFINITION 3. We say that equation (1) is *asymmetrical* (*asymmetrical by* f_r) if it is not symmetrical (not symmetrical by f_r).

DEFINITION 4. We say that equation (1) is *reducible* if $r[\bar{k}_{ij}] = n-1$.

Remark. We always have $r[\bar{k}_{ij}] \leqslant n-1$. It follows from the fact that

$$\bar{k}_{i1} = k_{i1} - k = k_{i1} - \sum_{j=1}^{n} k_{ij} = - \sum_{j=2}^{n} k_{ij} = - \sum_{j=2}^{n} \bar{k}_{ij}.$$

DEFINITION 5. We say that equation (1) is *non-reducible* if $r[\bar{k}_{ij}]$ $\leqslant n-2$.

According to our definitions it is e.g.

1° $f_1(x_1 + x_2) = f_1(x_1)f_2(x_2)$ asymmetrical and reducible;

2° $f_1(x_1 + x_2) = f_1(x_1)f_2(x_2) + f_1(x_2)f_2(x_1)$ symmetrical and reducible;

3° all symmetrical equations (1) with $n > 2$ are non-reducible;

4° $f_1(x_1 + x_2 + x_3) = f_1(x_1)f_2(x_3)f_3(x_2) + f_1(x_2)f_2(x_3)f_3(x_1) + f_1(x_3)f_2(x_2)f_3(x_1)$ asymmetrical, symmetrical by f_1, reducible;

5° $f_1(x_1 + x_2 + x_3 + x_4) = f_1(x_1)[f_2(x_2)f_3(x_3)f_4(x_4) + f_2(x_3)f_3(x_4)f_4(x_2) + \\ + f_2(x_4)f_3(x_2)f_4(x_3)]$
asymmetrical by f_1, non-reducible;

6° $f_1(x_1 + x_2 + x_3 + x_4) = f_1(x_1)[f_2(x_3)f_3(x_4) + f_2(x_4)f_3(x_3)]f_4(x_2) + \\ + f_1(x_2)[f_2(x_3)f_3(x_4) + f_2(x_4)f_3(x_3)]f_4(x_1) + \\ + f_1(x_3)[f_2(x_2)f_3(x_4)f_4(x_1) + f_2(x_4)f_3(x_1)f_4(x_2)] + \\ + f_1(x_4)[f_2(x_1)f_3(x_3)f_4(x_2) + f_2(x_3)f_3(x_2)f_4(x_1)]$

asymmetrical, symmetrical by f_1, non-reducible.

DEFINITION 6. We say that the solution $f_1(x), \ldots, f_n(x)$ of equation (1) is *trivial* if $f_1(x) \equiv 0$.

DEFINITION 7. We say that the solution $f_1(x), \ldots, f_n(x)$ of equation (1) is *non-trivial* if $f_1(x) \not\equiv 0$.

THEOREM I. *Every solution of equation* (1) *satisfies the following conditions*:

(i) $\qquad f_1(na) = kf_1(a) \ldots f_n(a) = kf_{ia}F_{ia} \qquad (i = 1, \ldots, n)$,

(ii) $\qquad\qquad F_1 = 1/k \qquad or \qquad f_{10} = 0$.

Proof. Putting in (1) $x_1 = \ldots = x_n = a$, we obtain (i). Hence (for $a = 0$ and for $i = 1$) (ii) follows. Q. E. D.

THEOREM II. *If* $f_1(x), \ldots, f_n(x)$ *is a non-trivial solution of equation* (1), *we have* $F_1 = f_{20} \ldots f_{n0} \neq 0$.

Proof. Suppose that $f_{i0} = 0$ for some $i > 1$. Putting in (1) $x_1 = \ldots = x_n = 0$, we obtain $f_{10} = 0$, and putting next $x_1 = x, x_2 = \ldots = x_n = 0$, we obtain $f_1(x) \equiv 0$. Therefore we must have $f_{i0} \neq 0$ for $i > 1$. Q. E. D.

THEOREM III. *If equation* (1) *is asymmetrical by* f_r, *each of its non-trivial solutions satisfies the condition*: $f_r(x) \neq 0$ *for every* x.

Proof. Equation (1) is asymmetrical by f_r, i.e. $k_{ir} \neq k_{jr}$ for some i, j. Suppose that $f_r(a) = 0$ for some a. Then putting in (1) $x_i = x, x_l = a$ $(l = 1, \ldots, n, l \neq i)$, we obtain

(4) $\qquad\qquad f_1(x + (n-1)a) = k_{ir}F_{ra}f_r(x)$.

Similarly we obtain

$$(4') \qquad f_1\big(x + (n-1)a\big) = k_{jr}F_{ra}f_r(x) .$$

It follows from (4) and (4') (after subtraction) that

$$(a) \qquad f_r(x) \equiv 0$$

or

$$(b) \qquad F_{ra} = 0 .$$

Case (a) for $r = 1$ contradicts the assumption of the theorem. For $r \neq 1$ we obtain in particular $f_{r0} = 0$, but this contradicts Theorem II.

In case (b) it follows from (4) or (4') that $f_1(y) \equiv 0$, but this contradicts the assumption of the theorem.

Thus we must have $f_r(x) \neq 0$ for every x. Q. E. D.

THEOREM IV. *If equation* (1) *is asymmetrical by* f_1 *and* $f_1(x), \ldots, f_n(x)$ *is its non-trivial solution, we must have* $f_1(x) \ldots f_n(x) \neq 0$ *everywhere.*

Proof. It follows from Theorem III that $f_1(x) \neq 0$ for every x. If for some a and $s \neq 1$ $f_{sa} = 0$, we obtain from Theorem I $f_{1na} = kf_{sa}F_{sa} = 0$. This contradicts the fact that $f_1(x) \neq 0$ and therefore $f_1(x) \ldots f_n(x) \neq 0$ for every x. Q. E. D.

THEOREM V. *If* $f_1(x), \ldots, f_n(x)$ *is a solution of reducible equation* (1) *and* $f_{10} \neq 0$, *then*

$$(5) \qquad f_i(x) = \frac{f_{i0}}{f_{10}} f_1(x) \qquad (i = 1, \ldots, n)$$

and the function $f_1(x)$ *satisfies the equation*

$$(6) \qquad f_1(x_1 + \ldots + x_n) = \frac{1}{f_{10}^{n-1}} f_1(x_1) \ldots f_1(x_n) .$$

Proof. Putting in (1) $x_i = x, x_j = 0 \ (j \neq i)$, we obtain the set of linear equations

$$f_i(x) = k_{i1}F_1 f_1(x) + k_{i2}F_2 f_2(x) + \ldots + k_{in}F_n f_n(x) \qquad (i = 1, \ldots, n)$$

with the unknowns $f_1(x), \ldots, f_n(x)$ or

$$(k_{i1}F_1 - 1)f_1(x) + k_{i2}F_2 f_2(x) + \ldots + k_{in}F_n f_n(x) = 0 \qquad (i = 1, \ldots, n) .$$

Notice that if $f_1(x), \ldots, f_n(x)$ is a solution of equation (1) and $f_{10} \neq 0$, then in view of Theorem II also $f_{i0} \neq 0 \ (i = 2, \ldots, n)$ and therefore $F_i \neq 0$ for $i = 1, \ldots, n$. Moreover, by Theorem I, we have $F_1 = 1/k$. Therefore

$$r \begin{bmatrix} k_{11}F_1-1 & k_{12}F_2 & \dots & k_{1n}F_n \\ k_{21}F_1-1 & k_{22}F_2 & \dots & k_{2n}F_n \\ \dots\dots\dots\dots\dots\dots\dots \\ k_{n1}F_1-1 & k_{n2}F_2 & \dots & k_{nn}F_n \end{bmatrix} = r \begin{bmatrix} \dfrac{k_{11}}{k}-1 & k_{12} & \dots & k_{1n} \\ \dfrac{k_{21}}{k}-1 & k_{22} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots \\ \dfrac{k_{n1}}{k}-1 & k_{n2}. & \dots & k_{nn} \end{bmatrix}$$

$$= r \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \dots & \bar{k}_{1n} \\ \bar{k}_{21} & \bar{k}_{22} & \dots & \bar{k}_{2n} \\ \dots\dots\dots\dots\dots \\ \bar{k}_{n1} & \bar{k}_{n2} & \dots & \bar{k}_{nn} \end{bmatrix} = n-1.$$

Without loss of generality we can assume that

$$\begin{vmatrix} \bar{k}_{22} & \dots & \bar{k}_{2n} \\ \dots\dots\dots \\ \bar{k}_{n2} & \dots & \bar{k}_{nn} \end{vmatrix} \neq 0.$$

Then

$$\begin{vmatrix} k_{22}F_2 & \dots & k_{2n}F_n \\ \dots\dots\dots\dots\dots \\ k_{n2}F_2 & \dots & k_{nn}F_n \end{vmatrix} \neq 0$$

and we have by Cramer's formulae

$$f_i(x) = \frac{1}{F_i} \cdot \frac{\begin{vmatrix} k_{22} & \dots & k_{2,i-1} & 1-k_{21}F_1 & k_{2,i+1} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ k_{n2} & \dots & k_{n,i-1} & 1-k_{n1}F_1 & k_{n,i+1} & \dots & k_{nn} \end{vmatrix}}{\begin{vmatrix} k_{22} & \dots & k_{2,i-1} & k_{2i} & k_{2,i+1} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ k_{n2} & \dots & k_{n,i-1} & k_{ni} & k_{n,i+1} & \dots & k_{nn} \end{vmatrix}} f_1(x).$$

But

$$\begin{vmatrix} k_{22} & \dots & k_{2,i-1} & 1-k_{21}F_1 & k_{2,i+1} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ k_{n2} & \dots & k_{n,i-1} & 1-k_{n1}F_1 & k_{n,i+1} & \dots & k_{nn} \end{vmatrix} = F_1 \begin{vmatrix} k_{22} & \dots & k_{2,i-1} & k-k_{21} & k_{2,i+1} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ k_{n2} & \dots & k_{n,i-1} & k-k_{n1} & k_{n,i+1} & \dots & k_{nn} \end{vmatrix}$$

$$= F_1 \begin{vmatrix} k_{22} & \dots & k_{2,i-1} & \sum_{j=2}^{n} k_{2j} & k_{2,i+1} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ k_{n2} & \dots & k_{n,i-1} & \sum_{j=2}^{n} k_{nj} & k_{n,i+1} & \dots & k_{nn} \end{vmatrix} = F_1 \begin{vmatrix} k_{22} & \dots & k_{2,i-1} & k_{2i} & k_{2,i+1} & \dots & k_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ k_{n2} & \dots & k_{n,i-1} & k_{ni} & k_{n,i+1} & \dots & k_{nn} \end{vmatrix}$$

and therefore

$$f_i(x) = \frac{F_1}{F_i} f_1(x) = \frac{f_{i0}}{f_{10}} f_1(x) \,.$$

Hence we obtain

$$f_1(x_{i_1}) f_2(x_{i_2}) \dots f_n(x_{i_n}) = \frac{F_1}{f_{10}^{n-1}} f_1(x_{i_1}) f_1(x_{i_2}) \dots f_1(x_{i_n})$$

$$= \frac{1}{k f_{10}^{n-1}} f_1(x_1) f_1(x_2) \dots f_1(x_n)$$

and therefore

$$f_1(x_1 + \dots + x_n) = \frac{1}{f_{10}^{n-1}} f_1(x_1) \dots f_1(x_n) \,.$$ Q.E.D.

Remark. Theorems I, II, III, IV are valid for x from an arbitrary group with the group operation "$+$".

THEOREM VI. *If a non-zero solution of the equation*

(7) $$f(x_1 + \dots + x_n) = \frac{1}{A^{n-1}} f(x_1) \dots f(x_n)$$

has a point of continuity, it must have the form

(8) $$f(x) = \varepsilon_n A e^{ax},$$

where a is an arbitrary constant, $\varepsilon_n = 1$ if n in (7) is an even number and $|\varepsilon_n| = 1$ if n in (7) is an odd number.

Proof. It is easy to see that $f_0 \neq 0$. (In the contrary case we should have $f(x) \equiv 0$.) Moreover, it follows from (7) that $f_0 = A$ if n is an even number and $|f_0| = |A|$ if n is an odd number, i.e. $f_0 = \varepsilon_n A$, where $\varepsilon_n = 1$ for even n and $|\varepsilon_n| = 1$ for odd n.

Now, let a be a point of continuity of the function $f(x)$. Putting in (7) $x_1 = \xi$, $x_2 = x - a$, $x_3 = \dots = x_n = 0$, we obtain

$$f(x - a + \xi) = \frac{\varepsilon_n^{n-2}}{A} f(\xi) f(x - a) \,.$$

If $\xi \to a$, $x = $ const, then $x - a + \xi \to x$, and since the right side of the last equality is continuous at a, $f(x)$ must be continuous at the arbitrary point x.

Making use of Aczél's method (cf. [1], page 140-144), we conclude that the function $f(x)$ has all the derivatives.

Differentiating (7) by x_n and then putting $x_1 = x$, $x_2 = \dots = x_n = 0$, we obtain

$$f'(x) = \frac{f_0^{n-2}}{A^{n-1}} f_0' f(x) \,.$$

Now, taking into account that $f_0 = \varepsilon_n A$, we conclude that

$$f(x) = \varepsilon_n A e^{ax},$$

where a is an arbitrary constant. Q. E. D.

Remark. Equation (7) has a very simple physical interpretation if $A = 1$, and $a < 0$: It is known that the probability of decay of the radioactive atom during the time x is equal for all the atoms of the given radioactive element. Denote this probability by $p(x)$. The probability that the atom does not decompose during the time x is $f(x) = 1 - p(x)$. Obviously, if the atom decomposes neither during the time x_1 nor during the time x_2 ... nor during the time x_n, it means that the atom does not decompose during the time $x_1 + x_2 + ... + x_n$. The probabilities of these occurrences must satisfy the equation

$$f(x_1 + x_2 + ... + x_n) = f(x_1) f(x_2) ... f(x_n) .$$

Hence $f(x) = e^{ax}$, and $p(x) = 1 - e^{ax}$. If we have N atoms of the radioactive element at the moment $t = 0$, we must have approximately $N(1 - e^{ax})$ atoms at the moment $t = x$. This agrees with experiment. (The constant a can be found experimentally.)

THEOREM VII. *If equation* (1) *is reducible, all its solutions with* $f_{10} \neq 0$ *such that at least one function* $f_i(x)$ *has a point of continuity must have form* (2).

Proof. It follows from Theorem V that the functions $f_i(x)$ satisfy (5) and the function $f_1(x)$ satisfies equation (6). Since $f_{10} \neq 0$, it follows from Theorem I that

$$(9) \qquad F_1 = f_{20} ... f_{n0} = \frac{1}{k} .$$

Since at least one function $f_i(x)$ has a point of continuity, it follows from (5) that $f_1(x)$ must have a point of continuity. Therefore we can apply Theorem VI to equation (6) and we obtain

$$f_1(x) = f_{10} e^{ax} .$$

On the basis of (5)

$$f_i(x) = f_{i0} e^{ax} \qquad \text{for} \quad i = 1, ..., n ,$$

i.e.

$$f_i(x) = A_i e^{ax} \qquad \text{for} \quad i = 1, ..., n ,$$

where the constants A_i satisfy, by virtue of (9), the condition

$$A_2 ... A_n = \frac{1}{k} . \qquad\qquad \text{Q. E. D.}$$

THEOREM VIII. *If equation* (1) *is reducible and asymmetrical by* f_1, *functions* (2) *are the only non-trivial solutions such that at least one function* $f_i(x)$ *has a point of continuity.*

Proof. Since equation (1) is asymmetrical by f_1, we have in view of Theorem III $f_{10} \neq 0$. Therefore we may apply Theorem VII and the proof is finished. Q. E. D.

Remark. Results of Theorem VIII are well known for the equation

$$f_1(x_1 + x_2) = f_1(x_1)f_2(x_2)$$

(cf. e.g. [1], page 47, 48).

The next two theorems show explicitly the connections between the type of asymmetry of equations (1) and their solutions.

In order to prove them we introduce the following

LEMMA. If $f_1(x), \ldots, f_n(x)$ is the solution of equation (1), the functions

$$(10) \qquad \varphi_i(x) = e^{-ax}f_i(x) \qquad (i = 1, \ldots, n)$$

satisfy the equation

$$(11) \qquad \varphi_1(x_1 + \ldots + x_n) = \sum_{(i_1, \ldots, i_n) \in P_k}' \varphi_1(x_{i_1}) \ldots \varphi_n(x_{i_n}).$$

If, moreover, $f_1(x), \ldots, f_n(x)$ have first derivatives and

$$(12) \qquad a = \frac{f_s'(\xi)}{f_s(\xi)}, \qquad where \qquad f_s(\xi) \neq 0 ,$$

then $\varphi_{s\xi}' = 0$ and

$$(13) \qquad \varphi_1^{(n-1)}(x + (n-1)\xi) = k_{ls}\Phi_{s\xi}\varphi_s(x) \qquad (l = 1, \ldots, n) .$$

Proof. Substituting (10) into (1), we obtain (11). Differentiating (10) and taking into account (12), we obtain $\varphi_{s\xi}' = 0$.

Since all the functions $\varphi_i(x)$ have first derivatives and satisfy (11), we can differentiate the right-hand side of (11) by n different variables and we conclude hence that the function $\varphi_1(x)$ has an nth derivative.

Differentiating (11) by $x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n$ and then putting $x_l = x, x_i = \xi$ $(i = 1, \ldots, n, i \neq l)$, we obtain by virtue of $\varphi_{s\xi}' = 0$ equalities (13). Q. E. D.

THEOREM IX. If equation (1) is asymmetrical by f_1, all its non-trivial solutions with first derivatives have form (2).

Proof. Let us fix some non-trivial, differentiable solution $f_1(x)$, $\ldots, f_n(x)$ of equation (1).

In view of Theorem IV we have $f_1(x) \neq 0, \ldots, f_n(x) \neq 0$ for every x. Let us fix an arbitrary ξ and write

$$(14) \qquad a_\xi = \frac{f_1'(\xi)}{f_1(\xi)} .$$

The functions

$$(15) \qquad \varphi_i(x) = e^{-a_\xi x}f_i(x) \qquad (i = 1, \ldots, n)$$

have first derivatives and, by the Lemma, satisfy equations (11) and (13) with $s = 1$. Since equation (1) is asymmetrical by f_1, equation (11) is asymmetrical by φ_1. Hence it follows that there exist p, q such that $k_{p1} \neq k_{q1}$, and making use of (13) for $l = p$ and for $l = q$, we obtain

$$(k_{p1} - k_{q1})\Phi_{1\xi}\varphi_1(x) \equiv 0 .$$

Since $k_{p1} \neq k_{q1}$ and $\varphi_1(x) \neq 0$, we must have $\Phi_{1\xi} = 0$. Thus by (13) $\varphi_1^{(n-1)}(x) \equiv 0$, i.e. $\varphi_1(x)$ is a polynomial of the order $\leqslant n-2$ and (in view of (15)) the function $f_1(x)$ is a product of a polynomial and the exponential function $e^{a_\xi x}$.

If we fix ξ' instead of ξ, we conclude analogously that $f_1(x)$ is represented as a product of a polynomial and the exponential function $e^{a_{\xi'}x}$. But the function $f_1(x)$ was fixed at the beginning of our considerations and therefore both these representations represent the same function. Hence it follows that $a_{\xi'} = a_\xi$, i.e. a_ξ does not depend on ξ. Denote it by a. Now, in view of (14), we have

$$f_1'(\xi) = af_1(\xi) \quad \text{for every } \xi.$$

Hence we conclude that

(16)
$$f_1(x) = A_1 e^{ax},$$

We look for non-trivial solutions and therefore we assume $A_1 \neq 0$.

To prove our theorem it suffices to show that

$$\frac{f_s'(\xi)}{f_s(\xi)} = a$$

for $s = 2, \ldots, n$ and for every ξ and that

$$f_{n0} = \frac{1}{k f_{20} \cdots f_{n-1,0}} .$$

Let us fix an arbitrary ξ and $s > 1$, and write

(17)
$$\beta = \frac{f_s'(\xi)}{f_s(\xi)} .$$

The functions

(18)
$$\widetilde{\varphi}_i(x) = e^{-\beta x} f_i(x) \quad (i = 1, \ldots, n)$$

satisfy the assumptions of the second part of the Lemma, and we have

(19)
$$\widetilde{\varphi}_1^{(n-1)}(x + (n-1)\xi) = k_{1s}\widetilde{\Phi}_{s\xi}\widetilde{\varphi}_s(x) .$$

On the basis of (16) and (18) we have

(20)
$$\widetilde{\varphi}_1(x) = A_1 e^{(a-\beta)x} .$$

Suppose that $\beta \neq \alpha$. Then in view of (20) and (19) we have

$$\widetilde{\varphi}_1^{(n-1)}(x) \neq 0, \qquad \widetilde{\Phi}_{s\xi} \neq 0, \qquad k_{1s} \neq 0,$$

$$\widetilde{\varphi}_s(x) = \frac{A_1}{k_{1s}\widetilde{\Phi}_{s\xi}} e^{(\alpha-\beta)(n-1)\xi}(\alpha-\beta)^{n-1}e^{(\alpha-\beta)x} = B_s e^{(\alpha-\beta)x}.$$

Hence and from (18) it follows that $f_s(x) = B_s e^{\alpha x}$, and by (17) $\beta = \alpha$, contrary to our assumption.

Thus we have proved that

$$\frac{f_s'(\xi)}{f_s(\xi)} = \alpha$$

for $s = 2, \ldots, n$ and for arbitrary ξ. Hence it follows that

$$f_i(x) = A_i e^{\alpha x} \qquad (i = 2, \ldots, n).$$

Substituting this into (1), we obtain the condition

$$f_1(x_1 + \ldots + x_n) = A_2 \ldots A_n \sum_{(i_1, \ldots, i_n)\, \epsilon P_k} f_1(x_{i_1}) e^{\alpha(x_{i_2} + \ldots + x_{i_n})}.$$

Hence and from (16) it follows that all the non-trivial solutions of equation (1) with first derivatives have form (2). Q. E. D.

THEOREM X. *If equation* (1) *is asymmetrical but symmetrical by* f_1, *all its non-trivial solutions with first derivatives have form* (2) *or* (3).

Proof. Suppose that $k_{pr} \neq k_{qr}$ $(r \neq 1)$ and that $f_1(x), \ldots, f_n(x)$ is a fixed differentiable solution of equation (1). By virtue of Theorem III $f_r(x) \neq 0$ for every x.

Fix an arbitrary ξ and take

$$\alpha_\xi = \frac{f_r'(\xi)}{f_r(\xi)}.$$

The functions

(21) $\qquad \varphi_i(x) = e^{-\alpha_\xi x} f_i(x) \qquad (i = 1, \ldots, n)$

satisfy the assumptions of the second part of the Lemma with $s = r$. Making use of (13) for $l = p$ and for $l = q$, we obtain after subtraction

$$(k_{pr} - k_{qr})\Phi_{r\xi}\varphi_r(x) \equiv 0.$$

Since $k_{pr} \neq k_{qr}$ and $\varphi_r(x) \neq 0$, it follows from the last equality that $\Phi_{r\xi} = 0$ and therefore

$$\varphi_1^{(n-1)}(x) \equiv 0,$$

i.e. $\varphi_1(x)$ is a polynomial of order $\leqslant n-2$. Thus, in view of (21), $f_1(x)$ is a product of a polynomial and an exponential function $e^{\alpha_\xi x}$. Such a representation must be unique and therefore $\alpha_\xi = $ const. Denote this constant by α. Now we may write

(22) $\qquad f_1(x) = \varphi_1(x) e^{\alpha x},$

where $\varphi_1(x)$ is a polynomial of order $\leqslant n-2$.

Now we shall prove that

(23)   *If* $f_s(x) = 0$, *then* $s = 1$ *and* $x = 0$.

In fact, suppose that $a \neq 0$ and $f_1(a) = 0$. Then it follows from Theorem I that $f_1(na) = 0$, $f_1(n^2a) = 0$, ... Since $f_1(x)$ has form (22), it may vanish only at a finite number of points. Therefore $a \neq 0$ and $f_1(n^\nu a) = 0$ for $\nu = 0, 1, ...$ is impossible and we must have $f_1(x) \neq 0$ for $x \neq 0$.

For $s > 1$ it follows from Theorem II that $f_{s0} \neq 0$. If we had $b \neq 0$ and $f_s(b) = 0$, it would follow from Theorem I that $f_1(nb) = 0$. Since we have proved it is impossible, (23) must be satisfied.

By virtue of (23) we may fix an arbitrary $s > 1$ and put

(24) $$\beta_\xi = \frac{f_s'(\xi)}{f_s(\xi)} .$$

For the functions

(25) $$\overset{\xi}{\varphi}_i(x) = e^{-\beta_\xi x} f_i(x) \qquad (i = 1, ..., n)$$

the Lemma implies

(26) $$\overset{\xi}{\varphi}_1^{(n-1)}\big(x + (n-1)\xi\big) = k_{ls} \overset{\xi}{\varPhi}_{s\xi} \overset{\xi}{\varphi}_s(x) \qquad (l = 1, ..., n).$$

Notice that by (22) and (25) we have

(27) $$\overset{\xi}{\varphi}_1(x) = \varphi_1(x) e^{(a-\beta_\xi)x} ,$$

where $\varphi_1(x)$ is a polynomial of order $\leqslant n-2$ which may vanish at the point $x = 0$ only.

If there exists l such that $k_{ls} = 0$ or if $\overset{\xi}{\varPhi}_{s\xi} = 0$ for every ξ, then, in view of (26), $\overset{\xi}{\varphi}_1^{(n-1)}(x) \equiv 0$ for every ξ, i.e. $\overset{\xi}{\varphi}_1(x)$ is a polynomial and we conclude from (27) that $\beta_\xi = a$. Since it is satisfied for every ξ, (24) implies

(28) $$f_s(x) \equiv A_s e^{ax} .$$

Now, suppose that $k_{ls} \neq 0$ for $l = 1, ..., n$ and $\overset{\xi}{\varPhi}_{s\xi} \neq 0$ for some ξ. Then it follows from (26) and (27) that

$$\overset{\xi}{\varphi}_s(x) = \frac{1}{k_{ls} \overset{\xi}{\varPhi}_{s\xi}} \overset{\xi}{\varphi}_1^{(n-1)}\big(x + (n-1)\xi\big) = \psi_s(x) e^{(a-\beta_\xi)x}$$

and by (25)

(29) $$f_s(x) = \psi_s(x) e^{ax} ,$$

where $\psi(x)$ is a polynomial of order $\leqslant n-2$.

This argumentation is true for $s = 2, ..., n$ and since (28) is a special case of (29), it follows from (29), (22), and from Theorem I that

$$\varphi_1(nx) = k\varphi_1(x)\psi_2(x) ... \psi_n(x) .$$

The order of the polynomial $\varphi_1(x)\psi_2(x) \ldots \psi_n(x)$ must be the same as the order of the polynomial $\varphi_1(nx)$. It is possible only if $\psi_s(x)$ are polynomials of the order 0, i.e. if $f_s(x)$ have form (28).

It follows from (22), (28) and from equation (1) that

$$\varphi_1(x_1 + \ldots + x_n) = A_2 \ldots A_n \sum_{i=1}^{n} k_{i1} \varphi_1(x_i)$$

but equation (1) is symmetrical by f_1 and therefore

$$\varphi_1(x_1 + \ldots + x_n) = A_2 \ldots A_n k_{11} \sum_{i=1}^{n} \varphi_1(x_i) \ .$$

Since $\varphi_1(x)$ is a polynomial, we can differentiate the last equality by two different variables and we conclude that it may be satisfied only if

(i) $$\varphi_1(x) = A_1, \qquad A_2 \ldots A_n = 1/k$$

or

(ii) $$\varphi_1(x) = A_1 x, \qquad A_2 \ldots A_n = 1/k_{11} \ . \ \cdot$$

In the case (i) solutions of equation (1) have form (2), in the case (ii) they have form (3). Q. E. D.

Solutions of symmetrical equations (1) are more complicated. E. g. it is known (cf. [2]) that the only non-trivial solutions of the equation

$$f_1(x_1 + x_2) = f_1(x_1)f_2(x_2) + f_1(x_2)f_2(x_1)$$

such that $f_1(x)$ and $f_2(x)$ have at least one point of continuity in common are the functions

$$f_1(x) = Ae^{ax}, \qquad f_2(x) = \tfrac{1}{2}e^{ax};$$
$$f_1(x) = Axe^{ax}, \qquad f_2(x) = e^{ax};$$
$$f_1(x) = Ae^{ax}\sinh bx, \qquad f_2(x) = e^{ax}\cosh bx;$$
$$f_1(x) = Ae^{ax}\sin bx, \qquad f_2(x) = e^{ax}\cos bx$$

$(A \neq 0, b \neq 0, a$ — arbitrary constants).

It is easy to verify that the equation

$$f_1(x_1 + x_2 + x_3) = f_1(x_1)f_2(x_2)f_3(x_3) + f_1(x_2)f_2(x_3)f_3(x_1) + f_1(x_3)f_2(x_1)f_3(x_2)$$

is satisfied by the functions

(i) $\quad f_1(x) = A_1 e^{ax}\cos bx$,

$\qquad f_2(x) = A_2 e^{ax}(\cos bx + \sqrt{3}\sin bx)$,

$\qquad f_3(x) = \dfrac{1}{3A_2} e^{ax}(\cos bx - \sqrt{3}\sin bx);$

(ii) $f_1(x) = A_1 e^{ax} \cos bx$,

$$f_2(x) = A_2 e^{ax} \left(\cos bx + \frac{1}{\sqrt{3}} \sin bx \right),$$

$$f_3(x) = \frac{1}{A_2} e^{ax} \left(\cos bx - \frac{1}{\sqrt{3}} \sin bx \right)$$

$(A_1 \neq 0, A_2 \neq 0, b, a$ — arbitrary constants).

The above examples show that symmetrical equations (1) may have many solutions different from (2) and (3). We shall not solve them in this paper.

References

[1] J. Aczél, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Berlin 1961.

[2] H. Świątak, *On the equation* $[\varphi(x+y)]^2 = [\varphi(x)g(y)+\varphi(y)g(x)]^2$, Zeszyty Naukowe U. J., Prace Mat. 10 (1965).