ANNALES
POLONICI MATHEMATICI
XXV (1971)

On the asymptotic behaviour of some sequences built of iterates

by J. DREWNIAK and M. KUCZMA (Katowice)

1. Let x be a real or complex variable and let f be a function fulfilling the following conditions:

(H) $f(x)$ is defined and continuous for $|x-a| < R$, $0 < |f(x)-a| < |x-a|$ for $0 < |x-a| < R$, and there exists the derivative $s = f'(a)$.

Several authors (cf. [1], [2] and [4]) have studied the behaviour of the sequence

$$y_n = \frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}, \quad n = 1, 2, \ldots, \tag{1}$$

where

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

and x_0 is an arbitrary point such that

$$0 < |x_0-a| < R. \tag{3}$$

In particular, the following results have been established:

(i) (Hamilton [2]) *If* $s \neq 1$, *then for every* x_0 *fulfilling* (3) $\lim y_n = |s|$ (1).

(ii) (Kuczma [4]) *If* $s = 1$, *then the limit* $\lim y_n$ *need not exist.*

However,

(iii) (Kuczma [4]) *If* $s = 1$ *and the limit* $\lim y_n$ *exists, then* $\lim y_n = 1$.

B. Choczewski [1] investigated the possible asymptotic behaviour of sequence (1) in the divergence case. Let u_n, $n = 1, 2, \ldots$, be an arbitrary sequence with positive terms and let us write

$$p_n = \prod_{i=1}^{n} u_i, \quad n = 1, 2, \ldots \tag{4}$$

(iv) (Choczewski [1]) *If* $0 < \limsup p_n < \infty$, *then*

(1) Here and in the sequel, unless otherwise stated, lim and limsup always refer to $n \to \infty$.

(*) *there exists a function f fulfilling conditions* (H) *with* $s = 1$, *and a point* x_0 *fulfilling* (3) *such that for the corresponding sequence* (1) *we have* $y_n \sim u_n$.

Here $y_n \sim u_n$ means that the sequences y_n and u_n are asymptotically equal, i.e., $\lim y_n/u_n = 1$.

In order to prove (iv) B. Choczewski used a device an essential part of which was the following observation.

Observation I. Let u_n and v_n be sequences of positive terms and let us define p_n and q_n by (4) and

$$q_n = \prod_{i=1}^{n} v_i, \qquad n = 1, 2, \ldots, \tag{5}$$

respectively. Then $u_n \sim v_n$ does not imply $p_n \sim q_n$.

Thus, if u_n does not fulfil the assumptions of (iv), we may try to find another sequence v_n such that $u_n \sim v_n$ and for sequence (5) we have

$$0 < \limsup q_n < \infty. \tag{6}$$

If we succeed, assertion (*) results from the transitivity of the relation $\sim$.

Thus B. Choczewski, though in fact has not gone so far, has almost proved

(v) (Almost Choczewski [1]) *If there exists a sequence* $\tilde{p}_n$, $n = 1, 2, \ldots$, *with positive terms, such that*

$$\lim \tilde{p}_{n+1}/\tilde{p}_n = 1 \tag{7}$$

and

$$0 < \limsup p_n/\tilde{p}_n < \infty, \tag{8}$$

where p_n *is given by* (4), *then* (*) *holds.*

In fact, it is enough to put $v_n = u_n \tilde{p}_{n-1}/\tilde{p}_n$, where $\tilde{p}_0 = 1$. Then by (7) $u_n \sim v_n$ and we have $q_n = p_n/\tilde{p}_n$ so that (6) results from (8).

Theorem (v) has the disadvantage that it involves another sequence, $\tilde{p}_n$, whose existence must be postulated. It would be nicer to have a condition expressed in terms of the sequence p_n itself. For it is clear from the investigations of B. Choczewski that sequence (4) plays an important rôle here. But any condition based on the asymptotic behaviour of the sequence p_n itself must turn out to be inadequate, since the asymptotic behaviour of sequence (4) is not invariant under the asymptotic equality for the sequence u_n. And here we come to a next observation.

Observation II. Let u_n, v_n be sequences of positive terms and let p_n, q_n be defined by (4) resp. (5). Then $u_n \sim v_n$ implies $\sqrt[n]{p_n} \sim \sqrt[n]{q_n}$.

Observation II follows from the well-known fact that if a sequence of positive terms converges to a positive limit, then the sequence of its geometric means also converges to the same limit.

Observation II suggests that the condition for (*) should be expressed in terms of $\sqrt[n]{p_n}$ rather than of p_n itself. Now, condition (7) implies that $\lim \sqrt[n]{\tilde{p}_n} = 1$, whence by (8)

$$\limsup \sqrt[n]{p_n} = 1. \tag{9}$$

It turns out that condition (9) is characteristic for those sequences u_n for which (*) holds. Namely, in section 2 we shall prove

(vi) *Let u_n be a sequence of positive terms and let p_n be defined by* (4). *Then* (*) *holds if and only if the sequence p_n fulfils condition* (9).

One may ask how wildly can behave sequence (1) in the case $s = 1$. The following result gives a certain answer to this question.

(vii) *A set E is the set of the points of accumulation of sequence* (1) *generated by a function f fulfilling* (H) *with $s = 1$ and by a point x_0 fulfilling* (3) *if and only if $E \subset \langle 0, \infty\rangle$, E is closed and*

$$\inf E \leqslant 1 \leqslant \sup E. \tag{10}$$

The purpose of the present paper is to prove assertions (vi) and (vii). The proofs will be supplied in sections 2 and 3.

Finally let us note that either of assertions (vi) and (vii) implies both (ii) and (iii). Since the present paper does not rely on [4], it yields new proofs of (ii) and (iii).

In fact, assertion (vi) implies (ii), (iii), (iv) and (v).

2. In this section we prove assertion (vi).

LEMMA 1. *If f fulfils* (H), *then for every sequence x_n defined by* (2) *with* (3) *we have*

$$\lim x_n = a. \tag{11}$$

Proof. The sequence $|x_n - a|$ is decreasing and bounded below (by zero) and hence converges to a limit ϱ, $0 \leqslant \varrho < R$. Further, there exists a subsequence x_{n_k} of x_n convergent to an x^* and, of course, $|x^* - a| = \varrho$. By the continuity of f the sequence $x_{n_k+1} = f(x_{n_k})$ converges to $f(x^*)$ and again we have $|f(x^*) - a| = \varrho$. But according to (H) this is not possible unless $\varrho = 0$, which is equivalent to (11).

LEMMA 2. *If f fulfils* (H), $s = 1$, *and the sequence y_n is given by* (1) *with* (2) *and* (3), *then*

$$\limsup \sqrt[n]{\prod_{i=1}^{n} y_i} = 1.$$

Proof. By (1)

$$\sqrt[n]{\prod_{i=1}^{n} y_i} = \sqrt[n]{|x_{n+1}-x_n|}\,/\sqrt[n]{|x_1-x_0|}. \tag{12}$$

The denominator on the left-hand side of (12) tends to 1 and, since by Lemma 1 $\lim|x_{n+1}-x_n| = 0$, we have

$$\limsup \sqrt[n]{|x_{n+1}-x_n|} \leqslant 1.$$

If we had $\limsup \sqrt[n]{|x_{n+1}-x_n|} < 1$, then there would exist positive constants M and $c < 1$ such that

$$|x_{n+1}-x_n| \leqslant Mc^n \quad \text{for } n = 0, 1, 2, \ldots \tag{13}$$

According to Lemma 1

$$x_n - a = \sum_{k=n}^{\infty} (x_k - x_{k+1}),$$

whence by (13)

$$|x_n - a| \leqslant M \sum_{k=n}^{\infty} c^k = \frac{M}{1-c} c^n, \quad n = 0, 1, 2, \ldots \tag{14}$$

On the other hand, since $s = 1$, we may find positive constants $\varepsilon < 1-c$ and δ such that

$$\frac{|f(x)-a|}{|x-a|} > c+\varepsilon \quad \text{for } |x-a| < \delta. \tag{15}$$

Further, there exists an N such that $|x_n - a| < \delta$ for $n \geqslant N$. Hence we get by (15)

$$|x_{n+1}-a| > (c+\varepsilon)|x_n - a| \quad \text{for } n \geqslant N,$$

or

$$|x_n - a| > K(c+\varepsilon)^n \quad \text{for } n = 0, 1, 2, \tag{16}$$

with a suitable constant $K > 0$. Relations (14) and (16) yield

$$(c+\varepsilon)^n < \frac{M}{K(1-c)} c^n, \quad n = 0, 1, 2, \ldots,$$

which is impossible, since $\varepsilon > 0$. Thus necessarily

$$\limsup \sqrt[n]{|x_{n+1}-x_n|} = 1,$$

which in view of (12) implies the assertion of the lemma.

LEMMA 3. *Let r_n, $n = 1, 2, \ldots$, be a sequence of real numbers such that*

(17) $$\limsup r_n = 0.$$

Then there exists a sequence m_n, $n = 1, 2, \ldots$, fulfilling the following conditions:

(18) $$m_n \quad \textit{is monotonic},$$

(19) $$m_{n+1} - m_n \quad \textit{is monotonic},$$

(20) $$\lim m_n = 0,$$

(21) $$r_n \leqslant m_n \quad \textit{for large } n,$$

(22) $$r_n = m_n \quad \textit{for infinitely many } n.$$

Proof. We shall distinguish three cases.

I. For infinitely many n we have $r_n > 0$.

Condition (17) implies that for every set S of positive integers containing at least one index j with $r_j > 0$ there exists $\sup\limits_{i \in S} r_i$ and this supremum is attained for a finite number of indices i. Making use of this remark we shall define two auxiliary sequences a_k, n_k. We put

(23) $$a_1 = \sup_{i \geqslant 1} r_i, \quad a_2 = \sup_{i > n_1} r_i, \quad a_{k+1} = \sup_{i \in S_k} r_i, \quad k = 2, 3, \ldots,$$

and

(24) $$n_k = \max\{i:\ r_i = a_k\}, \quad k = 1, 2, \ldots,$$

where for $k \geqslant 2$

(25) $$S_k = \left\{ i > n_k:\ \frac{a_{k-1} - a_k}{n_k - n_{k-1}} \geqslant \frac{a_k - r_i}{i - n_k} \right\}.$$

In order to prove that the above definitions are correct we must show that for every $k \geqslant 2$ the set S_k contains indices i such that $r_i > 0$. We shall do this, proving at the same time by induction the inequalities

(26) $$a_{k+1} < a_k, \quad k = 1, 2, \ldots,$$

(27) $$n_{k+1} > n_k, \quad k = 1, 2,$$

Suppose that we have already defined the sequences a_j and n_j for $j = 1, \ldots, k$, $k \geqslant 2$, where a_j decrease and n_j increase ([2]). Thus $(a_{k-1} - a_k)/(n_k - n_{k-1})$ is a fixed positive number, and therefore $(a_{k-1} - a_k)/(n_k - n_{k-1}) > a_k/(i - n_k)$ for i sufficiently large. Hence

$$\frac{a_{k-1} - a_k}{n_k - n_{k-1}} > \frac{a_k - r_i}{i - n_k}$$

([2]) Inequalities (26) and (27) for $k = 1$ are easily checked.

for large i such that $r_i > 0$, and in the present case there are infinitely many such i. Thus we may define a_{k+1} by (23) and n_{k+1} as $\max\{i:\ r_i = a_{k+1}\}$ (cf. (24)).

Since the supremum $\sup\limits_{i\in S_k} r_i$ is realized, we have $a_{k+1} = r_i$ for some $i\in S_k$, i.e. $i > n_k$. Hence (27) follows. To prove the inequality $a_{k+1} \leqslant a_k$ for $k \geqslant 3$ we shall show that $S_k \subset S_{k-1}$; for $k = 2$ it similarly results from the obvious inclusion $S_2 \subset \{i:\ i > n_1\}$.

Let $i\in S_k$, $k \geqslant 3$. Then $i > n_k > n_{k-1}$ and

$$(a_k - r_i)(n_k - n_{k-1}) \leqslant (a_{k-1} - a_k)(i - n_k)$$
$$= (a_{k-1} - a_k)(i - n_{k-1}) - (a_{k-1} - a_k)(n_k - n_{k-1}),$$

whence

$$(a_{k-1} - r_i)(n_k - n_{k-1}) \leqslant (a_{k-1} - a_k)(i - n_{k-1}),$$

i.e.

$$\frac{a_{k-1} - a_k}{n_k - n_{k-1}} \geqslant \frac{a_{k-1} - r_i}{i - n_{k-1}}.$$

By the definition of a_k and n_k we have in view of the above relation

$$\frac{a_{k-2} - a_{k-1}}{n_{k-1} - n_{k-2}} \geqslant \frac{a_{k-1} - a_k}{n_k - n_{k-1}} \geqslant \frac{a_{k-1} - r_i}{i - n_{k-1}},$$

which means that $i\in S_{k-1}$. Thus $S_k \subset S_{k-1}$, whence $a_{k+1} \leqslant a_k$. But the equality is impossible in view of (24) and (27). Hence $a_{k+1} < a_k$, which completes the induction.

Relation (24) implies also that

$$r_{n_k} = a_k, \qquad k = 1, 2, \ldots \tag{28}$$

Further, it follows from (23), (25) and (28)

$$\frac{a_{k-1} - a_k}{n_k - n_{k-1}} \geqslant \frac{a_k - a_{k+1}}{n_{k+1} - n_k}, \qquad k = 2, 3, \ldots \tag{29}$$

Now we put

$$\begin{aligned} m_n &= a_2 + (n_2 - n)\frac{a_1 - a_2}{n_2 - n_1} \quad \text{for } 1 \leqslant n \leqslant n_2, \\ m_n &= a_{k+1} + (n_{k+1} - n)\frac{a_k - a_{k+1}}{n_{k+1} - n_k} \quad \text{for } n_k < n \leqslant n_{k+1},\ k = 2, 3, \ldots \end{aligned} \tag{30}$$

We have by (30) for $1 \leqslant n < n+1 \leqslant n_2$ $(k = 1)$ and $n_k \leqslant n < n+1 \leqslant n_{k+1}$ $(k \geqslant 2)$

$$m_{n+1} - m_n = -\frac{a_k - a_{k+1}}{n_{k+1} - n_k} < 0 \tag{31}$$

(cf. (26) and (27)). Hence for $1 \leqslant n-1 < n < n+1 \leqslant n_2$ $(k = 1)$ and $n_k \leqslant n-1 < n < n+1 \leqslant n_{k+1}$ $(k \geqslant 2)$

$$(m_{n+1}-m_n)-(m_n-m_{n-1}) = 0. \tag{32}$$

On the other hand, for $n = n_k$, $k \geqslant 2$, we have

$$m_{n+1}-m_n = -\frac{a_k-a_{k+1}}{n_{k+1}-n_k}, \quad m_n-m_{n-1} = -\frac{a_{k-1}-a_k}{n_k-n_{k-1}}, \tag{33}$$

whence by (29)

$$(m_{n+1}-m_n)-(m_n-m_{n-1}) \geqslant 0. \tag{34}$$

Now, (18) results from (31) and (33), and (19) follows from (32) and (34). Further, relations (30) and (28) yield

$$m_{n_k} = a_k = r_{n_k}, \quad k = 1, 2, \ldots, \tag{35}$$

and since evidently $a_k > 0$, relations (17), (18) and (35) imply condition (20). Condition (22) is a consequence of (35). It remains to prove (21).

Let us take an n, $n_k < n \leqslant n_{k+1}$, $k \geqslant 2$ [3]. In view of (30) we have

$$m_n \geqslant a_{k+1}. \tag{36}$$

If $n \in S_k$, then the inequality

$$r_n \leqslant m_n \tag{37}$$

results from (23) and (36). If $n \notin S_k$, then according to (25) and (29)

$$\frac{a_k-r_n}{n-n_k} > \frac{a_{k-1}-a_k}{n_k-n_{k-1}} \geqslant \frac{a_k-a_{k+1}}{n_{k+1}-n_k},$$

whence by (30)

$$r_n < a_k-(n-n_k)\frac{a_k-a_{k+1}}{n_{k+1}-n_k} = a_{k+1}+(n_{k+1}-n)\frac{a_k-a_{k+1}}{n_{k+1}-n_k} = m_n.$$

II. For almost all n (i.e., possibly with an exception of a finite number of indices n) we have $r_n \leqslant 0$, and for infinitely many n we have $r_n = 0$.

Then the sequence $m_n = 0$, $n = 1, 2, \ldots$, evidently fulfils conditions (18) through (22).

III. We have $r_n < 0$ for $n \geqslant N$.

Condition (17) implies that for arbitrary fixed $n > N$ and $a < 0$ the supremum

$$\sup_{j>n} \frac{r_j-a}{j-n} \tag{38}$$

---

[3] A similar argument shows that (37) holds also for $1 \leqslant n \leqslant n_2$. Thus in case I relation (37) is valid for all n.

exists and is attained for a finite set of indices j. In fact, by (17) there are $j > n$ such that $r_j - a > 0$, whereas $\limsup_{j\to\infty} (r_j - a)/(j-n) = 0$.

Now we define sequences a_k and n_k by (28) and

$$\text{(39)}\qquad \begin{aligned} n_1 &= N, \\ n_{k+1} &= \sup\left\{i:\ \frac{r_i - a_k}{i - n_k} = \sup_{j>n_k}\frac{r_j - a_k}{j-n_k}\right\}, \qquad k = 1, 2, \ldots \end{aligned}$$

(note that in view of (28) $a_k < 0$), and we define the sequence m_n by (30).

Relation (27) evidently holds. Moreover, since in view of the previous remarks supremum (38) is attained by $r_j > a$, it follows from (39) that

$$\text{(40)}\qquad a_{k+1} = r_{n_{k+1}} > a_k.$$

Further, we have in view of (39), (27) and (28)

$$\frac{a_{k+1} - a_{k-1}}{n_{k+1} - n_{k-1}} = \frac{r_{n_{k+1}} - a_{k-1}}{n_{k+1} - n_{k-1}} \leqslant \sup_{j>n_{k-1}} \frac{r_j - a_{k-1}}{j - n_{k-1}} = \frac{r_{n_k} - a_{k-1}}{n_k - n_{k-1}} = \frac{a_k - a_{k-1}}{n_k - n_{k-1}},$$

and since

$$\frac{a_{k+1} - a_k}{n_{k+1} - n_k} = \frac{a_k - a_{k-1}}{n_k - n_{k-1}} - \left[\frac{a_k - a_{k-1}}{n_k - n_{k-1}} - \frac{a_{k+1} - a_{k-1}}{n_{k+1} - n_{k-1}}\right]\frac{n_{k+1} - n_{k-1}}{n_{k+1} - n_k},$$

we get

$$\text{(41)}\qquad \frac{a_{k+1} - a_k}{n_{k+1} - n_k} \leqslant \frac{a_k - a_{k-1}}{n_k - n_{k-1}}.$$

Relations (40), (41) and (30) imply that conditions (31)-(35) are fulfilled, with the inequalities in (31) and (34) turned into opposite ones. Consequently the sequence m_n fulfils conditions (18), (19), (20) and (22). Further, we have for $n > N$, say $n_k < n \leqslant n_{k+1}$, $k \geqslant 1$,

$$\frac{r_n - a_k}{n - n_k} \leqslant \sup_{j>n_k} \frac{r_j - a_k}{j - n_k} = \frac{r_{n_{k+1}} - a_k}{n_{k+1} - n_k} = \frac{a_{k+1} - a_k}{n_{k+1} - n_k}$$

(cf. (39) and (28)), whence by (30)

$$r_n \leqslant a_k + (n - n_k)\frac{a_{k+1} - a_k}{n_{k+1} - n_k} = a_{k+1} + (n_{k+1} - n)\frac{a_k - a_{k+1}}{n_{k+1} - n_k} = m_n.$$

Consequently condition (21) is fulfilled, too.

This completes the proof.

Corollary. *The sequence m_n fulfils the condition*

$$\text{(42)}\qquad \lim n(m_{n+1} - m_n) = 0.$$

Proof. By (20) the series $\sum(m_{n+1}-m_n)$ converges and hence ([3], § 14, Theorem 80) relation (42) results in view of (19).

Let us note also the following consequence of the mean-value theorem.

LEMMA 4. *A d, $0 < d < 1$, being fixed, there exists a positive constant C such that the inequality*

$$\left|\log\frac{1+x}{1+y}\right| \leqslant C|x-y|$$

holds for arbitrary $x, y \in \langle -d, d\rangle$.

Now we proceed to give

Proof of assertion (vi). It follows from Lemma 2 and Observation II that (*) implies (9). Conversely, let us assume that relation (9) holds. Then the sequence

$$r_n = \sqrt[n]{p_n}-1$$

fulfils (17) and we put

$$\tilde{p}_n = (1+m_n)^n,$$

where m_n is the sequence fulfilling conditions (18)-(22) of Lemma 3. (It follows from (20) that $\tilde{p}_n > 0$ for large n and we may change, if necessary, a finite number of values of $\tilde{p}_n$ to make $\tilde{p}_n > 0$ for all n.) Then by (21) and (22)

$$\limsup\frac{p_n}{\tilde{p}_n} = \limsup\left(\frac{1+r_n}{1+m_n}\right)^n = 1,$$

and thus condition (8) is fulfilled. On the other hand,

$$\frac{\tilde{p}_{n+1}}{\tilde{p}_n} = \left(\frac{1+m_{n+1}}{1+m_n}\right)^n(1+m_{n+1}), \tag{43}$$

and since by (20) $m_{n+1}, m_n \in \langle -d, d\rangle$ for large n, we obtain by Lemma 4 and by (42)

$$\lim\log\left(\frac{1+m_{n+1}}{1+m_n}\right)^n = 0. \tag{44}$$

Now, relation (7) results from (43), (44) and (20) and consequently (*) follows in virtue of (v).

3. In the present section we are going to prove assertion (vii). If E is the set of the points of accumulation of sequence (1) generated by a function f fulfilling (H) with $s = 1$, and by a point x_0 fulfilling (3), then obviously E is a closed set contained in $\langle 0, \infty\rangle$, and relation (10) results from Lemma 2. To prove the converse implication, we shall make use of the following

LEMMA 5. *If we are given numbers λ, μ such that*

$$0 < \lambda < 1 < \mu < \infty$$

and

$$\lambda^i \mu^j \neq 1 \quad \text{for all positive integers } i, j,$$

then for an arbitrary non-empty interval $(a, b) \subset (0, \infty)$ and arbitrary number $M \in (0, \infty)$ there exist positive integers k, l such that

$$a < \lambda^k M \mu^l < b.$$

The above lemma is a consequence of the well-known fact that if the numbers ω_1, ω_2 are positive and incommeasurable, then the set of the numbers of the form $n\omega_1 - m\omega_2$, where n, m run over the set of positive integers, is dense on the real axis.

Now let E be an arbitrary closed set contained in $\langle 0, \infty \rangle$ and fulfilling condition (10). Let E_0 be a finite or countable subset of E, dense in E:

$$\bar{E}_0 = E. \tag{45}$$

In order to prove (vii) it is enough to construct a sequence u_n fulfilling (9) (where p_n is given by (4)) and such that its set of the points of accumulation contains E_0 and is contained in E. (Note that asymptotically equal sequences have equal sets of the points of accumulation.)

We may write $E_0 = \{w_1, w_2, \ldots\}$. Let us form the triangular table

$$\begin{cases} w_1 \\ w_1 \ w_2 \\ w_1 \ w_2 \ w_3 \\ \ldots \end{cases} \tag{46}$$

(If the set E_0 is finite, table (46) has a finite number of columns, but the number of rows is always infinite.) If a w_i is 0 or ∞, then we replace the corresponding column by the sequence $1/n$ resp. n, n being the number of the row. Next we number the elements of table (46) in the order of rows, thus arriving at a sequence z_n such that $0 < z_n < \infty$, and the set of the points of accumulation of the sequence z_n contains E_0 and is contained in E (and hence, in view of (45), in fact equals E).

By (10) there exist $\lambda, \mu \in E$ such that $0 \leqslant \lambda \leqslant 1 \leqslant \mu \leqslant \infty$. We choose two sequences, λ_n and μ_n, in such a manner that $\lim \lambda_n = \lambda$, $\lim \mu_n = \mu$,

$$0 < \lambda_n < 1 < \mu_n < \infty, \tag{47}$$

and the sets of integral ($\neq 0$) powers of λ_n and μ_n are disjoint:

$$\{\lambda_n^\varkappa \colon n = 1, 2, \ldots; \varkappa = \pm 1, \pm 2, \ldots\} \cap \\ \cap \{\mu_n^\varkappa \colon n = 1, 2, \ldots; \varkappa = \pm 1, \pm 2, \ldots\} = \emptyset. \tag{48}$$

The last condition may be achieved, for instance, if we choose λ_n rational and μ_n transcendental.

Conditions (47) and (48) guarantee that for λ_n and μ_n the assumptions of Lemma 5 are fulfilled. Let a_n, b_n be monotonic sequences such that

$$0 < a_n < 1 < b_n < \infty$$

and

$$\lim a_n = \lim b_n = 1. \tag{49}$$

If we are given a sequence M_n of positive numbers, then by Lemma 5 we may find sequences k_n, l_n of positive integers such that

$$a_n < \lambda_n^{k_n} M_n \mu_n^{l_n} < b_n. \tag{50}$$

We shall define a sequence M_n, and at the same time a sequence c_n, by induction. We put

$$\begin{aligned} M_1 &= z_1, & c_1 &= \lambda_1^{k_1} M_1 \mu_1^{l_1}, \\ M_n &= c_{n-1} z_n, & c_n &= \lambda_n^{k_n} M_n \mu_n^{l_n}, \end{aligned} \tag{51}$$

where the positive integers k_n, l_n are chosen according to (50). Finally we define m_n as the number of factors of c_n, i.e.,

$$m_n = n + \sum_{i=1}^{n} (k_i + l_i), \qquad n = 1, 2, \ldots \tag{52}$$

(which may be proved by induction) and we assume $m_0 = 0$.

Now we put

$$u_i = \begin{cases} \lambda_n & \text{for } m_{n-1} < i \leqslant m_{n-1} + k_n, \\ z_n & \text{for } i = m_{n-1} + k_n + 1, \\ \mu_n & \text{for } m_{n-1} + k_n + 1 < i \leqslant m_n, \end{cases} \tag{53}$$

$n = 1, 2, \ldots$ Since the sequences λ_n and μ_n converge to elements of E, we have not introduced any new points of accumulation, and thus the set of the points of accumulation of the sequence u_n, like that of the sequence z_n, coincides with the set E. In order to complete the proof we need only to show that the sequence u_n fulfils condition (9).

Let us define p_n by (4). We have

$$p_{m_n} = \prod_{i=1}^{m_n} u_i = c_n. \tag{54}$$

Indeed, for $n = 1$ we have in view of (53), (52) and (51)

$$p_{m_1} = \prod_{i=1}^{m_1} u_i = \lambda_1^{k_1} z_1 \mu_1^{l_1} = \lambda_1^{k_1} M_1 \mu_1^{l_1} = c_1 .$$

Assuming (54) true for $n-1$, we have by (53), (52) and (51)

$$p_{m_n} = p_{m_{n-1}} \prod_{i=m_{n-1}+1}^{m_n} u_i = c_{n-1} \lambda_n^{k_n} z_n \mu_n^{l_n} = \lambda_n^{k_n} M_n \mu_n^{l_n} = c_n ,$$

and consequently (54) is valid for all $n = 1, 2, \ldots$ Hence we get in virtue of (50) and (49)

$$\lim p_{m_n} = 1 . \tag{55}$$

Now let us take an arbitrary p_j. We distinguish two cases.

(α) $j = m_{n-1} + k$, $0 < k \leqslant k_n$, for a certain $n \geqslant 1$. Then by (54) and (53)

$$p_j = p_{m_{n-1}} \prod_{i=m_{n-1}+1}^{j} u_i = c_{n-1} \lambda_n^{k} < c_{n-1} < b_{n-1} ,$$

since $\lambda_n < 1$.

(β) $j = m_{n-1} + k_n + 1 + l$, $0 \leqslant l \leqslant l_n$, for a certain $n \geqslant 1$. Then by (54) and (53)

$$p_j = p_{m_{n-1}} \prod_{i=m_{n-1}+1}^{j} u_i = c_{n-1} \lambda_n^{k_n} z_n \mu_n^{l} \leqslant c_{n-1} \lambda_n^{k_n} z_n \mu_n^{l_n} = c_n < b_n < b_{n-1} ,$$

since $\mu_n > 1$ and the sequence b_n is monotonic. Thus in either case we arrive at the estimation

$$p_j < b_{n-1} \quad \text{for } m_{n-1} < j \leqslant m_n ,\ n = 1, 2 , \tag{56}$$

Conditions (55) and (56) with (49) prove that

$$\limsup p_n = 1 ,$$

which implies (9) and completes the proof.

4. It follows from (H) that $f(a) = a$; in other words, a is a root of the equation

$$f(x) - x = 0 . \tag{57}$$

By Lemma 1 we have

$$a = x_0 + \sum_{n=1}^{\infty} (x_n - x_{n-1}) , \tag{58}$$

and formula (58) may be regarded as an algorithm for the approximate solution of equation (57). Now Hamilton's theorem (i) says that in the case $s \neq 1$ the convergence of the series in (58) is geometrical with the ratio $|s|$. Our results show that, in general, no such estimation of the rapidity of convergence of the series in (58) is possible in the case $s = 1$.

On the other hand, B. Choczewski [1] has given some additional conditions which assure that in the case $s = 1$ the limit $\lim y_n$ exists (and then, necessarily, $\lim y_n = 1$).

References

[1] B. Choczewski, *Note on the convergence of iterates*, Ann. Polon. Math. 24 (1971), p. 229-232.

[2] H. J. Hamilton, *Roots of equations by functional iteration*, Duke Math. J. 13 (1946), p. 113-121.

[3] K. Knopp, *Theorie und Anwendung der unendlichen Reihen*, 4. Auflage, Berlin, Heidelberg 1947.

[4] M. Kuczma, *On the convergence of iterates*, Ann. Polon. Math. 20 (1968), p. 195-198.

Reçu par la Rédaction le 27. 1. 1970