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SEPARABILITY OF THE CHARACTERISTIC POLYNOMIAL

1. Introduction. Two-dimensional ($2-D$) linear systems have received extensive attention in the last few years. Roesser [3] has extended the linear time-discrete state-space model from single-dimensional time to two-dimensional space. In $2-D$ linear time-discrete systems described by Roesser's model (see [3]) the separability of the characteristic polynomial has a very important meaning. Reyer [2] has presented a method for scaling, summation, multiplication, division and separation of multivariate polynomials using the array representation of multivariate polynomials.

The purpose of this paper is to formulate the necessary and sufficient condition for the separability of the characteristic polynomial and to give a simple procedure for checking separability and for determining polynomial factors.

2. Preliminaries. The state matrix of a $2-D$ linear system (in Roesser's model) is a partitioned matrix

(1) $$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} \in R^{n_1,n_1}, \quad A_{12} \in R^{n_1,n_2}, \quad A_{21} \in R^{n_2,n_1}, \quad A_{22} \in R^{n_2,n_2}, \quad A \in R^{n,n}$$

and $n = n_1 + n_2$.

The definition of the characteristic function of the matrix (1) (characteristic polynomial) is the following (see [3]):

Definition 1. The determinant of the matrix

$$\begin{bmatrix} I_{n_1} x_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} x_2 - A_{22} \end{bmatrix}$$

for the matrix (1) is called the $2-D$ *characteristic polynomial* (the $2-D$

characteristic function of the matrix), i.e.

$$f(x_1, x_2) = \det\begin{bmatrix} I_{n_1}x_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}x_2 - A_{22} \end{bmatrix}. \tag{2}$$

The $2-D$ characteristic function of the matrix is a polynomial of two variables which belongs to the ring $R[x_1, x_2]$. From (2) it is easy to notice that

$$f(x_1, x_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{ij} x_1^i x_2^j, \tag{3}$$

where

$$a_{n_1 n_2} = 1. \tag{4}$$

Definition 2. The $2-D$ characteristic polynomial $f(x_1, x_2)$ is called *separable* if and only if there exist polynomials

$$P_1(x_1) = \sum_{i=0}^{n_1} a_i x_1^i, \quad a_{n_1} = 1, \tag{5}$$

$$P_2(x_2) = \sum_{j=0}^{n_2} b_j x_2^j, \quad b_{n_2} = 1, \tag{6}$$

such that

$$f(x_1, x_2) = P_1(x_1) P_2(x_2). \tag{7}$$

The notion of separability refers, in general, to the possibility of separation of variables. Therefore, the polynomial $f(x_1, x_2) = \hat{P}_1(x_1)\hat{P}_2(x_2)$, in which $\hat{a}_{n_1} \neq 1$ and $\hat{b}_{n_2} \neq 1$, is a separable one. Taking into account (4) we obtain $\hat{a}_{n_1}\hat{b}_{n_2} = 1$. Multiplying the polynomial $\hat{P}_1(x_1)$ by $\hat{b}_{n_2}$ and $\hat{P}_2(x_2)$ by $\hat{a}_{n_1}$ we get the form (7) satisfying (5) and (6). Therefore, Definition 2 is not less general because we can always normalize the polynomial factors so that $a_{n_1} = 1$ and $b_{n_2} = 1$ if the polynomial $f(x_1, x_2)$ is separable.

3. Separability of the $2-D$ characteristic polynomial.

LEMMA 1. *If the characteristic polynomial* (3) *is separable then there are exactly one polynomial* $P_1(x_1)$ *and exactly one polynomial* $P_2(x_2)$ *satisfying* (7).

Proof. Let us assume that there are polynomials $Q_1(x_1) \neq P_1(x_1)$, $Q_2(x_2) \neq P_2(x_2)$ such that $f(x_1, x_2) = Q_1(x_1)Q_2(x_2)$. It is easy to notice that $\deg P_1(x_1) = \deg Q_1(x_1) = n_1$ and $\deg P_2(x_2) = \deg Q_2(x_2) = n_2$.

It appears from the assumptions that

$$f(x_1, x_2) = P_2(x_2)x_1^{n_1} + P_2(x_2)a_{n_1-1}x_1^{n_1-1} + \ldots + P_2(x_2)a_0 \tag{8}$$

and

(9) $$f(x_1, x_2) = Q_2(x_2)x_1^{n_1} + Q_2(x_2)q_{n_1-1}x_1^{n_1-1} + \ldots + Q_2(x_2)q_0.$$

The relations (8) and (9) can be written in the form

(10) $$f(x_1, x_2) = A_{n_1}x_1^{n_1} + A_{n_1-1}x_1^{n_1-1} + \ldots + A_0,$$

where $A_i \in R[x_2]$ $(i = 1, 2, \ldots, n_1)$ and $A_{n_1} = P_2(x_2)$, and

(11) $$f(x_1, x_2) = Q_{n_1}x_1^{n_1} + Q_{n_1-1}x_1^{n_1-1} + \ldots + Q_0,$$

where $Q_i \in R[x_2]$ $(i = 1, 2, \ldots, n_1)$ and $Q_{n_1} = Q_2(x_2)$.

From (10) and (11) and the algebra of one-variable polynomials (see [2], p. 165–166) we see that $A_{n_1} = Q_{n_1}$. Therefore,

(12) $$P_2(x_2) = Q_2(x_2).$$

Analogously (or from (8), (9) and (12)) we can write

$$P_1(x_1) = Q_1(x_1).$$

This completes the proof.

Let us write the polynomial (3) separating the terms which contain the factor $x_1^{n_1}$, in other words, the terms (3) for which $i = n_1$ and $j = 0, 1, 2, \ldots, n_2$. We obtain

(13) $$f(x_1, x_2) = x_1^{n_1}(x_2^{n_2} + a_{n_1n_2-1}x_2^{n_2-1} + \ldots + a_{n_10}) + \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2} a_{ij}x_1^i x_2^j.$$

If we denote by $g_2(x_2)$ the coefficient forming the polynomial of variable x_2 and standing at the element $x_1^{n_1}$, i.e.

(14) $$g_2(x_2) = x_2^{n_2} + a_{n_1n_2-1}x_2^{n_2-1} + \ldots + a_{n_10},$$

then the relation (13) is

(15) $$f(x_1, x_2) = x_1^{n_1}g_2(x_2) + u_2(x_1, x_2),$$

where $u_2(x_1, x_2)$ is the second term of (13) and it does not contain any element with the factor $x_1^{n_1}$.

Analogously, changing variables, we can write

$$f(x_1, x_2) = x_2^{n_2}g_1(x_1) + u_1(x_1, x_2),$$

where

(16) $$g_1(x_1) = x_1^{n_1} + a_{n_1-1,n_2}x_1^{n_1-1} + \ldots + a_{0n_2}$$

and

$$u_1(x_1, x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2-1} a_{ij}x_1^i x_2^j.$$

THEOREM 1. *If the characteristic polynomial given by* (3) *is separable then*

$$P_1(x_1) = g_1(x_1) \tag{17}$$

and

$$P_2(x_2) = g_2(x_2), \tag{18}$$

where the polynomials $g_1(x_1)$ and $g_2(x_2)$ are given by (14) *and* (16).

Proof. We can write from the assumption and from (5)

$$f(x_1, x_2) = P_1(x_1)P_2(x_2) = (x_1^{n_1} + a_{n_1-1}x_1^{n_1-1} + \ldots + a_0)P_2(x_2)$$
$$= x_1^{n_1}P_2(x_2) + u_2(x_1, x_2).$$

The comparison of the above result with (15) gives (18).

In the same way we can see that (17) holds. This completes the proof.

LEMMA 2. *The polynomial factors $g_1(x_1)$ and $g_2(x_2)$ given by the equations* (14) *and* (16) *satisfy the following relations*:

$$g_1(x_1) = \det(I_{n_1}x_1 - A_{11}), \tag{19}$$

$$g_2(x_2) = \det(I_{n_2}x_2 - A_{22}). \tag{20}$$

Proof. Let us calculate the coefficient standing at the factor $x_1^{n_1}$ in the polynomial $f(x_1, x_2)$. For this purpose we write the determinant (2) in the form

$$\det\left[\begin{array}{cccc|c} x_1-a_{11} & -a_{12} & \ldots & -a_{1n_1} & \\ 0-a_{21} & x_1-a_{22} & \ldots & -a_{2n_1} & -A_{12} \\ \ldots & \ldots & \ldots & \ldots & \\ 0-a_{n_11} & -a_{n_12} & \ldots & x_1-a_{n_1n_1} & \\ \hline & & -A_{21} & & I_{n_2}x_2-A_{22} \end{array}\right]. \tag{21}$$

Denoting by $A_{21}^{(i)}$ $(0 \leqslant i \leqslant n_1)$ the matrix A_{21} in which the first i columns have been replaced by zero columns, we can write the determinant (21) as the sum of the following two determinants (see [1], p. 98):

$$\det\left[\begin{array}{cccc|c} x_1 & 0-a_{12} & \ldots & -a_{1n_1} & \\ 0 & x_1-a_{22} & \ldots & -a_{2n_1} & -A_{12} \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0-a_{n_11} & \ldots & x_1-a_{n_1n_1} & \\ \hline & & -A_{21}^{(1)} & & I_{n_2}x_2-A_{22} \end{array}\right] +$$

$$+\det\left[\begin{array}{cccc|c} -a_{11} & -a_{12} & \cdots & -a_{1n_1} & \\ -a_{21} & x_1-a_{22} & \cdots & -a_{2n_1} & -A_{12} \\ \cdots & \cdots & \cdots & \cdots & \\ -a_{n_1 1} & -a_{n_1 2} & \cdots & x_1-a_{n_1 n_1} & \\ \hline & -A_{21} & & & I_{n_2}x_2-A_{22} \end{array}\right].$$

Our purpose is to calculate the factor standing at $x_1^{n_1}$, therefore, we may take no account of the second term of the above sum. We write the first term again as a sum of two determinants. Now n_1 such steps produce the following form of the determinant which contains the sought factor standing at $x_1^{n_1}$:

$$(22)\quad \det\left[\begin{array}{cccc|c} x_1 & 0 & \cdots & 0 & \\ 0 & x_1 & \cdots & 0 & -A_{12} \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & x_1 & \\ \hline & -A_{21}^{(n_1)}=0 & & & I_{n_2}x_2-A_{22} \end{array}\right]$$

$$= x_1^{n_1}\det(I_{n_2}x_2-A_{22}).$$

Comparing the factor standing at the element $x_1^{n_1}$ in (5) with the one in (22) we conclude that (20) holds. In the same way it can be seen that (19) holds too. This completes the proof.

From Theorem 1 and Lemma 1 it appears that the separable $2-D$ characteristic polynomial must have the form

$$f(x_1, x_2) = \det(I_{n_1}x_1-A_{11})\det(I_{n_2}x_2-A_{22})$$

and the polynomials (5) and (6) satisfy the equations

$$P_1(x_1) = \det(I_{n_1}x_1-A_{11}) \quad \text{and} \quad P_2(x_2) = \det(I_{n_2}x_2-A_{22}).$$

THEOREM 2. *The characteristic polynomial $f(x_1, x_2)$ given by* (2) *is separable if and only if*

$$(23)\qquad \det[I_{n_2}-(I_{n_2}x_2-A_{22})^{-1}A_{21}(I_{n_1}x_1-A_{11})^{-1}A_{12}] = 1.$$

Proof. From (2) it follows that

$$(24)\quad f(x_1, x_2) = \det\begin{bmatrix} I_{n_1}x_1-A_{11} & -A_{12} \\ -A_{21} & I_{n_2}x_2-A_{22} \end{bmatrix}$$

$$= \det(I_{n_1}x_1-A_{11})\det[I_{n_2}x_2-A_{22}-A_{21}(I_{n_1}x_1-A_{11})^{-1}A_{12}]$$

$$= \det(I_{n_1} x_1 - A_{11}) \det(I_{n_2} x_2 - A_{22})$$
$$\times \det[I_{n_2} - (I_{n_2} x_2 - A_{22})^{-1} A_{21} (I_{n_1} x_1 - A_{11})^{-1} A_{12}].$$

From Theorem 1, Lemmas 1 and 2, and equation (24) it follows that (23) is the necessary and sufficient condition for the separability of the $2-D$ characteristic polynomial (2). This completes the proof.

Remark 1. The expression

$$(I_{n_2} x_2 - A_{22})^{-1} A_{21} (I_{n_1} x_1 - A_{11})^{-1} A_{12} = 0$$

is a sufficient condition for the separability of the polynomial (2).

Remark 2. The necessary and sufficient condition (23) can be written in the following equivalent forms:

1. $\det[I_{n_2} - A_{21}(I_{n_1} x_1 - A_{11})^{-1} A_{12} (I_{n_2} x_2 - A_{22})^{-1}] = 1,$
2. $\det[I_{n_1} - (I_{n_1} x_1 - A_{11})^{-1} A_{12} (I_{n_2} x_2 - A_{22})^{-1} A_{21}] = 1,$
3. $\det[I_{n_1} - A_{12}(I_{n_2} x_2 - A_{22})^{-1} A_{21} (I_{n_1} x_1 - A_{11})^{-1}] = 1.$

4. Separability of the $n-D$ characteristic polynomial. The $n-D$ characteristic polynomial is defined as follows:

$$(25) \quad f(x_1, x_2, \ldots, x_n) = \det \begin{bmatrix} I_{n_1} x_1 - A_{11} & -A_{12} & \ldots & -A_{1n} \\ -A_{21} & I_{n_2} x_2 - A_{22} & \ldots & -A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -A_{n1} & -A_{n2} & \ldots & I_{n_n} x_n - A_{nn} \end{bmatrix}.$$

From (25) we have

$$(26) \qquad f(x_1, x_2, \ldots, x_n) = \sum_{i_1=0}^{n_1} \ldots \sum_{i_n=0}^{n_n} a_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n},$$

where $a_{n_1 \ldots n_n} = 1$.

In this case separability means that there exist polynomials $P_i(x_i)$ $(i = 1, 2, \ldots, n)$ such that

$$(27) \qquad f(x_1, x_2, \ldots, x_n) = P_1(x_1) \ldots P_n(x_n).$$

For the above polynomial it can be proved that if the characteristic polynomial (26) is separable then there exists one and only one set of normalized polynomials $P_i(x_i)$ $(i = 1, 2, \ldots, n)$ such that (27) holds (analogy of Lemma 1).

Presenting the polynomial (26) in the form (analogous to (15))

$$(28) \quad f(x_1, x_2, \ldots, x_n) = x_1^{n_1} \ldots x_{k-1}^{n_{k-1}} x_{k+1}^{n_{k+1}} \ldots x_n^{n_n} g_k(x_k) + u_k(x_1, \ldots, x_n)$$

we can prove (analogy of Theorem 1) that the polynomial factors $g_k(x_k)$ of a separable polynomial must fulfil the relations

$$(29) \qquad P_i(x_i) = g_i(x_i), \quad i = 1, \ldots, n.$$

Similarly as in Lemma 2, calculating the factors standing at $x_1^{n_1} \dots x_{k-1}^{n_{k-1}} x_{k+1}^{n_{k+1}} \dots x_n^{n_n}$ $(k = 1, 2, \dots, n)$ we receive the relations

$$P_i(x_i) = \det(I_{n_1} x_i - A_{ii}), \quad i = 1, 2, \dots, n. \tag{30}$$

Therefore, Lemmas 1 and 2 and Theorem 2 are true (after a suitable modification of notation) for the $n-D$ characteristic polynomial. So we can formulate the following theorem:

THEOREM 3. *The characteristic polynomial given by relation* (25) *is separable if and only if* $\det Q = 1$, *where*

$$Q = \begin{bmatrix} I_{n_1} & -A_{12}(I_{n_2}x_2 - A_{22})^{-1} & \dots & -A_{1n}(I_{n_n}x_n - A_{nn})^{-1} \\ -A_{21}(I_{n_1}x_1 - A_{11})^{-1} & I_{n_2} & \dots & -A_{2n}(I_{n_n}x_n - A_{nn})^{-1} \\ \dots & \dots & \dots & \dots \\ -A_{n1}(I_{n_1}x_1 - A_{11})^{-1} & -A_{12}(I_{n_2}x_2 - A_{22})^{-1} & \dots & I_{n_n} \end{bmatrix}.$$

Proof. From (25) we can write

$$f(x_1, \dots, x_n) = \det \begin{bmatrix} I_{n_1} & -A_{12}(I_{n_2}x_2 - A_{22})^{-1} & \dots & -A_{1n}(I_{n_n}x_n - A_{nn})^{-1} \\ -A_{21}(I_{n_1}x_1 - A_{11})^{-1} & I_{n_2} & \dots & -A_{2n}(I_{n_n}x_n - A_{nn})^{-1} \\ \dots & \dots & \dots & \dots \\ -A_{n1}(I_{n_1}x_1 - A_{11})^{-1} & -A_{12}(I_{n_2}x_2 - A_{22})^{-1} & \dots & I_{n_n} \end{bmatrix} \tag{31}$$

$$\times \det \begin{bmatrix} I_{n_1}x_1 - A_{11} & & \dots & 0 \\ 0 & I_{n_2}x_2 - A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_{n_n}x_n - A_{nn} \end{bmatrix}$$

$$= \det Q \det(I_{n_1}x_1 - A_{11}) \det(I_{n_2}x_2 - A_{22}) \dots \det(I_{n_n}x_n - A_{nn}).$$

From Lemmas 1 and 2 and Theorem 1 and relation (31) we easily obtain that $\det Q = 1$ is the necessary and sufficient condition for the separability of the characteristic polynomial (25). This completes the proof.

5. Algorithm. The presented necessary and sufficient condition for the separability of an $n-D$ characteristic polynomial $(n = 2, 3, \dots)$ requires the calculation of n inverse matrices. For a high value of n or for large dimensions of these matrices a lot of calculations are necessary.

The above considerations led us to formulate an algorithm which allows to check whether a given characteristic polynomial is separable. Simultaneously, during its realization we obtain forms of the polynomial factors $P_i(x_i)$.

Algorithm.

1. Determine the characteristic polynomial $f(x_1, \ldots, x_n)$ from relation (25).

2. Set the polynomials $g_i(x_i)$ $(i = 1, 2, \ldots, n)$ either by transforming the obtained polynomial to the form (28) or by using the relations (30) and (29).

3. Calculate the polynomial $F(x_1, x_2, \ldots, x_n) = g_1(x_1)g_2(x_2) \ldots g_n(x_n)$.

4. If $f(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$ then the characteristic polynomial $f(x_1, \ldots, x_n)$ is separable.

Example. Test the separability of the characteristic polynomial for A with

$$A_{11} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 6 & 6 \\ 0 & 7 & 7 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

Step 1: From (25) we obtain

$$f(x_1, x_2) = \det \begin{bmatrix} x_1-1 & -3 & -3 & -4 & -5 \\ -1 & x_1-1 & 0 & 0 & 0 \\ 1 & 0 & x_1-1 & 0 & 0 \\ 0 & -6 & -6 & x_2-1 & 2 \\ 0 & -7 & -7 & 2 & x_2-1 \end{bmatrix}$$

$$= x_1^3 x_2^2 - 2x_1^3 x_2 - 3x_1^2 x_2^2 - 3x_1^3 + 6x_1^2 x_2 + 3x_1 x_2^2$$
$$+ 9x_1^2 - 6x_1 x_2 - x_2^2 - 9x_1 + 2x_2 + 3.$$

Step 2:

$$g_1(x_1) = x_1^3 - 3x_1^2 + 3x_1 - 1, \quad g_2(x_2) = x_2^2 - 2x_2 - 3.$$

Step 3:

$$F(x_1, x_2) = g_1(x_1)g_2(x_2) = x_1^3 x_2^2 - 2x_1^3 x_2 - 3x_1^3 - 3x_1^2 x_2^2 + 6x_1^2 x_2$$
$$+ 9x_1^2 + 3x_1 x_2^2 - 6x_1 x_2 - 9x_1 - x_2^2 + 2x_2 + 3.$$

Step 4:

$$f(x_1, x_2) = F(x_1, x_2).$$

We conclude that the considered polynomial is a separable one.

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