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On some classes of holomorphic vector functions

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Abstract. In the article we consider some classes of analytic functions transforming the unit circle into a Banach space. The idea of construction of such classes arose from a deep analysis of works of Globevnik and Vidav [3] and of Suffridge [6].

Function spaces that are considered in this work are generalizations of Carathéodory's classes.

There are used, among others, the following denotations: $K(p, r)$ denotes the open ball with the centre p and radius r in a metric space, $K(r) = \{z \in C: |z| < r\}$, $\mathscr{P}$ denotes the class of Carathéodory functions and M_n the class of functions of the form

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t),$$

where μ is a non-decreasing simple function with the number of steps not greater than $n+1$ and $\mu(0) = 0$, $\mu(2\pi) = 1$. X^* denotes the conjugate space to a given linear topological space X.

1. Definition of the class $C(X, {}^*P, a_0)$ and examples. Let $\langle X, \|\ \|\rangle$ be complex Banach space. The set ${}^*P \subset X^*$ will be called *normable*, if there exist two positive numbers c, C such that the inequality

$$\sup\{|x^*(x)|;\ x^* \in {}^*P,\ \|x^*\| \leqslant C\} \geqslant c\|x\|$$

is true for every $x \in X$.

If a set ${}^*P$ is normable, then the functional $\|\ \|_1$ defined by the equality

$$\|x\|_1 = \sup\{|x^*(x)|:\ x^* \in {}^*P,\ \|x^*\| \leqslant C\} \tag{1}$$

is a norm equivalent to the norm $\|\ \|$ and

$$\sup\{|x^*(x):\ x^* \in {}^*P,\ \|x^*\|_1 \leqslant 1\} = \|x\|_1.$$

In the space X with the norm $\|\ \|_1$ the set ${}^*P$ is also normable, but with constants equal to 1.

In this work every considered Banach space with a given normable set will be treated as a space with the norm defined by (1).

DEFINITION 1. Let X be a complex Banach space, ${}^*P$ a normable set in it and a_0 a fixed point from X such that $\|a_0\| = 1$ and for every $x^* \in {}^*P$ $x^*(a) > 0$. We shall denote by $C(X, {}^*P, a_0)$ the set of all holomorphic functions f in $K(1)$ with values in the space X such that $f(0) = a_0$ and for every functional $x^* \in {}^*P$, $\operatorname{re} x^*(f(z)) > 0$ for $z \in K(1)$.

EXAMPLE 1. Let $X = C^n$, ${}^*P$ be the set of functionals of the form

$$x_i^*(x) = x_i \quad \text{for } x = (x_1, \ldots, x_n) \in C^n,$$

where $i = 1, 2, \ldots, n$ and $a_0 = (1, 1, \ldots, 1)$. This space $C(X, {}^*P, a_0)$ will be denoted by $C(1)$.

EXAMPLE 2. Let $X = m$ (the space of all bounded sequences with complex terms) and ${}^*P$ be the set of functionals such that

$$x_i^*(x) = x_i \quad \text{for } x = \{x_i\} \in m,$$

where $i = 1, 2, \ldots$ Let $a_0 = \{\xi_j\}$, $\xi_j = 1$ for $j = 1, 2, \ldots$ By $C(2)$ we shall denote the space $C(X, {}^*P, a_0)$ with the above defined X, ${}^*P$ and a_0.

EXAMPLE 3. Let $X = c_0$ (the set of all complex sequences converging to 0) and ${}^*P$ be the set of functionals of the form

$$x_i^*(x) = x_i \quad \text{for } x = \{x_i\} \in c_0,$$

$i = 1, 2, \ldots$, and $a_0 = \{1/i\}$. We shall denote this space by $C(3)$.

EXAMPLE 4. Let $X = c$ (the space of all converging complex sequences) and ${}^*P$ be the set of functionals of the form

$$x_i^*(x) = x_i \quad \text{for } x = \{x_i\} \in c,$$

$i = 1, 2, \ldots$ Let $a_0 = \{\xi_j\}$, where $\xi_j = 1$ for $j = 1, 2, \ldots$

We shall denote this space by $C(4)$.

EXAMPLE 5. Let $X = C([a, b])$ (the space of all continuous complex functions defined on $[a, b]$), and let ${}^*P$ be the set $\{x_t^*\colon t \in [a, b]\}$ of functionals defined by

$$x_t^*(x) = x(t) \quad \text{for } x \in C([a, b]).$$

Let a_0 be the constant function equal to 1. We shall denote this space by $C(5)$.

2. Basic properties of the class $C(X, {}^*P, a_0)$.

THEOREM 1. *If $f \in C(X, {}^*P, a_0)$ and $z \in K(1)$, then*

$$f(z) \in K\left(\frac{1+r^2}{1-r^2} a_0, \frac{2r}{1-r^2}\right), \quad \textit{where } r = |z|.$$

Proof. For every functional $x^* \in {}^*P$ fulfilling the condition $\|x^*\| \leqslant 1$ and every $f \in C(X, {}^*P, a_0)$ the function $x^*(f)/x^*(a_0)$ belongs to $\mathscr{P}$. Hence follows the inequality

$$\left|\frac{x^*(f(z))}{x^*(a_0)} - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2}, \quad \text{where } r = |z|.$$

Next, we have

$$\left| x^*\left(f(z)-a_0\frac{1+r^2}{1-r^2}\right)\right| \leqslant \frac{2r}{1-r^2}.$$

Using the fact that $*P$ is a normable set with constants equal to 1 we infer that

$$\left\| f(z)-a_0\frac{1+r^2}{1-r^2}\right\| \leqslant \frac{2r}{1-r^2}, \quad \text{where } r = |z|.$$

THEOREM 2. *If* $f \in C(X, *P, a_0)$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *for* $z \in K(1)$, *then* $\|a_n\| \leqslant 2$ *for* $n = 1, 2, \ldots$

Proof. For every functional $x^* \in *P$ fulfilling the inequality $\|x^*\| \leqslant 1$ we can write

$$x^*(f(z)) = \sum_{n=0}^{\infty} x^*(a_n) z^n \quad \text{and} \quad \frac{x^*(f)}{x^*(a_0)} \in \mathscr{P}.$$

Hence

$$\left|\frac{x^*(a_n)}{x^*(a_0)}\right| \leqslant 2 \quad \text{and} \quad |x^*(a_n)| \leqslant 2 \quad \text{for } n = 1, 2, \ldots$$

Thus $\|a_n\| \leqslant 2$ for $n = 1, 2, \ldots$, because of $a_n = \sup\{|x^*(a_n)|: x^* \in *P, \|x^*\| \leqslant 1\}$.

THEOREM 3. *The set* $C(X, *P, a_0)$ *is convex and closed in the space of continuous functions of* $K(1)$ *into a Banach space* X *with the topology of almost uniform convergence.*

Proof. We omit the proof of convexity because of its simplicity.

Let $\{f_n\}$ be an almost uniformly convergent sequence of functions from $C(X, *P, a_0)$. According to Theorem 1, this sequence is almost uniformly bounded in $K(1)$. The limit f of $\{f_n\}$ is a holomorphic function in $K(1)$ in view of the Vitali theorem for holomorphic vector functions (see [1], p. 250).

Let x^* be an arbitrary functional from $*P$. The sequence $\{f_n\}$ is almost uniformly convergent, and so is the sequence $\{x^*(f_n)/x^*(a_0)\}$. The terms of this sequence are Carathéodory functions; thus, according to the compactness of the class of Carathéodory functions, the limit $x^*(f)/x^*(a_0)$ is also a Carathéodory function. Hence

$$\mathrm{re}\left(\frac{x^*(f(z))}{x^*(a_0)}\right) > 0 \quad \text{for } z \in K(1).$$

Thus $\mathrm{re}\big(x^*(f(z))\big) > 0$ for $z \in K(1)$ and $x^* \in *P$ because of $x^*(a_0) > 0$.

It is obvious that $f(0) = a_0$ and, according to Definition 1, $f \in C(X, *P, a_0)$. This shows that the set $C(X, *P, a_0)$ is closed.

Basing on theorems included in the article [4] one can prove

THEOREM 4. *Let $\lambda_0, \lambda_1, \ldots, \lambda_n$ be arbitrary complex numbers different from 0. Conclude the functional $G(f) = \|\sum_{i=0}^{n} \lambda_i f^{(i)}(0)\|$ defined on $C(X, {}^*P, a_0)$. Then there exists a function $f_0 \in C(X, {}^*P, a_0)$ of the form $f_0 = a_0 \varphi^*$, where $\varphi^* \in M_n$, and such that*

$$G(f_0) = \sup\{G(f): f \in C(X, {}^*P, a_0)\}.$$

3. Characterization of extremal points in $C(X, {}^*P, a_0)$. At the beginning, let us introduce the following notation:

$E(A)$ will denote the set of all extremal points in a convex set $A \subset C(X, {}^*P, a_0)$, and $\mathscr{E}$ the set of extremal points in $\mathscr{P}$.

THEOREM 5. *Let f be from a convex set $A \subset C(X, {}^*P, a_0)$. If $x^*(f)/x^*(a_0) \in \mathscr{E}$ for every functional $x^* \in {}^*P$, then $f \in E(A)$.*

Proof. Suppose that $f \notin E(A)$. There exist different functions $f_1, f_2 \in A$ such that $f = \lambda f_1 + (1-\lambda) f_2$ for some $\lambda \in (0, 1)$.

Since for some $z_0 \in K(1)$ we have $f_1(z_0) \neq f_2(z_0)$, then there exists $x_0^* \in {}^*P$ such that

$$x_0^*(f_1(z_0)) \neq x_0^*(f_2(z_0)),$$

and hence $x_0^*(f_1) \neq x_0^*(f_2)$. This relation and the equality

$$\frac{x_0^*(f)}{x_0^*(a_0)} = \lambda \frac{x_0^*(f_1)}{x_0^*(a_0)} + (1-\lambda) \frac{x_0^*(f_2)}{x_0^*(a_0)}$$

imply that $x_0^*(f)/x_0^*(a_0)$ is not an extremal point in $\mathscr{P}$.

The contradiction ends the proof.

DEFINITION 2. We shall say that a convex set $A \subset C(X, {}^*P, a_0)$ *has the property* $\mathscr{W}$ if for each function $f \in A$ and functional $x^* \in {}^*P$ the equality

$$\frac{x^*(f)}{x^*(a_0)} = \lambda\varphi_1 + (1-\lambda)\varphi_2 \quad \text{for } \varphi_1, \varphi_2 \in \mathscr{P} \text{ and } 0 < \lambda < 1$$

implies that there exist two functions f_1, f_2 in A such that

$$f = \lambda f_1 + (1-\lambda) f_2$$

and

$$\frac{x^*(f_i)}{x^*(a_0)} = \varphi_i \quad \text{for } i = 1, 2.$$

THEOREM 6. *If f belongs to a convex set $A \subset C(X, {}^*P, a_0)$ having the property $\mathscr{W}$, then $f \in E(A)$ if and only if*

$$\frac{x^*(f)}{x^*(a_0)} \in \mathscr{E} \quad \textit{for every functional } x^* \in {}^*P.$$

Proof. The sufficiency follows immediately from Theorem 5. The necessity can be proved in a similar way to that used in the proof of Theorem 5.

One can easily prove the following theorem.

THEOREM 7. *The spaces* $C(1)$, $C(2)$, $C(3)$, $C(4)$ *have the property* $\mathscr{W}$.

DEFINITION 3. We shall say that a mapping f from $K(1)$ into c *belongs to* $A_{C(4)}$ iff it has the form

$$f = f(z) = \left\{ \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu_n(\theta) \right\},$$

where $\{\mu_n\}$ is a convergent sequence of non-decreasing functions on the interval $[0, 2\pi]$ which are left continuous and fulfil the conditions $\mu_n(0) = 0$, $\mu_n(2\pi) = 1$ for $n \in N$.

Remark. The set $A_{C(4)}$ is a convex subset of $C(4)$.

This is a consequence of Helly's second theorem. Moreover, one can prove that $A_{C(4)}$ has the property $\mathscr{W}$.

DEFINITION 4. We shall say that a map f from $K(1)$ into $C([a, b])$ *belongs to* $A_{C(5)}$ iff it can be represented in the form

$$f = f(z)(t) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta, t),$$

where $\mu = \mu(\theta, t)$ is a function defined on $[0, 2\pi] \times [a, b]$ fulfilling the following conditions:

for every fixed $t \in [a, b]$ μ is non-decreasing, left continuous and $\mu(\theta, t) = 0$, $\mu(2\pi, t) = 1$;

for every fixed $\theta \in [0, 2\pi]$, μ is continuous in $[a, b]$.

Remark. The set $A_{C(5)}$ is a proper convex subset of $C(5)$.

This fact follows immediately from Helly's second theorem.

THEOREM 8. *The set* $A_{C(5)}$ *has the property* $\mathscr{W}$.

Before we prove this theorem we shall give three lemmas.

LEMMA 1. *Let* $\theta_1, \theta_2 \in [0, 2\pi]$, $\theta_1 < \theta_2$ *and* $\theta_3 \in (\theta_1, \theta_2)$. *If there are given continuous functions defined on* $[a, b]$:

$$f(\theta_1, t),\ f(\theta_2, t),\ f(\theta_3, t),\ f_1(\theta_1, t),\ f_2(\theta_1, t),\ f_1(\theta_2, t),\ f_2(\theta_2, t)$$

fulfilling the following conditions:

$$f(\theta_i, t) = \tfrac{1}{2}\big(f_1(\theta_i, t) + f_2(\theta_i, t)\big), \quad f_1(\theta_i, t) \leqslant f(\theta_i, t) \leqslant f_2(\theta_i, t),$$

$$0 \leqslant f(\theta_1, t) \leqslant f(\theta_3, t) \leqslant f(\theta_2, t) \leqslant 1, \quad 0 \leqslant f_i(\theta_1, t) \leqslant f_i(\theta_2, t) \leqslant 1$$

for $i = 1, 2$, *then there exist functions* $f_1(\theta_3, t)$, $f_2(\theta_3, t)$ *such that*

$$f(\theta_3, t) = \tfrac{1}{2}\big(f_1(\theta_3, t) + f_2(\theta_3, t)\big), \quad f_i(\theta_1, t) \leqslant f_i(\theta_3, t) \leqslant f_i(\theta_2, t),$$

$$f_1(\theta_3, t) \leqslant f(\theta_3, t) \leqslant f_2(\theta_3, t)$$

for all $t \in [a, b]$ *and* $i = 1, 2$.

Proof. Let us denote

$$A_1 = \{t;\ f_2(\theta_1, t) \geqslant f(\theta_3, t) \wedge f_1(\theta_2, t) \geqslant f(\theta_3, t) \wedge f_2(\theta_1, t) \geqslant f_1(\theta_2, t)\},$$
$$A_2 = \{t;\ f_1(\theta_2, t) \geqslant f(\theta_3, t) \wedge (f_2(\theta_1, t) \leqslant f(\theta_3, t) \vee f_2(\theta_1, t) \leqslant f_1(\theta_2, t))\},$$
$$A_3 = \{t;\ f_2(\theta_1, t) \leqslant f(\theta_3, t) \wedge f_1(\theta_2, t) \leqslant f(\theta_3, t) \wedge f_2(\theta_2, t) \leqslant f_1(\theta_2, t)\},$$
$$A_4 = \{t;\ f_2(\theta_1, t) \leqslant f(\theta_3, t) \wedge f_1(\theta_2, t) \leqslant f(\theta_3, t) \wedge f_1(\theta_2, t) \leqslant f_2(\theta_1, t)\},$$
$$A_5 = \{t;\ f_1(\theta_2, t) \leqslant f(\theta_3, t) \wedge f_2(\theta_1, t) \geqslant f(\theta_3, t)\}.$$

It is easy to show that $\bigcup_{i=1}^{5} A_i = [a, b]$. Let $f_2(\theta_3, t)$ be defined in the following way:

$$f_2(\theta_3, t) \stackrel{\mathrm{df}}{=} \begin{cases} f_2(\theta_1, t) & \text{for } t \in A_1, \\ f_1(\theta_2, t) & \text{for } t \in A_2, \\ 2f(\theta_3, t) - f_1(\theta_2, t) & \text{for } t \in A_3, \\ 2f(\theta_3, t) - f_2(\theta_1, t) & \text{for } t \in A_4, \\ f_2(\theta_1, t) & \text{for } t \in A_5, \end{cases}$$

and $f_1(\theta_3, t) \stackrel{\mathrm{df}}{=} 2f(\theta_3, t) - f_2(\theta_3, t)$.

We show by an elementary calculation that the functions $f_1(\theta_3, t)$, $f_2(\theta_3, t)$ defined above fulfil all required conditions.

LEMMA 2. *Let f_1, f_2 be real functions defined in some neighbourhood $U(t_0, \delta)$ of $t_0 \in \boldsymbol{R}$. If f_1, f_2 are lower semi-continuous at t_0 and if the function*

$$f(t) = \tfrac{1}{2}(f_1(t) + f_2(t)) \quad \textit{for } t \in U(t_0, \delta)$$

is continuous at t_0, then the functions f_1 and f_2 are also continuous at t_0.

LEMMA 3. *Let $\mu(\theta, t)$ be a real function defined on the set $[0, 2\pi] \times [a, b]$, fulfilling the following conditions:*

for every $t \in [a, b]$ the function $\mu(\theta, t)$ is non-decreasing, left continuous and $\mu(0, t) = 0$, $\mu(2\pi, t) = 1$;

for every $\theta \in (0, 2\pi)$ the function $\mu(\theta, t)$ is continuous in $[a, b]$.

Moreover, suppose that for some $t_0 \in [a, b]$ there are two different left continuous functions $\tilde{\mu}_1(\theta, t_0)$, $\tilde{\mu}_2(\theta, t_0)$ defined on $[0, 2\pi]$ such that $\tilde{\mu}_i(0, t_0) = 0$, $\tilde{\mu}_i(2\pi, t_0) = 1$ for $i = 1, 2$ and

$$\mu(\theta, t_0) = \tfrac{1}{2}(\tilde{\mu}_1(\theta, t_0) + \tilde{\mu}_2(\theta, t_0)) \quad \textit{for } \theta \in [0, 2\pi].$$

Under these assumptions there are two different functions $\mu_1(\theta, t)$, $\mu_2(\theta, t)$ defined on $[0, 2\pi] \times [a, b]$ fulfilling the two conditions corresponding to those imposed on the function $\mu(\theta, t)$ and

$$\mu(\theta, t) = \tfrac{1}{2}(\mu_1(\theta, t) + \mu_2(\theta, t)) \quad \textit{for } (\theta, t) \in [0, 2\pi] \times [a, b].$$

Proof. Since $\tilde{\mu}_1 \neq \tilde{\mu}_2$, there exists a point $\theta_0 \in [0, 2\pi]$ such that

$0 < \mu(\theta_0, t_0) < 1$. The function μ is continuous with respect to t, and so there exists $\delta > 0$ such that

$$0 < \mu(\theta_0, t) < 1 \quad \text{for } t \in (t_0 - \delta, \ t_0 + \delta).$$

Thus there exists a positive number ε such that

$$\varepsilon < \mu(\theta_0, t) < 1 - \varepsilon \quad \text{for } t \in (t_0 - \delta, \ t_0 + \delta).$$

Let $\mu_2(\theta_0, t)$ be a continuous function defined in $[a, b]$ and fulfilling the conditions:

$$\mu_2(\theta_0, t) = \mu(\theta_0, t) \quad \text{for } t \in [a, t-\delta] \cup [t_0 + \delta, b],$$
$$0 < \mu_2(\theta_0, t) - \mu(\theta_0, t) < \varepsilon \quad \text{for } t \in (t_0 - \delta, t_0 + \delta),$$

and let

$$\mu_1(\theta_0, t) = 2\mu(\theta_0, t) - \mu_2(\theta_0, t) \quad \text{for } t \in [a, b].$$

One can easily prove that these functions fulfil the following conditions:

$$\begin{gathered} 0 \leqslant \mu_1(\theta_0, t) \leqslant \mu(\theta_0, t) \leqslant \mu_2(\theta_0, t) \leqslant 1 \quad \text{for } t \in [a, b], \\ \mu_1(\theta_0, t_0) < \mu(\theta_0, t_0) < \mu_2(\theta_0, t_0), \quad \mu(\theta_0, t) = \tfrac{1}{2}(\mu_1(\theta_0, t) + \mu_2(\theta_0, t)). \end{gathered} \tag{2}$$

Using Lemma 1 we shall construct the inquired functions $\mu_i(\theta, t)$ $(i = 1, 2)$ defined on the set $Q \times [a, b]$, where Q consists of θ_0 and all rational numbers from the interval $[0, 2\pi]$. Let $0, 2\pi, \theta_0, q_1, q^2, \ldots$ be the sequence of all different elements of Q. Let

$$\mu_1(0, t) = \mu_2(0, t) = \mu(0, t) = 0, \quad \mu_1(2\pi, t) = \mu_2(2\pi, t) = \mu(2\pi, t) = 1$$

for $t \in [a, b]$.

We first define the functions μ_1, μ_2 at the points (q_1, t), $t \in [a, b]$.

Assume that $q_1 \in (0, \theta_0)$. Then, taking in Lemma 1, $\theta_1 = 0$, $\theta_2 = \theta_0$, $\theta_3 = q_1$, $f = \mu$, $f_1 = \mu_1$, $f_2 = \mu_2$, we see that all assumptions of this lemma are fulfilled. Now we define

$$\mu_1(q_1, t) = f_1(q_1, t), \quad \mu_2(q_1, t) = f_2(q_1, t) \quad \text{for } t \in (a, b),$$

where $f_1(q_1, t)$, $f_2(q_1, t)$ are functions from the assertion of Lemma 1.

Next assume that we have defined the functions μ_1 and μ_2 on the set $\{0, 2\pi, \theta_0, q_1, \ldots, q_n\} \times [a, b]$. Write: $q_{-2} = 0$, $q_{-1} = 2\pi$, $q_0 = \theta_0$. Then denote

$$\theta_1 = \max\{q_i\colon q_i - q_{n+1} < 0, \ i = -2, -1, 0, 1, \ldots, n\},$$
$$\theta_2 = \min\{q_i\colon q_i - q_{n+1} > 0, \ i = -2, -1, 0, 1, \ldots, n\}.$$

Let us apply Lemma 1 to the points θ_1, θ_2, q_{n+1}, taking $f = \mu$, $f_1 = \mu_2$, $f_2 = \mu_2$. Assume

$$\mu_1(q_{n+1}, t) = f_1(q_{n+1}, t), \quad \mu_2(q_{n+1}, t) = f_2(q_{n+1}, t)$$

for $t \in [a, b]$.

By induction we have defined the functions μ_1, μ_2 on the set $Q\times[a, b]$. The functions just constructed are non-decreasing and left continuous on Q for every fixed $t\in[a, b]$ and they are continuous on $[a, b]$ for every fixed $\theta\in Q$.

Now extend the functions μ_1, μ_2 onto $[0, 2\pi]\times[a, b]$ in the following way: let

$$\mu_i(x, t) \stackrel{\text{df}}{=} \lim_{\substack{q\nearrow x\\ q\in Q}} \mu_i(q, t), \qquad i = 1, 2,$$

for an arbitrary $x\in[0, 2\pi]$. We obtain functions $\mu_i(\theta, t)$ defined on $[0, 2\pi]\times[a, b]$ with the following properties:

(a) $\mu_i(\theta, t)$ $(i = 1, 2)$ is left continuous and non-decreasing function on $[0, 2\pi]$ for every $t\in[a, b]$, and

$$\mu_i(0, t) = 0, \qquad \mu_i(2\pi, t) = 1;$$

(b) $\mu_i(\theta, t)$ $(i = 1, 2)$ is a continuous function on $[a, b]$ for every $\theta\in[0, 2\pi]$;

(c) $\mu(\theta, t) = \frac{1}{2}\big(\mu_1(\theta, t)+\mu_2(\theta, t)\big)$ for $(\theta, t)\in[0, 2\pi]\times[a, b]$.

Properties (a), (c) follow immediately from the construction of those functions. We need only show (b).

Let θ be an arbitrary number from $[0, 2\pi]$ and let $\{q_n\}$ be a non-decreasing sequence of numbers from Q which is convergent to θ. Then

$$\mu_i(\theta, t) = \lim_{n\to\infty} \mu_i(q_n, t) \qquad \text{for } t\in[a, b].$$

The sequences $\{\mu_i(q_n, t)\}$ $(i = 1, 2)$ are non-decreasing sequences of continuous (with respect to t) functions. In view of Theorem 9 from [5] (p. 388) the limit functions are lower semicontinuous. From property (c) and Lemma 2 it follows that functions μ_1, μ_2 are continuous with respect to t. Since $\mu_1(\theta_0, t_0) \neq \mu_2(\theta_0, t)$, then $\mu_1 \neq \mu_2$.

Hence the functions μ_1, μ_2 defined above have the asserted properties.

Proof of Theorem 8. Assume that f is extremal in $A_{C(5)}$ and that for some $t_0\in[a, b]$ the function $f(z)(t_0)$ is not extremal in $\mathscr{P}$. Then, according to the definition of $A_{C(5)}$,

$$f(z)(t) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta, t)$$

and

$$\mu(\theta, t_0) = \lambda\mu_1(\theta, t_0)+(1-\lambda)\,\mu_2(\theta, t_0) \tag{3}$$

for some $\lambda \in (0, 1)$ and $\theta \in [0, 2\pi]$, where μ is defined on $[0, 2\pi] \times [a, b]$ and fulfils the following conditions:

$\mu(\theta, t)$ is non-decreasing, left continuous in $[0, 2\pi]$ and $\mu(0, t) = 0$, $\mu(2\pi, t) = 1$ for each fixed $t \in [a, b]$;

$\mu(\theta, t)$ is continuous in $[a, b]$ for every fixed $\theta \in [0, 2\pi]$, and $\mu_1(\theta, t_0)$, $\mu_2(\theta, t_0)$ are left continuous in $[0, 2\pi]$ and

$\mu_i(0, t_0) = 0$, $\mu_i(2\pi, t_0) = 1$ for $i = 1, 2$.

Modifying the functions $\mu_1(\theta, t_0)$, $\mu_2(\theta, t_0)$ we can obtain functions $\tilde{\mu}_1(\theta, t_0)$, $\tilde{\mu}_2(\theta, t_0)$ such that equality (3) holds for them with $\lambda = \frac{1}{2}$.

Hence the assumptions of Lemma 3 are fulfilled; therefore there exist functions $\nu_1(\theta, t)$, $\nu_2(\theta, t)$ defined on $[0, 2\pi] \times [a, b]$ and fulfilling the assertion of the lemma.

The functions

$$f_1 = f_1(z)(t) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu_1(\theta, t)$$

and

$$f_2 = f_2(z)(t) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu_2(\theta, t)$$

belong to $A_{C(5)}$ and the equality $f = \frac{1}{2}(f_1 + f_2)$ holds, but $f_1 \neq f_2$. Thus the function f is not extremal in $A_{C(5)}$, which contradicts the assumption.

Remark. In this part of the work we have proved that, the spaces $C(1)$, $C(2)$, $C(3)$, $C(4)$, $A_{C(4)}$, $A_{C(5)}$ having the property $\mathscr{W}$, the extremal points in them are completely characterized by Theorem 6.

4. On a certain subclass of $C(X, {}^*P, a_0)$.

DEFINITION 5. Let W_{r_0} be a fixed convex compact subset of X. By $C(W_{r_0}, X, {}^*P, a_0)$ we shall denote the set of those mappings $f \in C(X, {}^*P, a_0)$ for which $f(\overline{K(r_0)}) \subset W_{r_0}$, where r_0 is a fixed number from $(0, 1)$.

We shall consider the space $C(W_{r_0}, X, {}^*P, a_0)$ as a topological space with the topology of almost uniform convergence.

THEOREM 9. *The space $C(W_{r_0}, X, {}^*P, a_0)$ is a convex and compact set in the space of continuous mappings of $K(1)$ into X with the topology of almost uniform convergence.*

Proof. The convexity of $C(W_{r_0}, X, {}^*P, a_0)$ follows from the convexity of the sets W_{r_0} and $C(X, {}^*P, a_0)$.

Let $\{f_n\}$ be a sequence of elements from $C(W_{r_0}, X, {}^*P, a_0)$ almost uniform convergent to f. In virtue of the closedness of $C(X, {}^*P, a_0)$ in the space of continuous mappings of $K(1)$ into the Banach space X, f belongs to

$C(X, {}^*P, a_0)$. Let z be an arbitrary point in $K(r_0)$. Since $f(z) = \lim_{n\to\infty} f_n(z)$ and W_{r_0} is a closed set, then $f(z)\in W_{r_0}$, and hence $f\in C(W_{r_0}, X, {}^*P, a_0)$. This proves that the set $C(W_{r_0}, X, {}^*P, a_0)$ is closed.

It is yet necessary to prove that this set is compact. Assume then the following notation:

$$C_{|\overline{K(r_0)}} = \{f(\overline{K(r_0)}): f\in C(W_{r_0}, X, {}^*P, a_0)\}.$$

It follows by Theorem 9.13.1 from [2] that the set $C_{|\overline{K(r_0)}}$ is a family of equicontinuous functions. In view of the Ascoli theorem (see for example [2], p. 164) the set $C_{|\overline{K(r_0)}}$ is conditionally compact in the space of continuous mappings of $K(r_0)$ into the space X. The set $C(W_{r_0}, X, {}^*P, a_0)$ being closed, so is the set $C_{|\overline{K(r_0)}}$ is, and hence it is compact.

In virtue of the principle of analytic extensions (see [2], p. 236) it follows that the transformation

$$C(W_{r_0}, X, {}^*P, a_0)\ni f\mapsto f|\overline{K(r_0)}\in C_{|\overline{K(r_0)}}$$

is a bijection from $C(W_{r_0}, X, {}^*P, a_0)$ onto $C_{|\overline{K(r_0)}}$. According to the Vitali theorem (see [1], p. 250), uniform convergence of elements of $C(W_{r_0}, X, {}^*P, a_0)$ in $\overline{K(r_0)}$ is equivalent to almost uniform convergence in $K(1)$.

This proves that $C(W_{r_0}, X, {}^*P, a_0)$ is compact.

EXAMPLE 6. Consider the space $C(4)$ and the set

$$W_{r_0} = \left\{x = \{\varphi_n\}\in c\colon \|x\| \leqslant \frac{1+r_0}{1-r_0} \text{ and } \bigwedge_{\varepsilon>0}\ \sup_{n,m>n_0(\varepsilon)} |\varphi_n-\varphi_m| < \varepsilon\right\},$$

where $n_0(\varepsilon)$ is some fixed non-increasing function defined on R^+ and such that $n_0(\varepsilon)\to\infty$ when $\varepsilon\to 0^+$ and $0<r_0<1$.

The set W_{r_0} is a convex and compact in the space of convergent complex sequences. We shall denote this space $C(W_{r_0}, X, {}^*P, a_0)$ by $\tilde{C}(4)$.

THEOREM 10. *Let*

$$V_{r_0} = \{x = \{\varphi_n\}\in W_{r_0}\colon \bigwedge_{\varepsilon>0}\ \sup_{n,m>n_0(\varepsilon)} |\varphi_n-\varphi_m| < \varepsilon\}.$$

If $f(\overline{K(r_0)})\subset V_{r_0}$ *and* $f = \{\psi_k\}$ *is an extremal point in* $\tilde{C}(4)$, *then* $\psi_k\in\mathscr{E}$ *for every positive integer* k.

Proof. Let $f = \{\psi_k\}$ be an extremal point in $\tilde{C}(4)$ and let $f(\overline{K(r_0)})\subset V_{r_0}$; moreover, suppose that there exists a positive integer i_0 such that ψ_{i_0} is not an extremal point in $\mathscr{P}$. Then there exist $\lambda\in(0, 1)$ and two different

Carathéodory functions $\psi_{i_0}^1$, $\psi_{i_0}^2$ such that

$$\psi_{i_0} = \lambda\psi_{i_0}^1 + (1-\lambda)\psi_{i_0}^2.$$

It is not difficult to prove that the function

$$h_\varepsilon(z) = \sup_{n,m > n_0(\varepsilon)} |\psi_n(z) - \psi_m(z)|$$

is continuous in $K(1)$. Hence

$$\sup_{z\in K(r_0)} [\sup_{n,m>n_0(\varepsilon)} |\psi_n(z) - \psi_m(z)|] < \varepsilon \quad \text{for every } \varepsilon > 0.$$

Let us introduce an auxiliary function

$$s(\varepsilon) = \varepsilon - \sup_{z\in K(r_0)} h_\varepsilon(z) \quad \text{for } \varepsilon > 0.$$

It is easy to see that $s(\varepsilon) > 0$.

Let $\varepsilon_0 = \inf\{\varepsilon\colon n_0(\varepsilon) \leqslant i_0\}$. Then $s(\varepsilon)$ assumes a finite numbers of different values for $\varepsilon \geqslant \varepsilon_0$, and this implies that

$$\delta_0 = \inf_{\varepsilon \geqslant \varepsilon_0} s(\varepsilon) > 0.$$

Let $\lambda_1, \lambda_2 \in [0, 1)$ be numbers such that

$$\sup_{z\in K(r_0)} |\psi_{i_0}(z) - \tilde{\psi}_{i_0}^j(z)| < \delta_0/2$$

for the functions $\tilde{\psi}_{i_0}^j = \lambda_j\psi_{i_0} + (1-\lambda_j)\psi_{i_0}^j$, $j = 1, 2$.

Now define $f_1 = \{f_n^1\}$, $f_2 = \{f_n^2\}$ in the following way:

$$f_n^1(z) = \begin{cases} \psi_n(z) & \text{for } n \neq i_0, \\ \tilde{\psi}_{i_0}^1(z) & \text{for } n = i_0, \end{cases}$$

$$f_n^2(z) = \begin{cases} \psi_n(z) & \text{for } n \neq i_0, \\ \tilde{\psi}_{i_0}^2(z) & \text{for } n = i_0. \end{cases}$$

One can easily verify that $f_1, f_2 \in \tilde{C}(4)$ and $f = \lambda' f_1 + (1-\lambda')f_2$ for some $\lambda' \in (0, 1)$. This proves that f is not an extremal point in $\tilde{C}(4)$, which contradicts our assumptions.

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