ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

On a new subclass of the class S

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Abstract. Let $S_{(\alpha,\beta)}$, $\alpha\epsilon(0, 2)$, $\beta\epsilon(-2, 0)$, $\alpha-\beta\leqslant 2$ denote the class of functions of the form $f(z) = z+a_2z^2+a_3z^3+\ldots$ regular and univalent in the unit disc K_1, where $K_r = \{z\colon |z|<r\}$, and satisfying the condition

$$\beta\frac{\pi}{2}<\arg\frac{zf'(z)}{f(z)}<\alpha\frac{\pi}{2}.$$

If $\alpha=1$ and $\beta=-1$, then $S_{(1,-1)}=S^*$.
If $\beta=-\alpha$, then $S_{(\alpha,-\alpha)}=S_\alpha$, $\alpha\epsilon(0, 1\rangle$.
If $\beta=\alpha-2$, then $S_{(\alpha,\alpha-2)}=\check{S}_\delta$,
S^*, S_α, $\check{S}_\delta$ being defined on pages 153, 154.

This paper gives the structural formula in the above-mentioned class, the set of variability of the functional $zf'(z)/f(z)$ and exact upper and lower estimation of the modulus of the expression $zf'(z)/f(z)$ for stated $z\epsilon K_1$.

Moreover, exact estimations of $|a_2|$ and $|a_3|$ in this class are given, and the extremal function is also found.

1. Introduction. Let S denote the class of functions $f(z)$ of the form

$$f(z) = z+a_2z^2+a_3z^3+\ldots \tag{1.1}$$

regular and univalent in the unit disc K_1, where $K_r=\{z\colon |z|<r\}$.

Let $S^*\subset S$ be the class of functions star-like w.r.t. the origin, that is satisfying the condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0 \quad \text{for} \quad z\epsilon K_1. \tag{1.2}$$

Let $S\alpha\subset S$ represent the class of functions $f(z)$ of the form (1.1), satisfying the condition

$$\left|\arg\frac{zf'(z)}{f(z)}\right|\leqslant\alpha\frac{\pi}{2} \quad \text{for} \quad z\epsilon K_1,\ 0<\alpha\leqslant 1. \tag{1.3}$$

This class has been investigated by several authors, including J. Stankiewicz [3].

Let

$$\check{S}\delta, \quad \delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

denote the class of functions $f(z)$ of the form (1.1) satisfying the condition

$$\operatorname{Re} e^{-i\delta} \frac{zf'(z)}{f(z)} > 0 \quad \text{for} \quad z \in K_1. \tag{1.4}$$

These functions are called δ-spiral functions.

Next, let P be the class of functions $p(z) = 1+p_1z+p_2z^2+\ldots$ regular in the unit disc K_1 and satisfying the condition

$$\operatorname{Re} p(z) > 0 \quad \text{for} \quad z \in K_1. \tag{1.5}$$

2. Class $S(\alpha, \beta)$. Consider the class $S_{(\alpha,\beta)}$, $\alpha \in (0, 2)$, $\beta \in (-2, 0)$, of functions $f(z)$ of the form (1.1) regular and univalent in the unit disc K_1 and satisfying the condition

$$\beta \frac{\pi}{2} < \arg \frac{zf'(z)}{f(z)} < \alpha \frac{\pi}{2}. \tag{2.1}$$

Furthermore, it is assumed that in the sequel $\alpha - \beta \leqslant 2$. In some cases this class coincides with the well-known ones, namely

$$S_{(1,-1)} = S^*,$$
$$S_{(\alpha,-\alpha)} = S_\alpha,$$
$$S_{(\alpha,\alpha-2)} = \check{S}_\delta.$$

Note that in the definition of the class $S_{(\alpha,\beta)}$ an idea of subordination occurs, because the expression $\dfrac{zf'(z)}{f(z)}$ is required to be in the univalent angle including the point $w = 1$.

Let $F(z)$ be the univalent function of the class P in K_1 mapping the unit disc K_1 onto the angle

$$\beta \frac{\pi}{2} < \arg F(z) < \alpha \frac{\pi}{2}, \quad \alpha \in (0, 2), \beta \in (-2, 0), F(0) = 1,$$

where $\arg F(z)$ is that unique branch which equals 0 in zero. If by $\varphi \dashv_1 \Phi$ is meant the subordination of the function φ to the majorant Φ in the unit disc K_1, then the class $S_{(\alpha,\beta)} = S(F)$ may be defined in the following way:

$$f(z) \in S(F) \Leftrightarrow \frac{zf'(z)}{f(z)} \dashv_1 F(z), \quad z \in K_1, \tag{2.2}$$

where

$$F(z) = \left[\frac{1+z}{1-z}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right) - i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}} \cdot e^{i\frac{\pi}{4}(\alpha+\beta)}.$$

The form of $F(z)$ easily follows from the normalization of the composition mappings

$$w = \omega^{\frac{\alpha-\beta}{2}} e^{i\frac{\pi}{4}(\alpha+\beta)} \quad \text{and} \quad \omega = \frac{1+z}{1-z}$$

(see [3]). And since each function $p(z) \in P$ satisfies the condition

$$p(z) \prec_1 \frac{1+z}{1-z},$$

then for every function $f \in S(F)$ there exists a $p(z) \in P$ such that

$$\text{(2.3)} \qquad \frac{zf'(z)}{f(z)} = \left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right) - i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}} e^{i\frac{\pi}{4}(\alpha+\beta)}.$$

The present paper gives the structural formula, the set of variability of the functional $\dfrac{zf'(z)}{f(z)}$ and sharp estimation of the modulus of the expression $\dfrac{zf'(z)}{f(z)}$ in the class $S_{(\alpha,\beta)}$. We also give the form of the extremal function and the estimations of a_2, a_3 in this class.

3. Estimations of some functionals in the class $S_{(\alpha,\beta)}$.

THEOREM I. *Function* $f(z)$ belongs *to the class* $S_{(\alpha,\beta)}$ *if and only if there exists a function* $p(z)$ *of the class* P *such that*

$$\text{(3.1)} \qquad f(z) = z\exp\left\{\int_0^z \frac{1}{z}\left[\left(p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right) - i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right)^{\frac{\alpha-\beta}{2}} e^{i\frac{\pi}{4}(\alpha+\beta)} - 1\right]dz\right\}.$$

Proof. It follows from definition (2.2) that

$$\frac{zf'(z)}{f(z)} = \left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right) - i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}} e^{i\frac{\pi}{4}(\alpha+\beta)}, \quad z \in K_1.$$

Hence

$$\frac{f'(z)}{f(z)}-\frac{1}{z}$$
$$=\frac{1}{z}\left[\left(p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right)^{\frac{\alpha-\beta}{2}}e^{i\frac{\pi}{4}(\alpha+\beta)}-1\right]$$

or

$$\ln\frac{f(z)}{z}$$
$$=\int_0^z\frac{1}{z}\left\{\left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}}e^{i\frac{\pi}{4}(\alpha+\beta)}-1\right\}dz.$$

Hence the structural formula for the class $S_{(\alpha,\beta)}$ is obtained.

$$f(z)$$
$$=z\exp\int_0^z\frac{1}{z}\left\{\left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}}e^{i\frac{\pi}{4}(\alpha+\beta)}-1\right\}dz.$$

THEOREM II. *In the class $S_{(\alpha,\beta)}$ for any fixed z, $|z|=r$, $r<1$ the set of variability of the expression*

$$\left[\frac{zf'(z)}{f(z)}\right]^{\frac{2}{\alpha-\beta}}$$

is the closed disc

$$\left|\left[\frac{zf'(z)}{f(z)}\right]^{\frac{2}{\alpha-\beta}}-\frac{1+r^2e^{i\pi\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}}{1-r^2}\right|\leqslant 2\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\frac{r}{1-r^2}. \tag{3.2}$$

The extremal function is of the form

$$f(z)=z\exp\left\{\int_0^z\frac{1}{z}\left[\left(\frac{1+ze^{i\pi\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}}{1-z}\right)^{\frac{\alpha-\beta}{2}}-1\right]dz\right\}. \tag{3.3}$$

Proof. It follows from the structural formula that

$$\frac{zf'(z)}{f(z)}=\left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}}e^{i\frac{\pi}{4}(\alpha+\beta)}$$

or

$$p(z) = \frac{\left[\dfrac{zf'(z)}{f(z)}\right]^{\frac{2}{\alpha-\beta}} e^{-i\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)} + i\sin\dfrac{\pi}{2}\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)}{\cos\dfrac{\pi}{2}\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)}.$$

Since the set of variability of $p(z)\in P$ for fixed z, $|z| = r$ is a disc whose diameter, lying on real-axis, has end-points

$$\frac{1-r}{1+r} \quad \text{and} \quad \frac{1+r}{1-r},$$

the proposition of Theorem II follows.

As a corollary from Theorem II we obtain the following sharp estimations:

$$\left[\frac{\sqrt{1+2r^2\cos\pi\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)+r^4}-2r\cos\dfrac{\pi}{2}\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)}{1+r^2}\right]^{\frac{\alpha-\beta}{2}} \tag{3.4}$$

$$\leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant \left[\frac{\sqrt{1+2r^2\cos\pi\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)+r^4}+2r\cos\dfrac{\pi}{2}\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)}{1-r^2}\right]^{\frac{\alpha-\beta}{2}},$$

$$\frac{1+r^2\cos\pi\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)-2r\cos\dfrac{\pi}{2}\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)}{1+r^2} \tag{3.5}$$

$$\leqslant \operatorname{Re}\left\{\left[\frac{zf'(z)}{f(z)}\right]^{\frac{2}{\alpha-\beta}}\right\} \leqslant \frac{1+r^2\cos\pi\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)+2r\cos\dfrac{\pi}{2}\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)}{1-r^2}.$$

In some cases the familiar estimations occur for the respective classes of functions.

THEOREM III. *If*

$$f(z) = z+a_2z^2+a_3z^3+\ldots \in S_{\alpha,\beta},$$

then

$$|a_2| \leqslant (\alpha-\beta)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right).$$

The extremal function is of the form (3.3).

Proof. Let

$$p(z) = 1+p_1z+p_2z^2+\ldots \in P$$

and let

$$f(z) = z\exp\left\{\int_0^z \frac{1}{z}\left[\left(p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right)^{\frac{\alpha-\beta}{2}} e^{i\frac{\pi}{4}(\alpha+\beta)}-1\right]dz\right\}.$$

Write

$$A(z) = \frac{\left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}} e^{i\frac{\pi}{4}(\alpha+\beta)}-1}{z}.$$

Thus

$$f(z) = z\exp\int_0^z A(z)\,dz.$$

Then

$$f''(z) = 2\exp\left\{\int_0^z A(z)\,dz\right\}A(z)+z\exp\left\{\int_0^z A(z)\,dz\right\}A^2(z)+z\exp\left\{\int_0^z A(z)\,dz\right\}A'(z).$$

Moreover,

$$\lim_{z\to 0}A(z) = \lim_{z\to 0}\frac{\frac{\alpha-\beta}{2}}{1}\times$$

$$\times\frac{\left[p(z)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)-i\sin\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\right]^{\frac{\alpha-\beta}{2}-1}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)p(z)\,e^{i\frac{\pi}{4}(\alpha+\beta)}}{1}$$

$$= \frac{\alpha-\beta}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)e^{i\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}p_1.$$

Thus

$$f''(0) = (\alpha-\beta)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)e^{i\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}p_1$$

or

$$a_2 = \frac{f''(0)}{2} = \frac{\alpha-\beta}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)e^{i\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}p_1.$$

Since $|p_1| \leqslant 2$, we have

$$|a_2| \leqslant (\alpha-\beta)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right).$$

For the function (3.3) we have

$$p(z) = \frac{1+z}{1-z} \quad \text{and} \quad p_1 = 2,$$

and thus the estimation $|a_2|$ is sharp.

THEOREM IV. *If*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in S_{(\alpha,\beta)},$$

hen we have the sharp estimation

(a) $|a_3| \leqslant \frac{\alpha-\beta}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)$ *for* $0 < \alpha-\beta \leqslant \frac{2}{3}$,

(b) $|a_3| \leqslant \frac{\alpha-\beta}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left[\left(\frac{3}{2}(\alpha-\beta)-1\right)\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)+1\right]$

for $\frac{2}{3} \leqslant \alpha-\beta \leqslant 2$.

The extremal functions are the following functions, respectively:

$$f_1(z) = z\exp\left\{\int_0^z \frac{1}{z}\left[\left(\frac{1+z^2 e^{i\pi\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}}{1-z^2}\right)^{\frac{\alpha-\beta}{2}}-1\right]dz\right\},$$

$$f_2(z) = z\exp\left\{\int_0^z \frac{1}{z}\left[\left(\frac{1+z\, e^{i\pi\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}}{1-z}\right)^{\frac{\alpha-\beta}{2}}-1\right]dz\right\}.$$

Proof. From the structural formula we obtain

$$a_3 = \frac{\alpha-\beta}{4}e^{i\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left[\frac{1}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left(\frac{3}{2}(\alpha-\beta)-1\right)p_1^2+p_2\right].$$

When the expression $\frac{3}{2}(\alpha-\beta)-1$ is positive, i.e., when $\frac{2}{3} < (\alpha-\beta) \leqslant 2$, the estimation (b) is obtained directly from the expression for a_3 obtained from the structural formula by using the inequality $|p_n| \leqslant 2$.

When the expression $\frac{3}{2}(\alpha-\beta)-1$ is negative, i.e., $0<\alpha-\beta<\frac{2}{3}$, the estimation (a) is obtained in the following way: if

$$w(z) = w_1 z + w_2 z^2 + \ldots$$

is a regular function in the unit disc K_1 and satisfying the assumptions of Schwarz's Lemma, then there exists a function

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

with a positive real part such that

$$w(z) = \frac{p(z)-1}{p(z)+1}.$$

After easy calculations we obtain

$$p_1 = 2w_1,\ p_2 = 2(w_1^2 + w_2).$$

The expression for a_3 is now as follows:

$$|a_3| = \frac{\alpha-\beta}{4}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left|p_2 - \frac{1}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left(1-\tfrac{3}{2}(\alpha-\beta)\right)p_1^2\right|$$

$$= \frac{\alpha-\beta}{4}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)|p_2 - \gamma p_1^2|,$$

where

$$0 < \gamma = \frac{1}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left(1-\tfrac{3}{2}(\alpha-\beta)\right) < \frac{1}{2}$$

or

$$|a_3| = \frac{\alpha-\beta}{4}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)|2w_1^2 + 2w_2 - 4\gamma w_1^2|$$

$$= \frac{\alpha-\beta}{2}\cos\frac{\pi}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)|w_2 + (1-2\gamma)w_1^2|.$$

Making use of the familiar inequality [1],

$$|w_2| \leqslant 1 - |w_1|^2$$

we obtain

$$|w_2 + (1-2\gamma)w_1^2| \leqslant |w_2| + (1-2\gamma)|w_1|^2 \leqslant 1 - 2\gamma|w_1|^2 \leqslant 1.$$

Hence

$$a_3 \leqslant \frac{a-\beta}{2} \cos \frac{\pi}{2} \left(\frac{a+\beta}{a-\beta} \right).$$

In particular: in the class S^* $|a_3| \leqslant 3$, in the class S_a

$$|a_3| \leqslant a \quad \text{for} \quad 0 < a < \tfrac{1}{3}$$

and

$$|a_3| \leqslant 3a^2 \quad \text{for} \quad \tfrac{1}{3} < a < 1,$$

and in the class $\breve{S}_\delta$

$$|a_3| \leqslant \cos \frac{\pi}{2} (a-1) \left[1 + 2 \cos \frac{\pi}{2} (a-1) \right].$$

Proving the case (a) the authors have followed some ideas from the remarks of Professor J. Krzyż as expressed in his review of A. Wesołowski's doctoral dissertation.

Finally, the authors would like to express their gratitude to Professor Z. Lewandowski — head of a seminar, conducted at the Institute of Mathematics of the Polish Academy of Science, which provided inspiration for the present paper.

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Reçu par la Rédaction le 18. 7. 1971