COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 2

PROJECTIVE LIMITS OF TOPOLOGICAL ALGEBRAS

BY

T. MÜLDNER (WARSZAWA)

It has been proved by Michael (see [1], Theorem 5.1) that every complete locally m-convex algebra is isomorphic to a projective limit of Banach algebras. In this paper we generalize this theorem.

1. We first recall some necessary definitions (see, e.g., [4]). By a *topological algebra* we mean a topological linear space (not necessarily Hausdorff) over complex or real scalars in which there is defined a jointly continuous multiplication. We define a *K-algebra* as a topological algebra which, as a topological linear space, belongs to the class K of topological linear spaces. In the sequel we shall consider the following classes:

LC — the class of locally convex topological Hausdorff algebras,

F^* — the class of metrizable topological Hausdorff algebras,

F — the class of complete metrizable topological Hausdorff algebras,

B_0 — the class of complete metrizable locally convex topological Hausdorff algebras.

By an *F-seminorm* in a topological algebra A we mean a mapping $\nu\colon A \to R_+$ such that

(a) $\nu(x+y) \leqslant \nu(x)+\nu(y)$ for $x, y \in A$,

(b) $\nu(\lambda x) \to 0$ for all x and λ scalar, $\lambda \to 0$,

(c) $\nu(\lambda x) \leqslant \nu(x)$ for λ scalar, $|\lambda| \leqslant 1$ and $x \in A$.

We can now formulate our theorems:

THEOREM 1. *Every complete topological Hausdorff algebra A is isomorphic to a projective limit of F-algebras.*

THEOREM 2. *Every complete locally convex topological Hausdorff algebra A is isomorphic to a projective limit of B_0-algebras.*

The proof of Theorem 1 is based upon the following lemma:

LEMMA. *Let X be an arbitrary topological algebra and let $\Phi(X)$ be a basis of closed balanced neighbourhoods of zero in X. If*

$$N = \bigcap_{U \in \Phi(X)} U,$$

then N is a closed ideal in X.

Proof. If X is a topological Hausdorff algebra, then $N = \{0\}$. Suppose that $N \neq \{0\}$, and let $0 \neq x, y \in N$, $0 \neq \alpha, \beta \in C$. For $U \in \Phi(X)$ choose $V \in \Phi(X)$ such that $V + V \subset U$. Clearly,

$$x, y \in (\alpha^2 + \beta^2)^{-1} V,$$

whence

$$\alpha x + \beta y \in \alpha(\alpha^2 + \beta^2)^{-1} V + \beta(\alpha^2 + \beta^2)^{-1} V \subset V + V \subset U,$$

and

$$\alpha x + \beta y \in N. \tag{1}$$

It follows that N is a linear subspace of X. Now let $0 \neq x \in N$ and $y \in X$. For $U \in \Phi(X)$ choose $V \in \Phi(X)$ such that $V^2 \subset U$. We can find $\lambda > 0$ such that $\lambda y \in V$. From (1) we have $x/\lambda \in V$, whence

$$xy = \frac{1}{\lambda} x \lambda y \in V^2 \subset U,$$

which proves the Lemma.

Proof of Theorem 1. In virtue of Theorem 1.3 of [2] there is a basis $\Phi(A)$ as in the Lemma. Let Ω be a family $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ of "elementary neighbourhood chains", i.e. of $\Phi_\lambda = \{U_n^\lambda\}_{n=1}^\infty$ such that, for $n = 1, 2, \ldots$,

1. $U_n^\lambda \in \Phi(A)$,
2. $U_{n+1}^\lambda + U_{n+1}^\lambda \subset U_n^\lambda$,
3. $(U_{n+1}^\lambda)^2 \subset U_n^\lambda$.

It follows from Theorem I.6.1 of [2] and the remark after this theorem (see also [3]) that the topology τ_λ, given by Φ_λ in the set A, is defined by the F-seminorm $|\cdot|_\lambda$. We set $N_\lambda = \bigcap_{n=1}^{\infty} U_n^\lambda$, $A_\lambda = A/N_\lambda$, and x_λ — a congruence class of $x \pmod{N_\lambda}$. It follows from the Lemma that N_λ is a closed (in the topology τ_λ) ideal and that A_λ, equipped with the topology I_λ, given by the F-norm $\|x_\lambda\|_\lambda = |x|_\lambda$, is an F^*-algebra. We define a partial order in the set Λ putting

$$\lambda \prec \mu \quad \text{iff} \quad U_n^\mu \subset U_n^\lambda \text{ for } n = 1, 2, \ldots$$

We shall show that this relation turns Λ into a directed set. Let $\lambda, \mu \in \Lambda$. We define inductively a chain $\Phi_\delta \in \Omega$. First, we find $U_1^\delta \in \Phi(A)$ such that $U_1^\delta \subset U_1^\lambda \cap U_1^\mu$. Suppose that we have defined $U_1^\delta, \ldots, U_n^\delta$ satisfying, for $1 < k \leqslant n$, the following conditions:

1_n. $U_k^\delta \subset U_k^\lambda \cap U_k^\mu$,

2_n. $U_k^\delta + U_k^\delta \subset U_{k-1}^\delta$,

3_n. $(U_k^\delta)^2 \subset U_{k-1}^\delta$,

4_n. $U_k^\delta \in \Phi(A)$.

We choose $V \in \Phi(A)$ such that $V^2 \subset U_n^\delta$, $W \in \Phi(A)$ such that $W+W \subset V \cap U_n^\delta$, and $U_{n+1}^\delta \in \Phi(A)$ such that $U_{n+1}^\delta \subset U_{n+1}^\lambda \cap U_{n+1}^\mu \cap W$. Then we have

1_{n+1}. $U_{n+1}^\delta \in \subset U_{n+1}^\lambda \cap U_{n+1}^\mu$.

2_{n+1}. $U_{n+1}^\delta + U_{n+1}^\delta \subset U_{n+1}^\lambda \cap U_{n+1}^\mu \cap W + U_{n+1}^\lambda \cap U_{n+1}^\mu \cap W \subset W + W \subset U_n^\delta$.

3_{n+1}. $(U_{n+1}^\delta)^2 \subset (U_{n+1}^\lambda \cap U_{n+1}^\mu \cap W)^2 \subset W^2 \subset (V \cap U_n^\delta)^2 \subset V^2 \subset U_n^\delta$.

It is obvious that if we put $\Phi_\delta = \{U_n^\delta\}_{n=1}^\infty$, then $\delta \in \Lambda$ and λ, $\mu \prec \delta$. Now it is easy to verify that, for every $\lambda \prec \mu$, the mapping

$$g_{\lambda\mu}\colon x_\mu \to x_\lambda$$

is a homomorphism from A_μ into A_λ. From the proof of Theorem I.6.1 of [2] it follows that, for $\lambda \prec \mu$, we have $|\cdot|_\lambda \leqslant |\cdot|_\mu$, so $g_{\lambda\mu}$ is a continuous mapping. Let

$$Y = \varprojlim \{A_\lambda, g_{\lambda\mu}\}$$

be the projective limit of algebras A_λ. The mapping

$$\varkappa\colon x \to \{x_\lambda\}_{\lambda \in \Lambda}$$

is clearly a 1-1 homomorphism from A into Y. We shall show that $\varkappa$ is onto.

Let H be an arbitrary non-void finite subset of Λ and let $z = \{z_\lambda\}_{\lambda \in \Lambda}$ be a fixed element of Y. We can find a $\beta \in \Lambda$ such that $\alpha \prec \beta$ for $\alpha \in H$. If we put $g_\lambda(x) = x_\lambda$ for $x \in A$, $\lambda \in \Lambda$, then we can choose $x_H \in A$ such that $g_\beta(x_H) = z_\beta$. It follows that

$$g_\alpha(x_H) = g_{\alpha\beta}\big(g_\beta(x_H)\big) = g_{\alpha\beta}(z_\beta) = z_\alpha \quad \text{for } \alpha \in H. \tag{2}$$

We are now going to show that $\{x_H\}$ is a Cauchy net. For an arbitrary $V \in \Phi(A)$ let

$$\Phi_{\lambda_0} = \{U_n^{\lambda_0}\}_{n=1}^\infty \in \Omega$$

be an elementary chain such that $U_1^{\lambda_0} = V$ (the existence of such a chain follows from the continuity of addition and multiplication). We put $H_0 = \{\lambda_0\}$. If $H_1, H_2 \supset H_0$, then $\lambda_0 \in H_1 \cap H_2$, and from (2) we get

$$g_{\lambda_0}(x_{H_1}) = z_{\lambda_0} = g_{\lambda_0}(x_{H_2}).$$

It follows that $x_{H_1} - x_{H_2} \in V$, so $\{x_H\}$ is a Cauchy net.

From the completeness of the algebra A it follows that there exists $y = \lim x_H$. Clearly, $y_\lambda = z_\lambda$ for $\lambda \in \Lambda$, so we have proved that $\varkappa$ is onto.

It can be easily verified that $\varkappa$ is a homeomorphic mapping. We denote by $\tilde{A}_\lambda$ the completion of A_λ with respect to the topology I_λ, and by $\tilde{g}_{\lambda\mu}\colon \tilde{A}_\mu \to \tilde{A}_\lambda$ the extension of $g_{\lambda\mu}$.

It is easy to see that Y is dense in $\varprojlim\{\tilde{A}_\lambda, \tilde{g}_{\lambda\mu}\}$, but Y is isomorphic to A, so it is complete. It follows that

$$Y \approx \varprojlim\{\tilde{A}_\lambda, \tilde{g}_{\lambda\mu}\},$$

which completes the proof of Theorem 1.

The proof of Theorem 2 is analogous (if A is a complete LC-algebra, then A_λ is a metric LC-algebra and $\tilde{A}_\lambda$ is a B_0-algebra).

Remark. It is easy to see that if A is a complete locally convex topological Hausdorff algebra such that, for every sequence $\{p_n\}$ of continuous seminorms, $\sup p_n$ is a continuous seminorm, then A is m-convex, and so is isomorphic to a projective limit of Banach algebras. Therefore, a complete locally convex topological Hausdorff algebra is isomorphic to a projective limit of non-normable B_0-algebras iff the sequences of continuous seminorms are unbounded.

Acknowledgement. The author is indebted to Professor W. Żelazko for his guidance and suggestions.

REFERENCES

[1] E. Michael, *Locally multiplicatively convex topological algebras*, Memoirs of the American Mathematical Society 11 (1952).

[2] H. Schaefer, *Topological vector spaces*, New York 1966.

[3] *Summer school on topological vector spaces*, Lecture Notes 331 (1973).

[4] W. Żelazko, *Selected topics in topological algebras*, Aarhus Lecture Notes 31 (1971).

INSTITUTE OF MATHEMATICAL MACHINES
WARSZAWA

Reçu par la Rédaction le 25. 4. 1974