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ON LOCALLY TREE-LIKE GRAPHS

Abstract. We deal with locally tree-like graphs and provide an upper bound to the number of edges in such graphs.

1. Introduction. In this paper we present an upper bound to the number of edges in locally tree-like graphs. For basic terminology and notation, see [2].

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is a finite set of vertices and $E(G)$ is a set of edges, i.e., two-element subsets of $V(G)$. By the *neighbourhood* $N(x, G)$ of vertex x in G we mean the subgraph of G induced by all vertices adjacent to x in G. A graph G is a *locally tree-like graph* if the neighbourhood of every vertex x of G is a tree. An essential role in the characterization of locally tree-like graphs is played by 2-trees, the generalizations of trees. The class of 2-trees is defined recursively in the following way:

(1) K_3 is a 2-tree.

(2) Let H be a 2-tree and let G be a graph obtained from H by adding one new vertex and two edges connecting it to two adjacent vertices in H. Then G is a 2-tree.

(3) The class of 2-trees contains no graphs except those described in (1) and (2).

The underlying theorem characterizing locally tree-like graphs with the minimal number of edges is a simple corollary to Theorem 9 in [3].

THEOREM 1. *The class of 2-trees consists of all connected locally tree-like graphs with the minimal number of edges equal to* $2n-3$, *where* n *is the number of vertices in the graph.*

A connected locally tree-like graph which is not a 2-tree and has more than $2n-3$ edges is presented in Fig. 1.

Since locally tree-like graphs cannot be too dense, Zelinka asked in [3] for determining an upper bound to the number of edges in such graphs. He proved in [3] the following

THEOREM 2 ([3]). *For any positive integer q there exists a connected locally tree-like graph in which the minimal degree of a vertex is greater than q.*

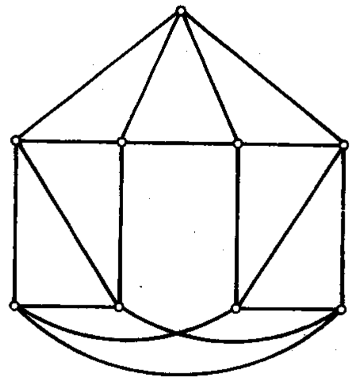

Fig. 1

The proof of Theorem 2 in [3] is based on the existence of a finite projective geometry PG(p) in which each line is incident with $p+1$ points and each point is incident with $p+1$ lines. We present here another proof of this theorem based only on the following proposition:

PROPOSITION 1. *For any integer $p \geqslant 1$ there exist an integer n and a family $\mathscr{B}_p$ of $(p+1)$-element subsets of $S = \{1, 2, \ldots, n\}$ such that*

(1) *the cardinality $|\mathscr{B}_p|$ of $\mathscr{B}_p$ is n,*

(2) *every element of S belongs to exactly $p+1$ members of this family.*

Proof of Proposition 1. For $p = 1$, we have $n = 3$ and

$$S = \{1, 2, 3\}, \quad \mathscr{B}_1 = \big\{\{1, 2\}, \{2, 3\}, \{1, 3\}\big\};$$

similarly for $p = 2$, $n = 7$ and

$$S = \{1, 2, \ldots, 7\},$$

$$\mathscr{B}_2 = \big\{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{5, 6, 7\}, \{1, 2, 7\}, \{1, 6, 7\}\big\}.$$

The families $\mathscr{B}_1$ and $\mathscr{B}_2$ satisfy the conditions of Proposition 1. Let p be an arbitrary integer, $p \geqslant 3$. Then let $n = p^2+p+1$ and $S = \{1, 2, \ldots, n\}$. The family

$$\mathscr{B}_p = \big\{\{i, i+1, \ldots, i+p\}: i = 1, 2, \ldots, p^2+1\big\} \cup \cup \big\{\{1, 2, \ldots, p+1-i, p^2+p+2-i, p^2+p+1-i, \ldots, p^2+p+1\}: i = 1, 2, \ldots, p\big\}$$

satisfies the assumptions of the proposition.

Proof of Theorem 2. The family $\mathscr{B}_p$ may be used to construct the graph desired in Theorem 2 in a similar manner as projective geometry PG(p) is used in [3]. The rest of the proof is the same as in [3].

From Theorem 2 it follows that for fixed q and n the number of edges in a connected locally tree-like graph G may be greater than $qn/2$, $n = |V(G)|$. Therefore, there exists no upper bound to the number of edges in G, which is a linear function of n.

2. Main results. We present now some properties of 2-trees and locally tree-like graphs useful in further considerations.

PROPOSITION 2. *If T is a 2-tree, then the graph T_1,*

$$T_1 = (V(T),\ E(T) \cup \{e\}),$$

where $e = \{u, v\}$, $u, v \in V(T)$ and $d_T(u, v) = 2$, is not a locally tree-like graph.

Proof. It is sufficient to notice that the addition of such an edge $e = \{u, v\}$, $u, v \in V(T)$, $d_T(u, v) = 2$, to the 2-tree T causes the appearing of a cycle in the neighbourhood of some vertex of the new graph T_1. Let w be the vertex adjacent to u and v. By the definition of 2-trees, the vertices u and w as well as w and v belong to some 3-clique in T. Let us assume that the vertices u, w, u' induce a 3-clique in T. If w, u', v also induce a 3-clique, then the vertex v has K_3 in its neighbourhood in the graph T_1 (see Fig. 2). Hence T_1 is not a locally tree-like graph.

Now, if w, u', v do not induce a 3-clique in T_1, then v and w cannot belong to the same clique as u and w. This situation is presented in Fig. 3.

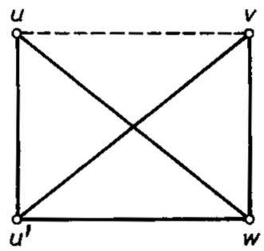

Fig. 2

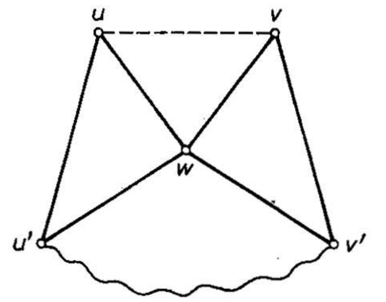

Fig. 3

Let us consider the neighbourhood of the vertex w in T and in T_1. The connectivity of the graph $N(w, T)$ implies the existence of a path between u and v in T; hence $N(w, T_1)$ contains a cycle. Therefore, again T_1 is not a locally tree-like graph.

The next property of connected locally tree-like graphs indicates that 2-trees are, in some sense, minimal graphs in the class of locally tree-like graphs.

THEOREM 3. *Every connected locally tree-like graph G contains a spanning subgraph isomorphic to a 2-tree.*

Proof. Let v be a vertex of G. Then the subgraph G_1 of G, induced by v and all its neighbours, is a 2-tree. If G is isomorphic to G_1, then we obtain our assertion. In the opposite case, G_1 may be extended to a 2-tree defined on the whole set of vertices of the graph G. For this purpose it suffices to show that if the 2-tree R is a subgraph of G and $V(G) \supsetneq V(R)$, and u is adjacent to R (i.e., u is adjacent to $w \in V(R)$), then there exists a 2-tree R' which is a subgraph of G and

$$V(R') \supset V(R) \cup \{u\}.$$

Essentially, the fact that w has a nonempty neighbourhood in R and $N(w, G)$ is connected implies the existence of a path P between vertices $w_1 \in V(N(w, R))$ and u, not containing the vertex w. We have then the situation illustrated in Fig. 4. The graph R' is obtained in such a way that we add to R the vertex u, all vertices of the path P not belonging to $V(R)$, the elements of $E(P) \setminus E(R)$ and edges connecting the vertex w with $V(P) \setminus V(R)$. Our assertion follows from the connectivity of G and the finiteness of the set $V(G)$.

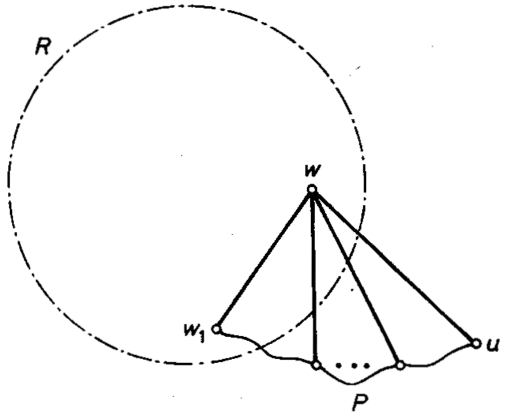

Fig. 4

Proposition 2 and Theorem 3 imply the following corollary:

COROLLARY 1. *If G is a connected locally tree-like graph, and a 2-tree T is its spanning subgraph, then $d_T(u, v) > 2$ for every $x = \{u, v\} \in E(G \setminus T)$.*

Hence we obtain

THEOREM 4. *Let G be a connected locally tree-like graph with n vertices. Then*

$$(1) \qquad |E(G)| \leqslant \binom{n}{2} - \max_T \big|\{\{u, v\}: u, v \in V(G),\ d_T(u, v) = 2,\ T \text{ a spanning 2-tree subgraph of } G\}\big|.$$

Inequality (1) gives the best possible upper bound to the number of edges in a locally tree-like graph in the following sense:

THEOREM 5. *There exists an infinite class of connected locally tree-like graphs for which* (1) *holds with equality.*

Proof. Let us consider a graph G defined in the following way:

$$V(G_n) = \{a, b, c, d, e, f, g, h_1, h_2, \ldots, h_n\}, \quad n \geqslant 0,$$

$$E(G_n) = \{\{a, b\}, \{a, d\}, \{a, g\}, \{b, c\}, \{b, g\}, \{b, e\}, \{c, d\}, \{c, f\}, \{a, e\}, \{c, g\}, \{d, f\}, \{d, e\}, \{e, f\}, \{f, g\}\} \cup \{\{h_i, g\}, \{h_i, f\}: i = 1, 2, \ldots, n\}$$

(see Fig. 5).

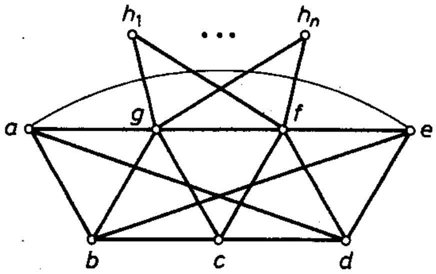

Fig. 5. The graph G_n

It is easy to check that G is a connected locally tree-like graph. Simultaneously, G has a spanning 2-tree T with diameter equal to 3 and G arises from T by adding all edges between the vertices remaining at distance 3 from each other. Hence the equality in (1) holds.

The following question remains. Let G be a connected locally tree-like graph with n vertices and let a 2-tree T be its spanning subgraph. Is it possible to express the number of pairs of vertices which are at distance 2 from each other in T as the function of n?

We now present another form of estimation (1) in which the right-hand side of the inequality is a function of n, $n = |V(G)|$. Let us first define a special kind of 2-trees.

A 2-tree on n vertices is called a *2-chain* if its degree sequence is of the following form:

$$(2, 3, \underbrace{4, 4, \ldots, 4}_{n-4}, 3, 2).$$

These special 2-trees are "extremal" graphs with respect to the number of pairs of vertices remaining at distance 2 from each other.

THEOREM 6. *The 2-chain on n vertices has the smallest possible number of pairs of vertices which are at distance 2 from each other among all 2-trees on n vertices, $n \geqslant 4$. This number is equal to $2n-7$.*

Proof. (A) First we show that the 2-chain G has $2n-7$ pairs of vertices which are at distance 2 from each other. It is easy to check the above formula for 2-chains with 4, 5, 6 or 7 vertices. For $n > 7$ we have (see Fig. 6)

$$d_G(i, i+3) = d_G(i, i+4) = 2, \quad i = 1, 2, 3,$$

$$d_G(4, 1) = d_G(4, 7) = d_G(4, 8) = 2,$$

$$d_G(i, i-3) = d_G(i, i+3) = d_G(i, i-4) = d_G(i, i+4) = 2, \quad i = 5, 6, \ldots, n-4,$$

$$d_G(n-3, n-7) = d_G(n-3, n-6) = d_G(n-3, n) = 2,$$

$$d_G(i, i-4) = d_G(i, i-3) = 2, \quad i = n-2, n-1, n.$$

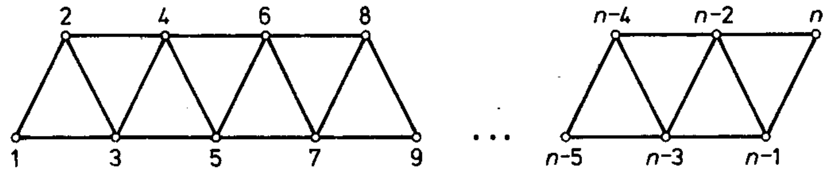

Fig. 6. The 2-chain on n vertices

From the above formulas it follows that the total number of pairs of vertices which are at distance 2 from each other is equal to

$$(6+3+4(n-4-5+1)+3+6)/2 = 2n-7.$$

(B) We prove now that $2n-7$ is the smallest number of pairs of vertices in a 2-tree G, $n = |V(G)|$, whose mutual distance is equal to 2. The proof is by induction on n.

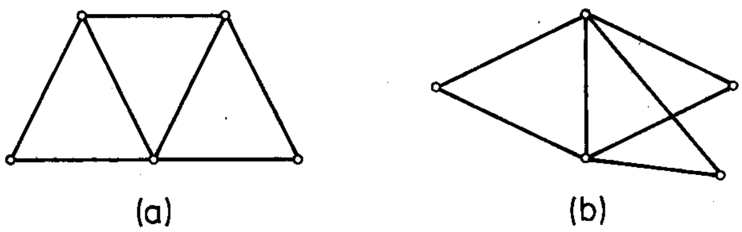

Fig. 7. The 5-vertex 2-trees

1. The 2-chain is a unique 2-tree with 4 vertices; hence our assertion is true for $n = 4$. For $n = 5$ the situation is shown in Fig. 7. We have $10-7 = 3$ pairs of vertices with the desired properties in subcase a, and 3 such pairs in subcase b. This fact confirms our assertion for $n = 5$. For $n = 6$ there exists only one 2-tree with n vertices, containing $12-7 = 5$ pairs of vertices which are at distance 2 from each other; it is the 2-chain. The other 6-vertex 2-trees (see Fig. 8) have 6 such pairs.

2. Let G be a 2-tree with k vertices. Choose in G a vertex v of degree 2 (the existence of such a vertex follows from the definition of a 2-tree). The

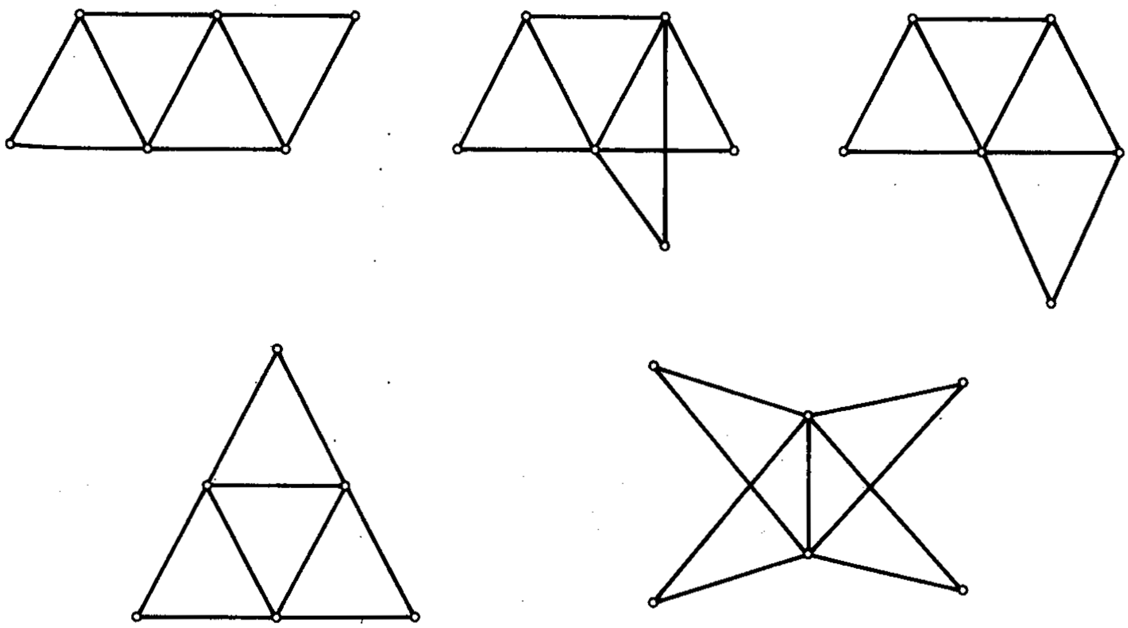

Fig. 8. The 6-vertex 2-trees

inductive assumption implies that the number of pairs of vertices remaining at distance 2 from each other in the 2-tree $G-v$ is greater than or equal to $2(k-1)-7$. The addition of the vertex v to the graph $G-v$ to obtain the graph G causes the increase of the number of pairs of vertices with the above property by at least 2; hence this number for the graph G amounts to at least

$$2(k-1)-7+2 = 2k-7.$$

Hence we may conclude the theorem for all $k \geqslant 4$.

Remark. An immediate consequence of the above statements is the fact that a connected locally tree-like graph G contains at most $n^2/2-5n/2+7$ edges, $n = |V(G)|$. It turns out [4] that Erdős and Simonovits have proved a stronger result for graphs without wheels (the class of these graphs includes the class of locally tree-like graphs). Their bound is equal to $n^2/4+n/4$ and is smaller than that mentioned above for almost all positive integers.

References

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