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THE k-POINT-ARBORICITY OF A GRAPH

BY

DON R. LICK (KALAMAZOO, MICHIGAN)

1. Introduction. Graphs having a certain property are often characterized in terms of a type of configuration or subgraph which they cannot have. For example, a graph is *totally disconnected* (or has *chromatic number one*) if it contains no lines; a graph is a *forest* (or has *point-arboricity one*) if and only if it contains no cycles. Chartrand et al. [3] defined a graph to have property P_n for a positive integer n if it contains no subgraph homeomorphic to the complete graph K_{n+1} or to the complete bipartite graph

$$K\left(\left[\frac{n+2}{2}\right], \left\{\frac{n+2}{2}\right\}\right).$$

For the first four natural numbers n, the graphs with property P_n are exactly the totally disconnected, acyclic (forests), outerplanar, and planar graphs, respectively. This unification suggested the extension of many results known to hold for one of the above four classes to one or more of the remaining classes. Chartrand et al. were very successful in this approach, but the methods of proof employed for values of n less than five generally did not extend to larger values.

The Chartrand et al. approach is somewhat topological in nature. In [9], White and the author adopted a more graph-theoretic viewpoint. They defined a graph to be *k-degenerate* for a non-negative integer k if each induced subgraph has minimum degree at most k, and if π_k denotes the class of all k-degenerate graphs, then π_0 and π_1 are exactly the classes of totally disconnected graphs and the forests, respectively, while the classes π_2 and π_5 properly contain all outerplanar graphs and planar graphs, respectively. The advantage of this approach is that many well-known results for chromatic number and point-arboricity (corresponding to the cases $k = 0$ and $k = 1$, respectively) have natural extensions for all larger values of k. However, some of the newer properties of chromatic number and point-arboricity do not carry over in general.

In this paper we take a different viewpoint. Instead of studying graphs with property P_n or k-degenerate graphs, we consider the class A_k of all k-acyclic graphs for a non-negative integer k. A_0 and A_1 again turn out to be exactly the classes of totally disconnected graphs and of forests, respectively. The advantage of this approach is that many of the newer, as well as the older, results for chromatic number and point-arboricity (corresponding to the cases $k = 0$ and $k = 1$, respectively) have natural extensions for all larger values of k.

In Sections 2 and 3 we provide basic definitions and establish some elementary properties for k-acyclic graphs. Section 4 presents the definition of the k-point-arboricity of a graph. These concepts generalize the chromatic number ($k = 0$) and the point-arboricity ($k = 1$) of a graph. The formula for the k-point-arboricity of the complete n-partite graphs is developed in Section 5. Section 6 deals with critical graphs and, finally, Section 7 provides bounds for the k-point-arboricity of a graph.

2. Preliminaries. Those definitions not provided here may be found in [1] or [6]. The graphs under consideration are *ordinary*, that is, finite undirected graphs without loops or multiple lines (*Michigan graphs*). The point set and the line set of the graph G are denoted by $V(G)$ and $E(G)$, respectively. The *degree* $d(v)$ of the point v of G is the number of lines incident with it. The largest degree among the points of G is called the *maximum degree* of G and is denoted by $\Delta(G)$; while the smallest degree among the points of G is called the *minimum degree* of G and is denoted by $\delta(G)$.

A *subgraph* of a graph G consists of a subset of the point set of G and a subset of the line set of G which together form a graph. The *subgraph* $\langle S\rangle$ *induced by the subset* S *of* $V(G)$ has point set S and contains all lines of $E(G)$ incident with two points of S. Two graphs are said to be *disjoint* if they have no points in common.

The *complete n-partite graph* $K(p_1, p_2, \ldots, p_n)$ has its point set V partitioned into n subsets V_i with $|V_i| = p_i$, $i = 1, 2, \ldots, n$, such that two points u and v are adjacent if $u \in V_h$ and $v \in V_j$, $h \neq j$. The sets V_i, $i = 1, 2, \ldots, n$, are called the *partite sets* of $K(p_1, p_2, \ldots, p_n)$. If $p_i = 1$, $i = 1, 2, \ldots, n$, then the graph is *complete* on n points and is denoted by K_n.

In [2], Beineke and Pippert defined *k-walk* in a graph as an alternating sequence of complete graphs K_k and K_{k+1},

$$K_k^0, K_{k+1}^1, K_k^1, K_{k+1}^2, \ldots, K_k^{n-1}, K_{k+1}^n, K_k^n, \tag{1}$$

beginning and ending with complete graphs K_k and such that the complete graph K_{k+1}^i contains the two complete graphs K_k^{i-1} and K_k^i, where $K_k^{i-1} \neq K_k^i$ (actually, K_k^{i-1} and K_k^i have $k-1$ points in common). This k-walk is called a *k-cycle* if $n > 2$, $K_k^0 = K_k^n$, and all other elements of sequence (1) are distinct. The *length* of a k-cycle is the number of complete graphs K_{k+1} that the k-cycle contains. Since K_{k+1} is a k-walk, the smallest k-cycle is K_{k+2} and it has length 3. We show this by providing its sequence (1), where $K_k^0 = K_k^3$, $k \geqslant 1$. Consider the complete graph K_{k+2} with the point set $\{v_1, v_2, \ldots, v_{k-1}, u_1, u_2, u_3\}$. Let

$$K_k^0 = \langle\{v_1, v_2, \ldots, v_{k-1}, u_1\}\rangle,$$
$$K_{k+1}^1 = \langle V(K_k^0)\cup\{u_2\}\rangle, \quad K_k^1 = K_{k+1}^1 - u_1,$$
$$K_{k+1}^2 = \langle V(K_k^1)\cup\{u_3\}\rangle, \quad K_k^2 = K_{k+1}^2 - u_2,$$
$$K_{k+1}^3 = \langle V(K_k^2)\cup\{u_1\}\rangle, \quad K_k^3 = K_{k+1}^3 - u_3 = K_k^0.$$

Thus, for each positive integer k, K_{k+2} is an example of a k-cycle.

PROPOSITION 1. *For any positive integer k, the complete $(k+1)$-partite graph $K(p_1, p_2, \ldots, p_{k+1})$, where $p_1 = p_2 = \ldots = p_{k-1} = 1$ and $p_k = p_{k+1} = 2$, is a k-cycle of length 4.*

3. k-acyclic graphs. Let k be a non-negative integer. A graph G is said to be *k-acyclic* if it has no k-cycles. (For $k = 0$, G has no lines.) We use the symbol A_k to denote the class of all k-acyclic graphs. It is easy to see that the complete graph K_{k+2} is $(k+1)$-acyclic, but not k-acyclic. Hence A_k is a proper subclass of A_{k+1} for each non-negative integer k.

A *totally disconnected graph* is one with no lines. Thus a graph is totally disconnected if and only if it is 0-acyclic. A *forest* is a graph without cycles, and these are exactly the 1-acyclic graphs (1-forests).

We now make some elementary observations about k-acyclic graphs.

PROPOSITION 2. (i) *If G is in A_k, then G is in A_n for each $n \geqslant k$.*

(ii) *For each graph G, there is a minimum non-negative integer k such that G is in A_k.*

(iii) *A graph G is in A_k if and only if each component of G is in A_k.*

(iv) *If G is in A_k, then each subgraph of G is in A_k.*

(v) *A graph G is in A_k if and only if each of its blocks is in A_k.*

4. k-point-arboricity of a graph. For any non-negative integer k, the *k-point-arboricity* $a_k(G)$ of the graph G is the minimum number of sets into which the point set $V(G)$ can be partitioned so that each set induces a k-acyclic subgraph of G. If $P = \{V_1, V_2, \ldots, V_n\}$ is a partition of $V(G)$ such that each set V_i, $i = 1, 2, \ldots, n$, induces a k-acyclic subgraph of G, then P is called a *k-acyclic partition* of G. The k-point-arboricity is then the minimum number of sets required in any k-acyclic partition of $V(G)$.

The 0-point-arboricity $a_0(G)$ is the extensively studied *chromatic number* of G, while the 1-point-arboricity $a_1(G)$ is the more recently investigated *point-arboricity* of G. For values of $k \geqslant 2$, the parameters $a_k(G)$ have not been considered as a topic of research. Each of the parameters $a_k(G)$ may be thought of as a coloring number, since it provides the minimum number of colors in any coloring of the points of G so that each color class induces a k-acyclic subgraph of G.

It is helpful, in determining $a_k(G)$, to note that we may restrict ourselves to connected graphs (see Proposition 2 (iii)).

PROPOSITION 3. *The value of* $a_k(G)$ *is the maximum of the values* $a_k(C_i)$ *for the components* C_i *of* G.

Since $a_k(G) \geqslant 1$ for each graph G and each non-negative integer k, and $a_n(G) \leqslant a_m(G)$ for each $n \geqslant m$, we make the following elementary observation:

PROPOSITION 4. *If* G *is a* k*-acyclic graph and* $m \geqslant k$, *then* $a_m(G) = 1$.

5. k-point-arboricity of the complete n-partite graphs. We now consider the k-point-arboricity of graphs in more detail. As one would expect, for most graphs G and for small values of k, the k-point-arboricity $a_k(G)$ is difficult to determine. However, for one important class of graphs, the complete n-partite graphs, the k-point-arboricity is easily calculated. For $k = 0$, the chromatic number or 0-point-arboricity of the complete n-partite graph is n. Chartrand et al. [5] exhibited a formula for $a_1(K(p_1, p_2, \ldots, p_n))$. We can now provide a formula for the k-point-arboricity, $k \geqslant 2$, of the complete n-partite graphs. For the remainder of this section, whenever we consider the complete n-partite graph $K(p_1, p_2, \ldots, p_n)$, we shall assume that the numbers $p_1, p_2, \ldots, p_n$ satisfy the inequality $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n$. We begin by making the following observations:

PROPOSITION 5. *Let* $k \geqslant 2$ *and let* G *denote the complete* n*-partite graph* $K(p_1, p_2, \ldots, p_n)$.

(i) *If* $n \leqslant k$, *then* $a_k(G) = 1$.

(ii) *If* $n = k+1$, *then*

$$a_k(G) = \begin{cases} 1 & \text{if } p_k = 1, \\ 2 & \text{if } p_k > 1. \end{cases}$$

(iii) *If* $k+2 \leqslant n \leqslant 2k$, *then* $a_k(G) = 2$.

(iv) *If* $n = 2k+1$, *then*

$$a_k(G) = \begin{cases} 2 & \text{if } p_k = 1, \\ 3 & \text{if } p_k > 1. \end{cases}$$

(v) *If* $n = 2k+2$, *then*

$$a_k(G) = \begin{cases} 2 & \text{if } p_{2k} = 1, \\ 3 & \text{if } p_{2k} > 1. \end{cases}$$

(vi) *If* $2k+3 \leqslant n \leqslant 3k$, *then* $a_k(G) = 3$.

For any real number r, we use the symbols $[r]$ and $\{r\}$ to denote the greatest integer not exceeding r and the least integer not less than r, respectively.

THEOREM 1. *Let $k \geqslant 2$, let $n \geqslant 2k+1$, and let h be an integer such that $0 \leqslant h \leqslant n$. Let G denote the complete n-partite graph $K(p_1, p_2, \ldots, p_n)$, where $p_i = 1$ if $i \leqslant h$, and $p_i > 1$ if $i > h$. Then*

$$a_k(G) = \max\left(\left\{\frac{n}{k+1}\right\}, \left\{\frac{n-[h/k]}{k}\right\}\right). \tag{2}$$

Proof. We begin the proof of this theorem by considering the special case where G is the complete n-partite graph $K(p_1, p_2, \ldots, p_n)$ with $p_1 = 1$ if $i \leqslant h$ and $p_i = 2$ if $i > h$. We prove that (2) holds in this case and then use this result to establish the general case. The proof for $k = 2$ needs to be handled in a slightly different manner, but it is very similar, and so we omit it here.

Let V_i, $i = 1, 2, \ldots, n$, be the partite sets of G as described in the definition of the complete n-partite graph. In this case $|V_i| = 1$ if $i \leqslant h$ and $|V_i| = 2$ if $i > h$.

We note that, for any $n \geqslant 2k+1$, if $h \geqslant kn/(k+1)$, then

$$\left\{\frac{n-[h/k]}{k}\right\} \leqslant \left\{\frac{n}{k+1}\right\},$$

and so we must show that $a_k(G) = \{n/(k+1)\}$. A k-acyclic partition of $V(G)$ into $\{n/(k+1)\}$ sets may be obtained by grouping the partite sets of G into groups of $k+1$, with k of them coming from the first h partite sets. Thus $a_k(G) \leqslant \{n/(k+1)\}$. For any partition of $V(G)$, if a set of the partition contains points from more than $k+1$ of the partite sets of G, then it induces a subgraph of G which contains a k-cycle. Thus $a_k(G) \geqslant \{n/(k+1)\}$.

We now use induction on n to prove formula (2). It follows from Proposition 5 (iv), (v) and (vi) that formula (2) holds for $n = 2k+1$, $2k+2$, and $2k+3$. Thus we assume that formula (2) holds for any m, $2k+1 \leqslant m < n$, and show that it also is valid for n. We consider the following cases:

(1) $h < k(n-k)/(k+1)$;

(2) $k(n-k)/(k+1) \leqslant h < kn/(k+1)$; and

(3) $kn/(k+1) \leqslant h$.

Case 3 was considered in the previous section, so we now treat cases 1 and 2.

Case 1. Assume that $h < k(n-k)/(k+1)$. Then

$$\left\{\frac{n-k}{k+1}\right\} \leqslant \left\{\frac{(n-k)-[h/k]}{k}\right\}.$$

Take the complete $(n-k)$-partite graph $H = K(p_1, p_2, \ldots, p_{n-k})$. The graph H is a subgraph of the graph G with exactly k fewer partite sets than G and each of the k partite sets, deleted from G to form H, contains exactly two points. Thus $a_k(H) \leqslant a_k(G) \leqslant a_k(H)+1$. The induction hypothesis implies that

$$a_k(H) = \left\{\frac{(n-k)-[h/k]}{k}\right\}.$$

We need to show that

$$a_k(G) = \left\{\frac{n-[h/k]}{k}\right\}, \tag{3}$$

so we assume that $a_k(G) = a_k(H) \equiv t$. Let U_i, $i = 1, 2, \ldots, t$, be any k-acyclic partition of $V(G)$ into t sets. If each of the sets U_i, $i = 1, 2, \ldots, t$, contains at most $k+2$ points, then $(k+2)t \geqslant 2n-h$, the number of points in G. But since

$$t = a_k(H) = \left\{\frac{(n-k)-[h/k]}{k}\right\},$$

we have the inequality

$$(k+2)\left\{\frac{(n-k)-[h/k]}{k}\right\} \geqslant 2n-h,$$

and solving for h we find that $h \geqslant kn/(k+1)$. Since, in this case, we assumed that $h < k(n-k)/(k+1)$, we have a contradiction. Thus at least one of the sets U_i, $i = 1, 2, \ldots, t$, contains at least $k+3$ points, say U_1. Then it follows from Proposition 1 that U_1 can only contain points from k of the partite sets of G. From this k-acyclic partition we will form a new k-acyclic partition W_i, $i = 1, 2, \ldots, t$, of $V(G)$ with t sets such that

$$W_1 = V_{n-k+1} \cup V_{n-k+2} \cup \ldots \cup V_n \equiv X.$$

If a point from one of the sets V_i, $i = 1, 2, \ldots, h$, is in U_1, then we exchange it with a point from a set V_j, $j = h+1, h+2, \ldots, n$, where $V_j \cap U_1 = \emptyset$. We have then constructed a new k-acyclic partition U_i', $i = 1, 2, \ldots, t$, of $V(G)$ with t sets such that U_1' contains only points of the sets V_j, $j = h+1, h+2, \ldots, n$. In view of the symmetry of the sets V_j, $j = h+1, h+2, \ldots, n$, we may assume that U_1' contains only points of X. We now add all the remaining points of X to U_1' to get a new k-acyclic partition W_i, $i = 1, 2, \ldots, t$, of $V(G)$ with t sets such that $W_1 = X$. Thus W_i, $i = 2, 3, \ldots, t$, is a k-acyclic partition of $V(H)$ with $t-1$ sets. This contradicts the fact that $a_k(H) = t$. Therefore, we must have (3).

Case 2. Assume that $k(n-k)/(k+1) \leqslant h < kn/(k+1)$. Then

$$\left\{\frac{n}{k+1}\right\} \leqslant \left\{\frac{n-[h/k]}{k}\right\} \quad \text{and} \quad \left\{\frac{(n-k)-[h/k]}{k}\right\} \leqslant \left\{\frac{n-k}{k+1}\right\}.$$

The induction hypothesis implies that $a_k(H) = \{(n-k)/(k+1)\}$. As in Case 1, $a_k(H) \leqslant a_k(G) \leqslant a_k(H)+1$, and so we have

$$\left\{\frac{n}{k+1}\right\} \leqslant \left\{\frac{n-[h/k]}{k}\right\} \leqslant \left\{\frac{n+1}{k+1}\right\} = \left\{\frac{n-k}{k+1}\right\}+1.$$

Thus we must show that $a_k(G) = \{(n-[h/k])/k\} = \{(n+1)/(k+1)\}$, and so we assume that $a_k(G) = a_k(H) = \{(n-k)/(k+1)\} \equiv t$. Let U_i, $i = 1, 2, \ldots, t$, be any k-acyclic partition of $V(G)$ into t sets. If each of the sets U_i, $i = 1, 2, \ldots, t$, contains at most $k+2$ points, then $(k+2)t \geqslant 2n-h$, the number of points in G. But since $t = \{(n-k)/(k+1)\}$, we have the inequality

$$(k+2)\left\{\frac{n-k}{k+1}\right\} \geqslant 2n-h,$$

and solving for h we find that $h \geqslant kn/(k+1)$. Since in this case we assumed that $h < kn/(k+1)$, we have a contradiction. Thus at least one of the sets U_i, $i = 1, 2, \ldots, t$, contains at least $k+3$ points, say U_1. Then, as in the proof of Case 1, there is a k-acyclic partition W_i, $i = 1, 2, \ldots, t$, of $V(G)$ into t sets, where $W_1 = X$. Thus W_i, $i = 2, 3, \ldots, t$, is a k-acyclic partition of $V(H)$ into $t-1$ sets. This contradicts the fact that $a_k(H) = t$. Therefore we must have (3).

We now return to the general case, that is, where p_i can have values other than 1 and 2. Let $G = K(p_1, p_2, \ldots, p_n)$, where $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n$. Let h be the maximum j such that $p_j = 1$ or let $h = 0$ if $p_1 > 1$. We consider two cases:

(4) $h \geqslant kn/(k+1)$, and

(5) $h < kn/(k+1)$.

Case 4. Assume that $h \geqslant kn/(k+1)$. We can find a k-acyclic partition of $V(G)$ into $\{n/(k+1)\}$ sets by grouping the partite sets of G into groups of $k+1$, with k of them coming from the first h sets. Thus $a_k(G) \leqslant \{n/(k+1)\}$. Since the complete n-partite graph $H = K(q_1, q_2, \ldots, q_n)$, where $q_i = 1$ if $i \leqslant h$ and $q_i = 2$ if $i > h$, is a subgraph of G, and $a_k(H) = \{n/(k+1)\}$, we have $a_k(G) \geqslant a_k(H) = \{n/(k+1)\}$.

Case 5. Assume that $h < kn/(k+1)$. Then we can obtain a k-acyclic partition of $V(G)$ into $\{(n-[h/k])/k\}$ sets as follows. For $i = 1, 2, \ldots, [h/k]$, let

$$U_i = V_{(i-1)k+1} \cup V_{(i-1)k+2} \cup \ldots \cup V_{ik} \cup V_{n-i+1}.$$

Then these $[h/k]$ sets U_i, $i = 1, 2, \ldots, [h/k]$, have used up exactly $(k+1)[h/k]$ of the partite sets of G. The remaining $n-(k+1)[h/k]$ partite sets can be grouped into $\{(n-(k+1)[h/k])/k\}$ sets each containing at most k of the remaining partite sets of G. Thus we have a k-acyclic partition of $V(G)$ into at most

$$\left[\frac{h}{k}\right]+\left\{\frac{n-(k+1)[h/k]}{k}\right\} = \left\{\frac{n-[h/k]}{k}\right\}$$

sets. Thus $a_k(G) \leqslant \{(n-[h/k])/k\}$. Since the complete n-partite graph H defined in Case 4 is a subgraph of G, and $a_k(H) = \{(n-[h/k])/k\}$, we have

$$a_k(G) \geqslant a_k(H) = \left\{\frac{n-[h/k]}{k}\right\}.$$

This completes the proof of Theorem 1.

We now have the following corollary providing the k-point-arboricity of the complete graphs:

COROLLARY 1 (a). *For the complete graph K_p with p points and for every non-negative integer k,*

$$a_k(K_p) = \left\{\frac{p}{k+1}\right\}.$$

Since every graph with p points can be considered as a subgraph of K_p, we obtain the following upper bound for the k-point-arboricity of a graph:

COROLLARY 1 (b). *For every graph G with p points and for every non--negative integer k,*

$$a_k(G) \leqslant \left\{\frac{p}{k+1}\right\}.$$

6. Critical graphs. In general, the bound given in Corollary 1 (b) is not particularly good. We shall improve this bound in Section 7 using a result from this section.

A graph G is said to be *n-critical with respect to k-point-arboricity* (or *with respect to a_k*) if $a_k(G) = n$ and $a_k(H) < a_k(G)$ for every proper subgraph H of G. Graphs which are critical with respect to a_0 (chromatic number) have been studied extensively; while in [4], [7], and [8], graphs critical with respect to a_1 (point-arboricity) were considered.

It is well known that any graph having chromatic number n contains an n-critical subgraph. Chartrand and Kronk [4] have established the analogue for point-arboricity. In the following proposition, we generalize these results for all values of k:

PROPOSITION 6. *If G is a graph with $a_k(G) = n$, then G contains a subgraph that is n-critical with respect to a_k.*

Any graph G that is n-critical with respect to chromatic number has the minimum degree $\delta(G) \geqslant n-1$. In [4], Chartrand and Kronk showed that every graph G which is n-critical with respect to point-

-arboricity satisfies $\delta(G) \geqslant 2(n-1)$. We now provide an analogue for all non-negative integers k.

THEOREM 2. *If G is a graph which is n-critical with respect to a_k, then $\delta(G) \geqslant (k+1)(n-1)$.*

Proof. There are no non-trivial graphs that are 1-critical with respect to a_k, and so we assume that $n \geqslant 2$. Let G be a graph which is n-critical with respect to a_k and assume that there is a point v of G with $d(v) < (k+1)(n-1)$. Since G is n-critical with respect to a_k, $a_k(G-v) = n-1$. Thus we can find a k-acyclic partition U_i, $i = 1, 2, \ldots, n-1$, of $V(G-v)$ into $n-1$ sets. Since $d(v) < (k+1)(n-1)$, at least one of these sets, say U_1, contains at most k points adjacent to v. Then $U_1 \cup \{v\}$ induces a k-acyclic subgraph of G, and so $U_1 \cup \{v\}, U_2, \ldots, U_{n-1}$ is a k-acyclic partition of G into $n-1$ sets. This contradicts our assumption that $a_k(G) = n$, and so $\delta(G) \geqslant (k+1)(n-1)$.

Any graph with at most $k+1$ points is k-acyclic. Thus, if G is a graph with $a_k(G) = n$, then G must have at least $(k+1)(n-1)+1$ points. This implies that any graph G which is n-critical with respect to a_k must also have at least $(k+1)(n-1)+1$ points. It is not difficult to show that the unique graph G with $a_k(G) = n$ and with $(k+1)(n-1)+1$ points is $G = K_{(k+1)(n-1)+1}$.

7. Bounds for the k-point-arboricity. As mentioned in Section 6, the upper bound for the k-point-arboricity given in Corollary 1 (b) is not particularly good. We now present an upper bound that is generally sharper, together with a lower bound. For a graph G and a non-negative integer k, let $M_k(G)$ denote the maximum number of points of G which induce a k-acyclic subgraph of G. The number $M_0(G)$ has been called the *point-independence number* of G.

PROPOSITION 7. *Let G be a graph with p points and let k be a non-negative integer. Then*

$$\left\{\frac{p}{M_k(G)}\right\} \leqslant a_k(G) \leqslant \left\{\frac{p-M_k(G)}{k+1}\right\}+1. \tag{4}$$

The proof of this result is omitted, since similar proofs can be found in [3].

In order to provide additional upper bounds for the k-point-arboricity of a graph, some further definitions are necessary. A set S of lines of G is called a *cutset* of G if $G-S$, the subgraph of G obtained by removing the lines of S, has more components than G. If $|S| = n$, then S is called an *n-cutset*. The minimum number of lines in any cutset of the connected graph G is called the *line-connectivity* of G, and is denoted by $\lambda(G)$. In [10], Matula defined the *strength* $\sigma(G)$ of the graph G as follows:

$$\sigma(G) = \max\{\lambda(H)\colon H \text{ is a subgraph of } G\}.$$

He then showed that $a_0(G) \leqslant 1+\sigma(G)$. The author [8] extended this result to point-arboricity $a_1(G) \leqslant 1+[\sigma(G)/2]$.

These two results are now generalized to k-point-arboricity. To do this, the following proposition is required:

PROPOSITION 8. *Let S be an n-cutset of the graph G and let G_1 and G_2 be disjoint induced subgraphs of $G-S$ such that $G = \langle V(G_1) \cup V(G_2)\rangle$. Then*

$$a_k(G) \leqslant \max\left(a_k(G_1),\, a_k(G_2),\, 1+\left[\frac{n}{k+1}\right]\right). \tag{5}$$

Proof. Let U_i, $i = 1, 2, \ldots, a_k(G_1)$, be a k-acyclic partition of G_1 ordered so that each of $U_1, U_2, \ldots, U_t$ is incident with at least $k+1$ lines of S, and each of $U_{t+1}, U_{t+2}, \ldots, U_{a_k(G_1)}$ is incident with at most k lines of S. Let W_i, $i = 1, 2, \ldots, a_k(G_2)$, be a k-acyclic coloring of G_2 such that the points of W_i are colored with the color i. We now color the points of G_1 in order to produce a k-acyclic coloring of G. For each j, $1 \leqslant j \leqslant a_k(G_1)$, let the color assigned to U_j be the minimum positive integer different from that color assigned to U_i, $1 \leqslant i < j$, and different from the color s if there are at least $k+1$ lines of S joining points of U_j to points of W_s.

Since there are at most $[n/(k+1)]-t+1$ sets of at least $k+1$ lines of S joining the set U_j to a set W_s, the choice of a color for U_j need only be from $j+[n/(k+1)]-t+1$ different colors. Hence the maximum number of colors required for the sets $U_1, U_2, \ldots, U_t$ is at most $1+[n/(k+1)]$. Furthermore, the color c associated with the set U_j, $t < j \leqslant a_k(G_1)$, satisfies the inequality $c \leqslant j$. Thus the maximum number of colors required for G_1 in this k-acyclic coloring of G is at most

$$\max\left(a_k(G_1),\, 1+\left[\frac{n}{k+1}\right]\right).$$

Therefore, this k-acyclic coloring of G requires at most

$$\max\left(a_k(G_1),\, a_k(G_2),\, 1+\left[\frac{n}{k+1}\right]\right)$$

colors, and inequality (5) is established.

We note that inequality (5) must produce an equality unless the maximum on the right-hand side is attained by $1+[n/(k+1)]$. This is the case we now study. The following theorem provides an inequality relating the k-point-arboricity and the line-connectivity of critical graphs:

THEOREM 3. *Let G be a graph which is $a_k(G)$-critical with respect to k-point-arboricity. Then*

$$a_k(G) \leqslant 1+\left[\frac{\lambda(G)}{k+1}\right]. \tag{6}$$

Proof. Let S be a $\lambda(G)$-cutset of G and let G_1 and G_2 be the components of $G-S$. It follows from Proposition 8 that

$$a_k(G) \leqslant \max\left(a_k(G_1), a_k(G_2), 1+\left[\frac{\lambda(G)}{k+1}\right]\right).$$

Since G is critical with respect to a_k, we have $a_k(G_1) < a_k(G)$, and $a_k(G_2) < a_k(G)$, and thus inequality (6) is established.

Theorem 2 now follows as a corollary to Theorem 3, since $\lambda(G) \leqslant \delta(G)$.

Clearly, (6) does not hold for arbitrary graphs, since two complete graphs $K_{(k+1)(n-1)+1}$ joined by a line have k-point-arboricity n, but line-connectivity one. We now provide an upper bound on the k-point-arboricity in terms of the strength of the graph.

THEOREM 4. *For any graph G,*

$$a_k(G) \leqslant 1+\left[\frac{\sigma(G)}{k+1}\right]. \tag{7}$$

Proof. Assume that G has the k-point-arboricity $a_k(G)$. Proposition 6 may be applied to find a subgraph H of G which is $a_k(G)$-critical with respect to a_k. Then Theorem 3 implies that $a_k(H) \leqslant 1+[\lambda(H)/(k+1)]$. Since $\lambda(H) \leqslant \sigma(G)$, inequality (7) is proved.

8. k-point-arboricity of planar graphs. We now consider the 2-point-arboricity and the 3-point-arboricity of planar graphs.

PROPOSITION 9. *If G is a planar graph, then*

$$a_2(G) \leqslant 2, \tag{8}$$

and this inequality is best possible.

Since any subgraph of a planar graph is a planar graph, and any planar graph has a point of degree at most 5, it follows that the strength of any planar graph is at most 5. Now inequality (8) follows from Theorem 4.

In order to show that inequality (8) is best possible, we shall make use of the class of graphs called *wheels*. For any integer $n \geqslant 4$, the *wheel* W_n is a 1-cycle of length $n-1$ and a point v not on the cycle such that each point of the cycle is adjacent to the point v. The graph K_4 is the wheel W_4. Then, for each integer $n \geqslant 4$, W_n is a planar graph which is a 2-cycle. Thus $a_2(W_n) = 2$.

Likewise, Theorem 4 implies that the 3-point-arboricity of a planar graph is at most 2. However, the following proposition improves this bound:

PROPOSITION 10. *If G is a planar graph, then $a_3(G) = 1$.*

In order to prove this result, we need only to show that a planar graph cannot contain a 3-cycle as a subgraph.

Let G be any planar graph and assume that the 3-cycle H is a subgraph of G. From the definition of 3-cycles it follows that if H has order p, then H contains a maximal 3-degenerate graph of order p as a proper subgraph (see [9]). Since any maximal 3-degenerate graph of order p has $3p-6$ lines, H must have at least $3p-5$ lines. But this contradicts the fact that a planar graph of order p contains at most $3p-6$ lines. Thus G does not have any 3-cycles as subgraphs, and so $a_3(G) = 1$.

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WESTERN MICHIGAN UNIVERSITY
CALIFORNIA STATE UNIVERSITY, LOS ANGELES
UNIVERSITY OF CALIFORNIA, IRVINE

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