ANNALES
POLONICI MATHEMATICI
XXXI (1975)

Existence and uniqueness of solutions of non-linear functional equations of r-th order

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Abstract. In this paper there are studied the existence, uniqueness and convergence of successive approximation problems for the non-linear functional equation of the order r

$$(*) \qquad x(t) = f\big(t, x(\beta_1(t)), \ldots, x(\beta_r(t))\big),$$

where an unknown function x is defined in the metric space (M_1, ϱ_1) and takes the values in the complete metric space (M_2, ϱ_2). This is done by use of the iteration method. Error estimations and a theorem on the continuous dependence of a solution on the right-hand side of equation $(*)$ are also established.

In the present paper we consider the non-linear functional equation of the order r

$$(1) \qquad x(t) = f\big(t, x(\beta_1(t)), \ldots, x(\beta_r(t))\big),$$

where an unknown function x is defined in the metric space (M_1, ϱ_1) and takes the values in the complete metric space (M_2, ϱ_2).

The particular case of equation (1) was considered by many authors, see: [1], [5]–[8] and [11], and under suitable additional assumptions on the spaces M_1, M_2, the special properties of the solutions of equation (1) such as monotonicity, convexity and regularity were also investigated. In particular, in papers [6]–[8] the existence of the solutions of equation (1) in special classes of functions was considered in the case $M_1 = M_2 = R$, R being a real number space. The question of the existence of convex solutions of equation (1) was considered in [1], [6] and [7] with suitable assumptions on M_1, M_2.

The case of continuous solutions of equation (1) under the assumption that β_i, $i = 1, \ldots, r$, are successive iterates of the function β_1, and $M_1 = R$, $M_2 = R^n$ was considered in [5].

However, in papers [3] and [9] the problem of the existence and uniqueness of solutions of functional equations of the form

$$f(t) = \sup_q F\big(t, q, f(T_1(t, q)), \ldots, f(T_r(t, q))\big),$$

with $M_1 = R^n$ and $M_2 = R^m$ was discussed. This equation in the case of the functions F and T_i, $i = 1, \ldots, r$, being independent on the parameter q is also a particular case of equation (1). The particular case of the integral-functional equation considered in [4] can also be reduced to equation (1).

Finally, in paper [1] equation (1) was discussed in the case when M_1 is a metric space and M_2 is a normed space with $\dim M_2 < \infty$. In this paper, under suitable assumptions, solutions are looked for in the class of functions fulfilling a Lipschitz condition.

Almost all authors use the well-known Banach fixed point theorem in proving the existence and uniqueness of results for cases similar to equation (1). Unfortunately this method involves a strong condition concerning the function f. This condition can be slightly weakened if it is supposed more on the functions β_i, $i = 1, \ldots, r$. Consequently in [1], [4], [6] and [8] conditions involving some relation between the Lipschitz constants of the function f and the estimations imposed on the functions β_i appear.

In this paper we shall consider the problem of the existence, uniqueness and convergence of successive approximations and the continuous dependence of solutions on the right-hand side of equation (1) in the general case indicated above. We use the method of iteration. The assumptions introduced in this paper seem to be so weak as it enables the method used here.

The particular cases of our theorems are the theorems established in [1], [6], [8] and [11].

The method used in this paper is very close to that used in [3] and [9], the general idea of which is presented in papers [10] and [12].

1. We introduce:

ASSUMPTION A. *Suppose that*

1° $f\colon M_1 \times M_2^r \to M_2$, $\beta_i\colon M_1 \to M_1$, *where r is a fixed positive integer number and (M_1, ϱ_1) is a metric space, (M_2, ϱ_2) is a complete metric space; t_0 is an arbitrarily fixed point of M_1,*

2° *there exists a non-decreasing function* $\omega\colon R_+^{r+1} \to R_+ \overset{\mathrm{df}}{=} [0, +\infty)$ *such that the function* $\omega_t\colon \omega_t(u_1, \ldots, u_r) \overset{\mathrm{df}}{=} \omega(t, u_1, \ldots, u_r)$, $u_i \in R_+$ *is continuous for each* $t \in R_+$, *the condition* $\omega(t, 0, \ldots, 0) \equiv 0$ *and*

$$(2)\qquad \varrho_2\big(f(t, x_1, \ldots, x_r), f(t, \bar{x}_1, \ldots, \bar{x}_r)\big) \leqslant \omega\big(\varrho_1(t, t_0), \varrho_2(x_1, \bar{x}_1), \ldots, \varrho_2(x_r, \bar{x}_r)\big),$$

for any $(t, x_1, \ldots, x_r)$, $(t, \bar{x}_1, \ldots, \bar{x}_r) \in M_1 \times M_2^r$ *hold true,*

3° *there exist non-decreasing functions* $a_i\colon R_+ \to R_+$, $i = 1, \ldots, r$, *and*

$$(3)\qquad \varrho_1\big(\beta_i(t), t_0\big) \leqslant a_i\big(\varrho_1(t, t_0)\big), \quad t \in M_1,\ i = 1, \ldots, r.$$

ASSUMPTION B. *Suppose that:*

1° *for a fixed function* $x_0\colon M_1 \to M_2$ *there exists the non-decreasing function* $\bar{u}\colon R_+ \to R_+$ *being a solution of the inequality*

$$\omega\big(c, u(\alpha_1(c)), \ldots, u(\alpha_r(c))\big) + h(c) \leqslant u(c), \quad c \in R_+, \tag{4}$$

where

$$h(c) = \sup_{t \in K(t_0, c)} \varrho_2\Big(x_0(t), f\big(t, x_0(\beta_1(t)), \ldots, x_0(\beta_r(t))\big)\Big),$$

$$K(t_0, c) \stackrel{\mathrm{df}}{=} [t\colon\ \varrho_1(t, t_0) \leqslant c,\ c \geqslant 0],$$

2° *in the class of functions satisfying the condition* $0 \leqslant u(c) \leqslant \bar{u}(c)$, $c \in R_+$, *the function* $u(c) \equiv 0$, $c \in R_+$ *is the only solution of the equation*

$$u(c) = \omega\big(c, u(\alpha_1(c)), \ldots, u(\alpha_r(c))\big). \tag{5}$$

Now let us construct the sequence $\{u_n\}$ by the relations

$$\begin{aligned} &u_0(c) = \bar{u}(c), \\ &u_{n+1}(c) = \omega\big(c, u_n(\alpha_1(c)), \ldots, u_n(\alpha_r(c))\big), \quad n = 0, 1, \ldots,\ c \in R_+. \end{aligned} \tag{6}$$

We have

LEMMA 1. *If conditions* 2°, 3° *of* A *and* B *are satisfied, then*

$$0 \leqslant u_{n+1}(c) \leqslant u_n(c) \leqslant \bar{u}(c), \quad n = 0, 1, \ldots, \tag{7}$$

$$u_n \rightrightarrows 0 \quad \textit{for } n \to \infty,\ c \in R_+,$$

where the sign $\rightrightarrows$ *denotes uniform convergence in any compact subset of* R_+.

Proof. We obtain relation (7) by induction. The convergence of the sequence $\{u_n\}$ is implied by (7). The limit of this sequence satisfies equation (5) and by Assumption B it must be identically equal zero. The uniform convergence of $\{u_n\}$ on compact subsets of R_+ follows from the monotonicity of sequence $\{u_n\}$ and of all the functions u_n.

Let us define the sequence $\{x_n\}$ by the relations

$$x_{n+1}(t) = f\big(t, x_n(\beta_1(t)), \ldots, x_n(\beta_r(t))\big), \quad n = 0, 1, \ldots,\ t \in M_1, \tag{8}$$

with a fixed x_0 (see Assumption B).

LEMMA 2. *If Assumptions* A *and* B *are satisfied, then*

$$\sup_{t \in K(t_0, c)} \varrho_2\big(x_n(t), x_0(t)\big) \leqslant \bar{u}(c), \quad n = 0, 1, \ldots,\ c \in R_+, \tag{9}$$

and

$$\sup_{t \in K(t_0, c)} \varrho_2\big(x_{n+p}(t), x_n(t)\big) \leqslant u_n(c), \quad n, p = 0, 1, \ldots,\ c \in R_+. \tag{10}$$

Proof. It is obvious that (9) holds for $n = 0$. If we suppose that (9) holds for some $n \geqslant 0$, then we have for $t \in K(t_0, c)$,

$$\varrho_2\big(x_{n+1}(t), x_0(t)\big) \leqslant \varrho_2\Big(f\big(t, x_n(\beta_1(t)), \ldots, x_n(\beta_r(t))\big), f\big(t, x_0(\beta_1(t)), \ldots \\ \ldots, x_0(\beta_r(t))\big)\Big) + \varrho_2\Big(x_0(t), f\big(t, x_0(\beta_1(t)), \ldots, x_0(\beta_r(t))\big)\Big)$$

$$\leqslant \omega\Big(\varrho_1(t, t_0), \varrho_2\big(x_n(\beta_1(t)), x_0(\beta_1(t))\big), \ldots, \varrho_2\big(x_n(\beta_r(t)), x_0(\beta_r(t))\big)\Big) + h(c)$$

$$\leqslant \omega\Big(c, \sup_{t\in K(t_0,\alpha_1(c))} \varrho_2\big(x_n(t), x_0(t)\big), \ldots, \sup_{t\in K(t_0,\alpha_r(c))} \varrho_2\big(x_n(t), x_0(t)\big)\Big) + h(c)$$

$$\leqslant \omega\big(c, \overline{u}(\alpha_1(c)), \ldots, \overline{u}(\alpha_r(c))\big) + h(c) \leqslant \overline{u}(c).$$

Hence

$$\sup_{t\in K(t_0,c)} \varrho_2\big(x_{n+1}(t), x_0(t)\big) \leqslant \overline{u}(c).$$

Now (9) follows by induction.

Now we prove (10). From (9) it follows that (10) holds for $n = 0$, $p = 0, 1, \ldots$ Further, if we suppose that (10) is true for $n, p \geqslant 0$, then for $t \in K(t_0, c)$,

$$\varrho_2\big(x_{n+p+1}(t), x_{n+1}(t)\big) \\ \leqslant \omega\Big(c, \sup_{t\in K(t_0,\alpha_1(c))} \varrho_2\big(x_{n+p}(t), x_n(t)\big), \ldots, \sup_{t\in K(t_0,\alpha_r(c))} \varrho_2\big(x_{n+p}(t), x_n(t)\big)\Big) \\ \leqslant \omega\big(c, u_n(\alpha_1(c)), \ldots, u_n(\alpha_r(c))\big) = u_{n+1}(c).$$

Now we obtain (10) by induction. Thus the proof of Lemma 2 is completed.

This enables formulation of the following:

THEOREM 1. *If Assumptions* A *and* B *are satisfied, then there exists a solution* $\overline{x}$ *of equation* (1), *being the limit of the sequence* $\{x_n\}$ *defined by* (8). *The estimations*

$$\sup_{t\in K(t_0,c)} \varrho_2\big(\overline{x}(t), x_0(t)\big) \leqslant \overline{u}(c), \quad c \in R_+, \tag{11}$$

and

$$\sup_{t\in K(t_0,c)} \varrho_2\big(\overline{x}(t), x_n(t)\big) \leqslant u_n(c), \quad n = 0, 1, \ldots, \ c \in R_+, \tag{12}$$

hold true. The solution $\overline{x}$ *of* (1) *is unique in the class of functions satisfying relation* (11).

Proof. The convergence of the sequence $\{x_n\}$ follows from Lemmas 1 and 2. If $p \to \infty$, then (10) yields estimation (12). Estimation (11) is implied by (12). By the estimation

$$0 \leqslant \sup_{t\in K(t_0,c)} \varrho_2\Big(f\big(t, \overline{x}(\beta_1(t)), \ldots, \overline{x}(\beta_r(t))\big), \overline{x}(t)\Big) \leqslant 2u_n(c), \quad n = 0, 1, \ldots, \ c \in R_+,$$

it follows that the function $\overline{x}$ satisfies equation (1).

To prove that the solution $\bar{x}$ is a unique solution of (1) in the class of functions satisfying relation (11) let us suppose that there exists another solution $\bar{\bar{x}}$ such that $\bar{\bar{x}}(t) \not\equiv \bar{x}(t)$ and

$$\sup_{t\in K(t_0,c)} \varrho_2\big(\bar{\bar{x}}(t), x_0(t)\big) \leqslant \bar{u}(c), \qquad c\in R_+ .$$

We get

$$\sup_{t\in K(t_0,c)} \varrho_2\big(\bar{\bar{x}}(t), x_n(t)\big) \leqslant u_n(c), \qquad n = 0, 1, \ldots,$$

by induction, and hence $\bar{\bar{x}}(t) = \bar{x}(t)$. This contradiction proves the uniqueness of $\bar{x}$ in the class indicated above. Thus the proof of Theorem 1 is completed.

2. In order to formulate a theorem on the continuous dependence of the solution of equation (1) on the right-hand side we consider the second equation

$$y(t) = g\big(t, y(\gamma_1(t)), \ldots, y(\gamma_r(t))\big), \tag{13}$$

where the functions g, γ_i, $i = 1, \ldots, r$, have the same properties as f, β_i, $i = 1, \ldots, r$.

Suppose that there exists a solution $\bar{y}$ of (13). Let

$$\bar{h}(c) = \sup_{t\in K(t_0,c)} \varrho_2\Big(f\big(t, \bar{y}(\beta_1(t)), \ldots, \bar{y}(\beta_r(t))\big), \bar{y}(t)\Big),$$

$$v_0(c) \geqslant \sup_{t\in K(t_0,c)} \varrho_2\big(\bar{x}(t), \bar{y}(t)\big),$$

$$v_{n+1}(c) = \omega\big(c, v_n(a_1(c)), \ldots, v_n(a_r(c))\big) + \bar{h}(c), \qquad n = 0, 1, \ldots, \quad c\in R_+ .$$

Now we shall prove the following:

THEOREM 2. *If Assumption A is satisfied and*

1° $\bar{x}$ *and* $\bar{y}$ *are solutions of equations* (1) *and* (13) *respectively,*

2° *there exists the limit* $\bar{v}$ *of the sequence* $\{v_n\}$, *then*

$$\sup_{t\in K(t_0,c)} \varrho_2\big(\bar{x}(t), \bar{y}(t)\big) \leqslant \bar{v}(c), \qquad c\in R_+ . \tag{14}$$

Proof. Put

$$v(c) = \sup_{t\in K(t_0,c)} \varrho_2\big(\bar{x}(t), \bar{y}(t)\big).$$

By Assumption A we get

$$\varrho_2\big(\bar{x}(t), \bar{y}(t)\big) \leqslant \varrho_2\Big(f\big(t, \bar{x}(\beta_1(t)), \ldots, \bar{x}(\beta_r(t))\big), f\big(t, \bar{y}(\beta_1(t)), \ldots, \bar{y}(\beta_r(t))\big)\Big) + \\ + \bar{h}(c) \leqslant \omega\big(c, v(a_1(c)), \ldots, v(a_r(c))\big) + \bar{h}(c).$$

Hence we have

$$v(c) \leqslant \omega\big(c, v(a_1(c)), \ldots, v(a_r(c))\big) + \bar{h}(c).$$

Since $v(c) \leqslant v_0(c)$, from the above consideration we get

$$v(c) \leqslant v_n(c), \quad n = 0, 1, \ldots,$$

by induction. Inequality (14) is implied by this inequality as $n \to \infty$.

Remark 1. Assumption 2° of Theorem 2 can be replaced by the following assumption: there exists a non-negative function w_0 satisfying the inequality

$$\omega\big(c, w_0(a_1(c)), \ldots, w_0(a_r(c))\big) + \max[\bar{h}(c), v_0(c)] \leqslant w_0(c).$$

Now in the class of functions satisfying the condition $0 \leqslant u(c) \leqslant w_0(c)$ there exists a function $\bar{w}$ being a solution of the equation

$$\omega\big(c, u(a_1(c)), \ldots, u(a_r(c))\big) + \bar{h}(c) = u(c), \quad c \in R_+.$$

Indeed, if we put

$$w_{n+1}(c) = \omega\big(c, w_n(a_1(c)), \ldots, w_n(a_r(c))\big) + \bar{h}(c), \quad n = 0, 1, \ldots, \ c \in R_+,$$

we see that $v(c) \leqslant w_n(c)$ and $w_{n+1}(c) \leqslant w_n(c)$, $n = 0, 1, \ldots$, $c \in R_+$, i.e. $w_n(c) \to \bar{w}(c)$. From this we obtain an assertion of Theorem 2 with $\bar{w}$ instead of $\bar{v}$.

3. We shall consider the function ω for which we are able to give effective conditions under which Assumption B is fulfilled.

Firstly we assume that

$$\omega(c, v_1, \ldots, v_r) = \sum_{i=1}^{r} l_i(c) v_i, \tag{15}$$

where $l_i\colon R_+ \to R_+$ and that they are non-decreasing functions.

It is easy to prove that inequality (4) with ω of the form (15) has a solution in the class of bounded, non-decreasing and non-negative function if h is bounded and

$$\sup_{c \in R_+} \sum_{i=1}^{r} l_i(c) < 1. \tag{16}$$

A condition of the form (16) was obtained in [11].

Now we formulate two lemmas which permit us to weaken condition (16).

Let

$$a_0(c) = c, \quad a_{n+1}^{i_1,\ldots,i_{n+1}}(c) \overset{\text{df}}{=} a_n^{i_1,\ldots,i_n}\big(a_{i_{n+1}}(c)\big),$$

$$l_0(c) = 1, \quad l_{n+1}^{i_1,\ldots,i_{n+1}}(c) \overset{\text{df}}{=} l_{i_{n+1}}(c)\, l_n^{i_1,\ldots,i_n}\big(a_{i_{n+1}}(c)\big), \quad i_n = 1, \ldots, r,$$

$$n = 0, 1, \ldots, \ c \in R_+,$$

$$\sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} \varphi_n^{i_1,\ldots,i_n} \overset{\text{df}}{=} \varphi_0 + \sum_{n=1}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} \varphi_n^{i_1,\ldots,i_n}.$$

In further considerations the case $r = +\infty$ is admitted.

The following lemmas can be proved in a similar way to Lemmas 9 and 10 in [3].

LEMMA 3. *For any function* $h\colon R_+ \to R_+$ *the condition*

$$\sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} l_n^{i_1,\ldots,i_n}(c)\, h\big(a_n^{i_1,\ldots,i_n}(c)\big) < +\infty, \qquad c \in R_+, \tag{17}$$

is necessary and sufficient for the equation

$$u(c) = \sum_{i=1}^{r} l_i(c)\, u\big(a_i(c)\big) + h(c), \qquad c \in R_+, \tag{18}$$

to have a non-negative solution $\bar{u}$.

If condition (17) *is fulfilled, then the function* $\bar{u}$,

$$\bar{u}(c) = \sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} l_n^{i_1,\ldots,i_n}(c)\, h\big(a_n^{i_1,\ldots,i_n}(c)\big), \qquad c \in R_+, \tag{19}$$

is a solution of equation (18), *and*

$$\lim_{n\to\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} l_n^{i_1,\ldots,i_n}(c)\, \bar{u}\big(a_n^{i_1,\ldots,i_n}(c)\big) = 0, \qquad c \in R_+.$$

There is no other solution of equation (18) *in the class of functions*

$$V \overset{\mathrm{df}}{=} \bigcup_{b\geqslant 0} [u\colon\ 0 \leqslant u(c) \leqslant b\bar{u}(c),\ c \in R_+].$$

We note that if

$$h_0(c) \overset{\mathrm{df}}{=} h(c),$$

$$h_{n+1}(c) \overset{\mathrm{df}}{=} \sum_{i=1}^{r} l_i(c)\, h_n\big(a_i(c)\big), \qquad n = 0, 1, \ldots, \quad c \in R_+,$$

then by the equality

$$h_n(c) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} l_n^{i_1\ldots i_n}(c)\, h\big(a_n^{i_1\ldots i_n}(c)\big), \quad n = 0, 1, \ldots, \quad c \in R_+,$$

condition (17) is equivalent to the condition

$$\sum_{n=0}^{\infty} h_n(c) < +\infty, \qquad c \in R_+, \tag{17'}$$

and consequently

$$\bar{u}(c) = \sum_{n=0}^{\infty} h_n(c).$$

However, we introduce $\bar{u}$ by formula (19) because under suitable assumptions it permits us to obtain further on the effective conditions in order that (17) or (17′) hold.

LEMMA 4. *If $\bar{u}$ is of the form* (19) *and the function $u \in V$ satisfies the inequality*

$$u(c) \leqslant \sum_{i=1}^{r} l_i(c)\, u\big(a_i(c)\big),$$

then $u(c) \equiv 0$, $c \in R_+$.

Remark 2 [9]. If $r = 1$, $a(c) \overset{\text{df}}{=} a_1(c)$, $l(c) \overset{\text{df}}{=} l_1(c)$, $c \in R_+$, then the sequences $\{a_n\}$, $\{l_n\}$ are of the form

$$a_0(c) = c, \quad a_{n+1}(c) = a\big(a_n(c)\big), \quad n = 0, 1, \ldots,$$

$$l_0(c) = 1, \quad l_{n+1}(c) = \prod_{i=0}^{n} l\big(a_i(c)\big), \quad c \in R_+,\ n = 0, 1, \ldots$$

In this case formulas (18) and (19) take the form

$$u(c) = l(c)\, u\big(a(c)\big) + h(c),$$

and

$$\bar{u}(c) = \sum_{n=0}^{\infty} l_n(c)\, h\big(a_n(c)\big).$$

From Theorem 1, Lemmas 3 and 4 we infer:

THEOREM 3. *If Assumption* A *and* (15) *are satisfied, and condition* (17) *holds with*

$$h(c) = \sup_{t \in K(t_0, c)} \varrho_2\Big(x_0(t), f\big(t, x_0(\beta_1(t)), \ldots, x_0(\beta_r(t))\big)\Big),$$

then there exists a solution $\bar{x}$ of equation (1) *with the following properties*

$$\sup_{t \in K(t_0, c)} \varrho_2\big(\bar{x}(t), x_0(t)\big) \leqslant \sum_{k=0}^{\infty} \sum_{i_1=1}^{r} \ldots \sum_{i_k=1}^{r} l_k^{i_1, \ldots, i_k}(c)\, h\big(a_k^{i_1, \ldots, i_k}(c)\big),$$

$$\sup_{t \in K(t_0, c)} \varrho_2\big(\bar{x}(t), x_n(t)\big) \leqslant \sum_{k=n}^{\infty} \sum_{i_1=1}^{r} \ldots \sum_{i_k=1}^{r} l_k^{i_1, \ldots, i_k}(c)\, h\big(a_k^{i_1, \ldots, i_k}(c)\big),$$

$$n = 0, 1, \ldots, \ c \in R_+.$$

The solution $\bar{x}$ of equation (1) *is unique in the class of functions*

$$X(M_1, M_2) \overset{\text{df}}{=} \bigcup_{b \geqslant 0} \Big[x\colon \sup_{t \in K(t_0, c)} \varrho_2\big(x(t), x_0(t)\big) \leqslant b\bar{u}(c),\ c \in R_+\Big].$$

Now Theorem 2 and Remark 1 imply:

THEOREM 4. *If Assumption* A *is satisfied and*

1° *$\bar{x}$ and $\bar{y}$ are solutions of equations* (1) *and* (13) *respectively,*

2° *condition* (17) *is satisfied with* $h(c) = \max[\bar{h}(c), v_0(c)]$, *where*

$$\bar{h}(c) = \sup_{t\in K(t_0,c)} \varrho_2\Big(f\big(t, \bar{y}(\beta_1(t)), \ldots, \bar{y}(\beta_r(t))\big), \bar{y}(t)\Big), \qquad c\in R_+,$$

then

$$\sup_{t\in K(t_0,c)} \varrho_2\big(\bar{x}(t), \bar{y}(t)\big) \leqslant \sum_{n=0}^{\infty}\sum_{i_1=1}^{r}\ldots\sum_{i_n=1}^{r} l_n^{i_1,\ldots,i_n}(c)\,\bar{h}\big(a_n^{i_1,\ldots,i_n}(c)\big).$$

From Theorem 3 it follows that condition (16) is weakened if l_i, a_i, h are continuous at zero and $a_i(0) = 0$, $i = 1, \ldots, r$. The continuity of the function h can be assured by the continuity of functions x_0, β_i, $i = 1, \ldots, r$, at t_0 and by the continuity of the function f at the point $(t_0, \eta, \ldots, \eta)$, where $\eta = \lim\limits_{t\to t_0} x_0(t)$.

To illustrate this fact we consider the following particular cases and we point out the conditions sufficient for condition (17) to be fulfilled.

(a) If $l_i(c) \leqslant l_i c$, $a_i(c) \leqslant a_i c$, $h(c) \leqslant h = \text{const}$, l_i, a_i are non-negative constants, and if $a_i < 1$, then condition (17) is fulfilled for any l_i, $c\in R_+$.

We note that now $l_n^{i_1,\ldots,i_n}(c) \leqslant c^n \prod\limits_{m=1}^{n} l_{i_m} a_{i_m}^{m-1}$.

Remark 3. In case (a) condition (16) is fulfilled locally and the method of successive extension can be applied in order to prove an non-local existence theorem.

If the function x_0 is continuous at t_0 and $x_0(t_0) = \eta$, $\eta = f(t_0, \eta, \ldots, \eta)$, then Theorem 3 gives a condition essentially weaker than condition (16). We observe this fact in the following cases:

(b) If $l_i(c) \leqslant l_i$, $a_i(c) \leqslant a_i c$, $h(c) \leqslant h c^p$, $p \geqslant 0$, l_i, a_i are non-negative constants, $i = 1, \ldots, r$, then condition (17) is fulfilled if

$$\sum_{i=1}^{r} l_i a_i^p < 1. \tag{20}$$

In this case we have $l_n^{i_1,\ldots,i_n}(c) \leqslant \prod\limits_{m=1}^{n} l_{i_m}$, $a_n^{i_1,\ldots,i_n}(c) \leqslant c \prod\limits_{m=1}^{n} a_{i_m}$.

We note that now the case $a_i > 1$, $i = 1, \ldots, r$, is not excluded.

Remark 4. A condition of the form (20) in the case $p = 1$ was obtained in [1]. In this paper it is assumed that f satisfies the Lipschitz condition also with respect to t and that a_i are Lipschitz coefficients of the functions β_i.

In the case $p = r = 1$ condition (20) was obtained also in [8].

Remark 5. The following example (see [13])

$$u(c) = \sum_{i=1}^{\infty} u\left(\frac{c}{2^i}\right) + hc^p, \quad c \in R_+, \tag{21}$$

shows that condition (20) is essential. For this equation condition (20) has the form $\sum_{i=1}^{\infty} (1/2^i)^p < 1$ and is fulfilled if $p > 1$. Then also $u(c) \equiv 0$ is the unique solution in the class of function $u(c) \leqslant \text{const} \cdot c^p$ of the homogeneous equation corresponding to equation (21), and therefore Assumption B is satisfied. But for $p = 1$ condition (20) is not fulfilled and in this case each function $u(c) = \text{const} \cdot c$ is a solution of the homogeneous equation corresponding to equation (21).

(c) If $l_i(c) \leqslant l_i$, $a_i(c) \leqslant a_i c^2$, $h(c) \leqslant hc$ and $a_i \cdot c < 1$, $i = 1, \ldots, r$, then condition (17) is fulfilled.

We note that now $l_n^{i_1, \ldots, i_n}(c) \leqslant c^{2^n} \prod_{m=1}^{n} a_{i_m}^{2^m}$.

Finally we consider the case for which condition (20) is not fulfilled but for which condition (17) holds.

(d) If $r = 1$, $a(c) \stackrel{\text{df}}{=} a_1(c)$, $l(c) \stackrel{\text{df}}{=} l_1(c)$ and $a(c) = c - ac^{s+1} \geqslant 0$, $a \geqslant 0$, $s > 0$, $l(c) = 1 + bc^q$, $b > 0$, $q > s$, $h(c) \leqslant Hc^p$, $H \geqslant 0$, $p > s$, then condition (17) (see Remark 1) is fulfilled.

Indeed, we have now (see [2])

$$0 \leqslant a_n(c) \leqslant K \frac{1}{n^{1/s}} \quad \text{or} \quad a_n(c) = O\left(\frac{1}{n^{1/s}}\right),$$

however,

$$l_{n+1}(c) = \prod_{i=1}^{n} l\left(1 + O\left(\frac{1}{i^{q/s}}\right)\right).$$

Since $q > s$, the sequence $\{l_n\}$ is convergent and therefore bounded, i.e. $|l_n(c)| \leqslant D$. Hence, and in view of $q > s$, we get

$$\sum_{n=0}^{\infty} l_n(c) h\big(a_n(c)\big) \leqslant D \sum_{n=0}^{\infty} h\big(a_n(c)\big) \leqslant DH \sum_{n=0}^{\infty} \left[O\left(\frac{1}{n^{1/s}}\right)\right]^p \leqslant G \sum_{n=0}^{\infty} \frac{1}{n^{p/s}} < +\infty.$$

4. Let R^n denote an n-dimensional real linear space. If $x \in R^n$ and $x = (x_1, \ldots, x_n)$, then $x \geqslant 0$ means that $x_i \geqslant 0$, $i = 1, \ldots, n$. However, $x \leqslant y$ when $y - x \geqslant 0$. If A is an $n \times n$ matrix, then $A \geqslant 0$ denotes that all elements of the matrix A are non-negative.

In the metric space M_2 we introduce a generalized metric d the values of which lie in R_+^n, such that there exist the numbers $b_1 > 0$, $b_2 > 0$ satisfying the condition

$$b_1 \|d(\xi, \eta)\|_{R^n} \leqslant \varrho_2(\xi, \eta) \leqslant b_2 \|d(\xi, \eta)\|_{R^n},$$

for any $\xi, \eta \in M_2$.

In this part of the paper we shall give theorems on the existence, uniqueness and continuous dependence of solution of equation (1) only in the case when the function f fulfils condition (2) with the linear comparative function ω.

THEOREM 5. *If conditions* 1°, 3° *of Assumption* A *are satisfied and*

1° *there exist non-negative, non-decreasing matrix-functions* L_i *defined for* $c \in R_+$ *such that*

$$d\big(f(t, x_1, \ldots, x_r),\ f(t, \bar{x}_1, \ldots, \bar{x}_r)\big) \leqslant \sum_{i=1}^{r} L_j\big(\varrho_1(t, t_0)\big) d(x_i, \bar{x}_i),$$

for $(t, x_1, \ldots, x_r), (t, \bar{x}_1, \ldots, \bar{x}_r) \in M_1 \times M_2^r$,

2° $\sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} L_n^{i_1,\ldots,i_n}(c) H\big(a_n^{i_1,\ldots,i_n}(c)\big) < +\infty$, *where*

$$H(c) = \sup_{t \in K(t_0, c)} d\Big(x_0(t), f\big(t, x_0(\beta_1(t)), \ldots, x_0(\beta_r(t))\big)\Big), \quad c \in R_+,$$

and

$$L_0(c) = I, \quad L_{n+1}(c) = L_{i_{n+1}}(c) L_n^{i_1,\ldots,i_n}\big(a_{i_{n+1}}(c)\big), \quad n = 0, 1, \ldots,$$

$c \in R_+$, *I being a unit matrix, then there exists a solution* $\bar{x}$ *of equation* (1), *being the limit of the sequence* $\{x_n\}$ *defined by* (8). *The estimations*

$$\sup_{t \in K(t_0, c)} d\big(\bar{x}(t), x_0(t)\big) \leqslant \sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} L_n^{i_1,\ldots,i_n}(c) H\big(a_n^{i_1,\ldots,i_n}(c)\big),$$

$$\sup_{t \in K(t_0, c)} d\big(\bar{x}(t), x_n(t)\big) \leqslant \sum_{k=n}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_k=1}^{r} L_k^{i_1,\ldots,i_k}(c) H\big(a_n^{i_1,\ldots,i_n}(c)\big),$$
$$n = 0, 1, \ldots,$$

hold true. The solution $\bar{x}$ *of* (1) *is unique in the class of functions* $X(M_1, M_2)$.

THEOREM 6. *If* $\bar{x}$ *and* $\bar{y}$ *are solutions of equations* (1) *and* (13) *respectively, and if the assumptions of Theorem* 5 *are fulfilled with* $H(c) = \max[\bar{H}(c), v_0(c)]$, *where*

$$\bar{H}(c) = \sup_{t \in K(t_0, c)} d\Big(f\big(t, \bar{y}(\beta_1(t)), \ldots, \bar{y}(\beta_r(t))\big), \bar{y}(t)\Big),$$

then

$$\sup_{t \in K(t_0, c)} d\big(\bar{x}(t), \bar{y}(t)\big) \leqslant \sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} L_n^{i_1,\ldots,i_n}(c) \bar{H}\big(a_n^{i_1,\ldots,i_n}(c)\big), \quad c \in R_+.$$

Remark 7. If we assume that $L_i(c) \leqslant L_i$, $L_i \geqslant 0$ in a constant matrix, $a_i(c) \leqslant a_i c$, $i = 1, \ldots, r$, $H(c) \leqslant Hc^p$, $p \geqslant 0$ and vector $H \geqslant 0$, then condition 2° of Theorem 5 is fulfilled if

$$\mu\Big(\sum_{i=1}^{r} a_i^p L_i\Big) < 1,$$

where $\mu(A)$ denotes the spectral radius of the matrix A.

5. We denote by $W(M_1, M_2)$ a class of functions defined in the metric space M_1 and taking the values in the complete metric space M_2. We assume $W(M_1, M_2)$ to have the property: if $x_n \in W(M_1, M_2)$ and $x_n \rightrightarrows x$, then $x \in W(M_1, M_2)$ ($\rightrightarrows$ denotes the uniform convergence in each closed ball).

We have:

Theorem 7. *If Assumption* A *and condition* (17) *are satisfied, and the functions* f, β_i, $i = 1, \ldots, r$ *are such that if* $x \in W(M_1, M_2)$ *and* $y(t) = f\big(t, x(\beta_1(t)), \ldots, x(\beta_r(t))\big)$, *then* $y \in W(M_1, M_2)$, *and if* $x_0 \in W(M_1, M_2)$, *then there exists a unique solution* $\bar{x} \in W(M_1, M_2)$ *of equation* (1) *with the following properties*

$$\sup_{t \in K(t_0, c)} \varrho_2\big(\bar{x}(t), x_0(t)\big) \leqslant \sum_{n=0}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} l_n^{i_1,\ldots,i_n}(c)\, h\big(a_n^{i_1,\ldots,i_n}(c)\big),$$

$$\sup_{t \in K(t_0, c)} \varrho_2\big(\bar{x}(t), x_n(t)\big) \leqslant \sum_{k=n}^{\infty} \sum_{i_1=1}^{r} \cdots \sum_{i_k=1}^{r} l_k^{i_1,\ldots,i_k}(c)\, h\big(a_k^{i_1,\ldots,i_k}(c)\big),$$

$$n = 0, 1, \ldots, \quad c \in R_+.$$

Conclusion 1. Let $W(M_1, M_2)$ be the class of all continuous functions at t_0. If we assume that the functions β_i, x_0, $i = 1, \ldots, r$, are continuous at t_0 and the function f is continuous at point $(t_0, v_1, \ldots, v_r)$, where $v_1, \ldots, v_r$ are arbitrary, then from Theorem 7 it follows that there exists a solution of equation (1) being continuous at t_0.

Conclusion 2. Let $W(M_1, M_2)$ denote the class of all continuous functions in M_1. If we assume that f, β_i, x_0 are continuous, then Theorem 7 asserts the existence of a solution of equation (1) being also a continuous function in M_1 (see [5], [7], [11]).

Conclusion 3. Let $W(M_1, M_2)$ be a class of functions fulfilling a Lipschitz condition with suitably chosen constants (see [1]). If additionaly we suppose that the functions f, β_i fulfil a Lipschitz condition with respect to t, then from Theorem 7 it follows that the solution $\bar{x}$ of equation (1) fulfils a Lipschitz condition.

Conclusion 4. Let now $M_1 = M_2 = R^n$ and $W(R^n, R^n)$ be a class of all non-decreasing functions. If we assume that the functions f and

β_i, $i = 1, \ldots, r$ are non-decreasing, then the solution of equation (1) given by Theorem 7 is also a non-decreasing function (see [1], [7]).

Conclusion 5. Let $W(R^n, R^n)$ be a class of all convex functions. Suppose that f, β_i, $i = 1, \ldots, r$, are non-decreasing and convex functions, then from Theorem 7 it follows that the solution of equation (1) is a convex function (see [1], [6], [7]).

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Reçu par la Rédaction le 24. 7. 1973