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BOOLEAN ALGEBRA AND MULTIVARIATE INTERPOLATION

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1. Introduction

The theory of distributive lattices was first applied by W. J. Gordon [27], [28] to problems of multivariate interpolation. Since then various applications of this generalization of the classical blending interpolation method [11], [14], [29] to different fields of applied mathematics such as the finite element method [12], [35], [36], [41], [42], computer aided geometric design [2], [3], interpolation [18], [19], [20], [21], approximation [32], [33], [37], and harmonic analysis [4], [6] have been found. In this paper we describe the basic constructions of Boolean algebras of multivariate interpolation projectors and the related remainder operators. We will illustrate the method of Boolean interpolation by considering bivariate polynomial and periodic spline interpolation. Furthermore we will discuss Boolean methods in multivariate interpolation of higher dimensions.

2. Parametric extensions

One basic construction in the Boolean theory of bivariate interpolation (and also multivariate interpolation) is the method of parametric extension of linear operators. A systematic functional analytic approach via tensor products is presented in the paper [25]. Let $k_1, \ldots, k_r$ be natural numbers satisfying $1 \leqslant k_1 < k_2 < \ldots < k_r$. This sequence is associated with r sets of interpolation points ordered in the following way:

$$x_1, \ldots, x_{k_1},$$
$$x_{k_1+1}, \ldots, x_{k_2},$$

$$x_{k_{r-1}+1}, \ldots, x_{k_r}.$$

We assume that $x_1, \ldots, x_{k_r}$ are distinct points in the real interval $[a, b]$. For each set $\{x_1, \ldots, x_{k_m}\}$ let

$$\{f_{1,m}, \ldots, f_{k_m,m}\} \subset \mathscr{C}[a, b] \quad (m = 1, \ldots, r)$$

be a set of associated fundamental functions:

$$f_{i,m}(x_\mu) = \delta_{i,\mu} \quad (i, \mu = 1, \ldots, k_m).$$

Similar constructions are assumed for the second variable. Let $1 \leqslant l_1 < l_2 < \ldots < l_r$ be an ordered sequence of positive integers which are related to r sets of interpolation points

$$y_1, \ldots, y_{l_1},$$
$$y_{l_1+1}, \ldots, y_{l_2},$$

$$y_{l_{r-1}+1}, \ldots, y_{l_r}.$$

Again the interpolation points $y_1, \ldots, y_{l_r}$ are supposed to be distinct and contained in a compact interval $[c, d]$. The related sets of fundamental functions are

$$\{g_{1,n}, \ldots, g_{l_n,n}\} \subset \mathscr{C}[c, d] \quad (n = 1, \ldots, r).$$

The sets of fundamental functions are used to construct *parametrically extended interpolation projectors* P_m and Q_n:

$$P_m(f)(x, y) = \sum_{i=1}^{k_m} f(x_i, y) f_{i,m}(x),$$

$$Q_n(f)(x, y) = \sum_{j=1}^{l_n} f(x, y_j) g_{j,n}(y)$$

where $f \in \mathscr{C}([a, b] \times [c, d])$. It is obvious that $P_m(f)$ and $Q_n(f)$ satisfy the following *interpolation conditions*:

$$P_m(f)(x_i, \bullet) = f(x_i, \bullet) \quad (i = 1, \ldots, k_m);$$

$$Q_n(f)(\bullet, y_j) = f(\bullet, y_j) \quad (j = 1, \ldots, l_n).$$

For the fundamental functions we make the additional hypotheses

$$\langle f_{1,m}, \ldots, f_{k_m,m} \rangle \subseteq \langle f_{1,m+1}, \ldots, f_{k_{m+1},m+1} \rangle \quad (m = 1, \ldots, r-1);$$

$$\langle g_{1,n}, \ldots, g_{l_n,n} \rangle \subseteq \langle g_{1,n+1}, \ldots, g_{l_{n+1},n+1} \rangle \quad (n = 1, \ldots, r-1).$$

These inclusions imply specific properties of the associated interpolation projectors which are listed below:

$$P_{m+i} P_m = P_m P_{m+i} = P_m \quad (1 \leqslant m < m+i \leqslant r);$$

$$Q_{n+j} Q_n = Q_n Q_{n+j} = Q_n \quad (1 \leqslant n < n+j \leqslant r).$$

We consider two important examples for which the additional hypotheses are valid.

Example 1. The fundamental functions are the Lagrange polynomials

$$\langle f_{1,m}, \ldots, f_{k_m,m} \rangle = \Pi_{k_m - 1},$$

$$\langle g_{1,n}, \ldots, g_{l_n,n} \rangle = \Pi_{l_n - 1}.$$

Example 2. We denote by $\mathrm{Sp}(x_1, \ldots, x_{k_m}; 2q-1)$ the space of 2π periodic splines of degree $2q-1$ with knots $\{x_1, \ldots, x_{k_m}\} \subset [0, 2\pi[$, $x_1 = 0$. Similarly, $\mathrm{Sp}(y_1, \ldots, y_{l_n}; 2q-1)$ is the space of 2π periodic splines of degree $2q-1$ with knots $\{y_1, \ldots, y_{l_n}\} \subset [0, 2\pi[$, $y_1 = 0$. It follows from the properties of spline functions that there exist fundamental functions $f_{i,m}$ and $g_{j,n}$ such that

$$\langle f_{1,m}, \ldots, f_{k_m,m} \rangle = \mathrm{Sp}(x_1, \ldots, x_{k_m}; 2q-1),$$

$$\langle g_{1,n}, \ldots, g_{l_n,n} \rangle = \mathrm{Sp}(y_1, \ldots, y_{l_n}; 2q-1)$$

[26]. Now the construction of the interpolation point sets imply that the relations

$$\mathrm{Sp}(x_1, \ldots, x_{k_m}; 2q-1) \subseteq \mathrm{Sp}(x_1, \ldots, x_{k_{m+1}}; 2q-1) \quad (m = 1, \ldots, r-1);$$

$$\mathrm{Sp}(y_1, \ldots, y_{l_n}; 2q-1) \subseteq \mathrm{Sp}(y_1, \ldots, y_{l_{n+1}}; 2q-1) \quad (n = 1, \ldots, r-1)$$

are true.

3. Lattices of interpolation projectors

It follows from the construction of P_m and Q_n that the *operator product* $P_m Q_n = Q_n P_m$ is again a projector on $\mathscr{C}([a, b] \times [c, d])$ having the explicit form

$$P_m Q_n(F)(x, y) = \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} f(x_i, y_j) f_{i,m}(x) g_{j,n}(y).$$

$P_m Q_n$ is called the *projector of tensor product interpolation*. It has the interpolation properties

$$P_m Q_n(f)(x_i, y_j) = f(x_i, y_j) \quad (i = 1, \ldots, k_m;\ j = 1, \ldots, l_n).$$

Since P_m and Q_n commute, its *Boolean sum* $P_m \oplus Q_n = P_m + Q_n - P_m Q_n$ is again a projector on $\mathscr{C}([a, b] \times [c, d])$. $P_m \oplus Q_n$ is called the *projector of Blending interpolation*. It possesses the transfinite interpolation conditions

$$(P_m \oplus Q_n)(f)(x_i, \bullet) = f(x_i, \bullet) \quad (i = 1, \ldots, k_m);$$

$$(P_m \oplus Q_n)(f)(\bullet, y_j) = f(\bullet, y_j) \quad (j = 1, \ldots, l_n).$$

$P_m Q_n$ and $P_m \oplus Q_n$ are special projectors obtained from the set of commuting projectors

$$\mathscr{L} = \{P_1, \ldots, P_r, Q_1, \ldots, Q_r\}.$$

They are special elements of a larger set of commuting projectors which will now be described. For this purpose we introduce operations on a given set $\mathscr{L}$ of commuting projectors. Let

$$\mathscr{L}' = \{PQ\colon\ P, Q \in \mathscr{L}\}$$

and

$$\mathscr{L}'' = \{P \oplus Q\colon\ P, Q \in \mathscr{L}'\}.$$

$\mathscr{L}'$ and $\mathscr{L}''$ are again sets of commuting projectors satisfying

$$\mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}''.$$

Thus, these constructions can be iterated. We define

$$\mathscr{L}_1 := \mathscr{L}'',$$

$$\mathscr{L}_{k+1} := \mathscr{L}_k'' \quad (k \in \mathbb{N})$$

which are again sets of commuting operators satisfying

$$\mathscr{L}_k \subseteq \mathscr{L}_{k+1} \quad (k = 1, 2, \ldots).$$

THEOREM 1. *Let* $\tilde{\mathscr{L}} := \bigcup_{k=1}^{\infty} \mathscr{L}_k$. *Then* $\tilde{\mathscr{L}}$ *is a set of commuting projectors containing* $\mathscr{L}$. *Moreover, the following relations are valid in* $\tilde{\mathscr{L}}$:

(1) $$PQ,\ P \oplus Q \in \tilde{\mathscr{L}} \quad \textit{for all } P, Q \in \tilde{\mathscr{L}};$$

(2) $$P(Q \oplus R) = (PQ) \oplus (PR) \quad \textit{for all } P, Q, R \in \tilde{\mathscr{L}};$$

(3) $$P \oplus (QR) = (P \oplus Q)(P \oplus R) \quad \textit{for all } P, Q, R \in \tilde{\mathscr{L}}.$$

Proof. Let $P, Q \in \tilde{\mathscr{L}}$. Then $P, Q \in \mathscr{L}_k$ for some k and we get

$$PQ \in \mathscr{L}_k' \subseteq \mathscr{L}_{k+1} \subseteq \tilde{\mathscr{L}},$$

$$P \oplus Q \in \mathscr{L}_k'' = \mathscr{L}_{k+1} \subseteq \tilde{\mathscr{L}}.$$

The remaining identities are easily checked by computation. ∎

$\tilde{\mathscr{L}}$ is a *partially ordered set* with respect to the order relation

$$P \leqslant Q \Leftrightarrow PQ = QP = P.$$

Thus, $\tilde{\mathscr{L}}$ is the smallest distributive lattice of commuting projectors containing $\mathscr{L}$ since $\inf\{P, Q\} = PQ \in \tilde{\mathscr{L}}$, $\sup\{P, Q\} = P \oplus Q \in \tilde{\mathscr{L}}$ for arbitrary $P, Q \in \tilde{\mathscr{L}}$.

It follows from the relations (2), (3) of Theorem 1 that the finiteness of $\mathscr{L}$ implies the finiteness of $\tilde{\mathscr{L}}$.

4. Chains of projectors

A basic principle of Boolean algebra in multivariate interpolation is the concept of a chain. Note that P_m $(m = 1, \ldots, r)$ and Q_n $(n = 1, \ldots, r)$ are chains in $\tilde{\mathscr{L}}$, i.e.

$$P_1 \leqslant P_2 \leqslant \ldots \leqslant P_r, \quad Q_1 \leqslant Q_2 \leqslant \ldots \leqslant Q_r.$$

We define a new family of projectors in $\tilde{\mathscr{L}}$ by

$$\begin{aligned} C_1 &= P_1 Q_1, \\ C_2 &= P_1 Q_2 \oplus P_2 Q_1, \\ &\ldots\ldots\ldots\ldots \\ C_r &= P_1 Q_r \oplus P_2 Q_{r-1} \oplus \ldots \oplus P_r Q_1. \end{aligned}$$

Since

$$\begin{aligned} C_r &= \sup\{P_1 Q_r, P_2 Q_{r-1}, \ldots, P_r Q_1\} \\ &\geqslant \sup\{P_1 Q_r, P_2 Q_{r-1}, \ldots, P_{r-1} Q_2\} \\ &\geqslant \sup\{P_1 Q_{r-1}, \ldots, P_{r-1} Q_1\} \\ &= C_{r-1} \end{aligned}$$

it follows that C_j $(j = 1, \ldots, r)$ is again a chain in $\tilde{\mathscr{L}}$:

$$C_1 \leqslant C_2 \leqslant \ldots \leqslant C_r.$$

Thus, we have constructed a new chain of nontrivial projectors from given chains which are defined as parametric extensions.

Since for $i = 1, \ldots, k_m$; $j = 1, \ldots, l_n$; $m + n = r + 1$ we have

$$C_r(f)(x_i, y_j) = P_m Q_n C_r(f)(x_i, y_j) = P_m Q_n(f)(x_i, y_j) = f(x_i, y_j)$$

it follows that $C_r(f)$ interpolates f at the union of the sets of interpolation points of $P_1 Q_r, \ldots, P_r Q_1$:

$$C_r(f)(x_i, y_j) = f(x_i, y_j) \quad (i = 1, \ldots, k_m;\ j = 1, \ldots, l_n;\ m + n = r + 1).$$

To determine the space of interpolants of C_r, i.e. $\mathrm{ran}(C_r)$, we note that

$$\mathrm{ran}(PQ) = \mathrm{ran}(P) \cap \mathrm{ran}(Q),$$
$$\mathrm{ran}(P \oplus Q) = \mathrm{ran}(P) + \mathrm{ran}(Q),$$

for $P, Q \in \tilde{\mathscr{L}}$. Thus we have

$$\mathrm{ran}(P_m Q_n) = \langle \{ f_{i,m} \otimes g_{j,n} \colon i = 1, \ldots, k_m;\ j = 1, \ldots, l_n \} \rangle$$

and

$$\mathrm{ran}(C_r) = \sum_{m+n=r+1} \mathrm{ran}(P_m Q_n).$$

For computational reasons it is important to have a sum representation for C_r.

THEOREM 2. *Let $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_r$ be chains in $\tilde{\mathscr{L}}$. Then the Boolean sum projector $C_r = \oplus_{m+n=r+1} P_m Q_n$ has the sum representation*

$$C_r = \sum_{j=1}^{r} P_j Q_{r+1-j} - \sum_{j=1}^{r-1} P_j Q_{r-j}.$$

In view of its importance we include the inductive

Proof. We have

$$\begin{aligned} C_r &= \bigoplus_{j=1}^{r} P_j Q_{r+1-j} \\ &= (P_1 Q_r \oplus P_2 Q_{r-1} \oplus \ldots \oplus P_{r-1} Q_2) \oplus P_r Q_1 \\ &= (P_1 Q_r \oplus \ldots \oplus P_{r-1} Q_2) + P_r Q_1 - (P_1 Q_1 \oplus \ldots \oplus P_{r-1} Q_1) \\ &= P_1 Q_r + \ldots + P_{r-1} Q_2 + P_r Q_1 - (P_1 Q_{r-1} + \ldots + P_{r-2} Q_2) - P_{r-1} Q_1 \\ &= \sum_{j=1}^{r} P_j Q_{r+1-j} - \sum_{j=1}^{r-1} P_j Q_{r-j}. \quad \blacksquare \end{aligned}$$

The sum representation of the *Boolean interpolation projector* C_r yields an explicit construction of the corresponding fundamental functions. Note first that (with $k_0 = 0$, $l_0 = 0$):

$$C_r(f)(x, y) = \sum_{m=1}^{r} \sum_{n=0}^{r-m} \sum_{i=1+k_{m-1}}^{k_m} \sum_{j=1+l_{r-m-n}}^{l_{r+1-m-n}} f(x_i, y_j) F_{i,j}(x, y)$$

which corresponds to the *disjoint decomposition of the interpolation points*:

$$\begin{aligned} &\bigcup_{m+n=r+1} \{(x_i, y_j) \colon i = 1, \ldots, k_m;\ j = 1, \ldots, l_n\} \\ &\quad = \bigcup_{m=1}^{r} \bigcup_{n=0}^{r-m} \{(x_i, y_j) \colon k_{m-1} < i \leqslant k_m;\ l_{r-m-n} < j \leqslant l_{r+1-m-n}\}. \end{aligned}$$

In view of the interpolation properties of $f_{i,r} \otimes g_{j,r}$ we have

$$F_{i,j}(x, y) = C_r(f_{i,r} \otimes g_{j,r})(x, y)$$

which implies the explicit formula

$$F_{i,j}(x, y) = \sum_{s=m}^{m+n} f_{i,s}(x) g_{j,r+1-s}(y) - \sum_{s=m}^{m+n-1} f_{i,s}(x) g_{j,r-s}(y)$$

$$(k_{m-1} < i \leqslant k_m;\ l_{r-m-n} < j \leqslant l_{r+1-m-n};\ 1 \leqslant m \leqslant r;\ 0 \leqslant n \leqslant r-m).$$

EXAMPLE 3. As fundamental functions $f_{i,m}$ and $g_{j,n}$ we choose polynomial Lagrange functions associated with the sets of interpolation points specified in the following way:

$$k_m = m, \quad l_n = n; \quad x_i = i, \quad y_j = j.$$

The set of interpolation points of the Boolean interpolant $C_r(f)$ is given by the triangle:

$$\{(i, j):\ 1 \leqslant i, j;\ i+j \leqslant r+1\}.$$

The space of interpolants is spanned by the monomials $x^i y^j$ $(0 \leqslant i, j; i+j \leqslant r-1)$:

$$\operatorname{ran}(C_r) = \sum_{m+n=r+1} \Pi_{m-1} \otimes \Pi_{n-1}.$$

The set of interpolation points for $r = 3$ is given in Figure 1. The cardinal functions are:

$$\begin{aligned}
F_{1,3}(x, y) &= f_{1,1}(x) g_{3,3}(y), \\
F_{1,2}(x, y) &= f_{1,1}(x) g_{2,3}(y) + f_{1,2}(x) g_{2,2}(y) - f_{1,1}(x) g_{2,2}(y), \\
F_{1,1}(x, y) &= f_{1,1}(x) g_{1,3}(y) + f_{1,2}(x) g_{1,2}(y) + f_{1,3}(x) g_{1,1}(y) \\
&\quad - f_{1,1}(x) g_{1,2}(y) - f_{1,2}(x) g_{1,1}(y); \\
F_{2,2}(x, y) &= f_{2,2}(x) g_{2,2}(y), \\
F_{2,1}(x, y) &= f_{2,2}(x) g_{1,2}(y) + f_{2,3}(x) g_{1,1}(y) - f_{2,2}(x) g_{1,1}(y); \\
F_{3,1}(x, y) &= f_{3,3}(x) g_{1,1}(y).
\end{aligned}$$

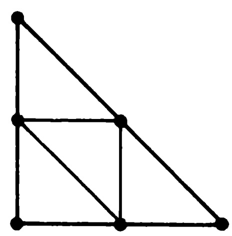

Fig. 1. Interpolation knots on a triangle

EXAMPLE 4. Again we choose polynomials as fundamental functions. The interpolation points are given by

$$k_m = 2m, \qquad l_n = 2n,$$

$$x_{2i-1} = 1+2(r-i), \quad x_{2i} = -x_{2i-1} \quad (i = 1, \ldots, r),$$

$$y_{2j-1} = 1+2(r-j), \quad y_{2j} = -y_{2j-1} \quad (j = 1, \ldots, r).$$

The space of interpolants for C_r satisfies

$$\operatorname{ran}(C_r) = \sum_{m+n=r+1} \Pi_{2m-1} \otimes \Pi_{2n-1}.$$

The set of interpolation points for $r = 2$ is illustrated in Figure 2.

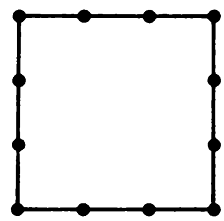

Fig. 2. Knots for Serendipity interpolation

EXAMPLE 5. Let

$$k_m = 2^m, \qquad h_m = 2\pi/k_m,$$

$$\{x_1, \ldots, x_{k_m}\} = \{ih_m\colon i = 0, \ldots, 2^m - 1\}$$

and

$$l_n = 2^n, \qquad \{y_1, \ldots, y_{l_n}\} = \{jh_n\colon j = 0, \ldots, 2^n - 1\}.$$

$L_m(x)$ is the unique periodic fundamental spline function of degree $2q-1$ with knots ih_m $(i = 0, \ldots, 2^m - 1)$, i.e.

$$L_m(ih_m) = \delta_{0,i} \quad (i = 0, \ldots, 2^m - 1).$$

The interpolation projectors P_m and Q_n are obtained by translation [26]:

$$P_m(f)(x, y) = \sum_{i=0}^{2^m-1} f(ih_m, y)\, L_m(x - ih_m),$$

$$Q_n(f)(x, y) = \sum_{j=0}^{2^n-1} f(x, jh_n)\, L_n(y - jh_n),$$

Figure 3 gives an illustration of the set of interpolation points for $C_3(f)$. We add a list of typical fundamental functions:

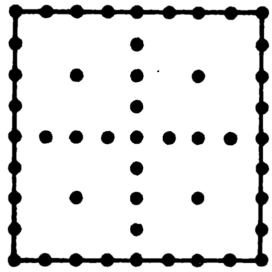

Fig. 3. Periodic spline interpolation

$$
\begin{aligned}
F_{1,1}(x, y) &= L_1(x)L_3(y)+L_2(x)L_2(y)+L_3(x)L_1(y) \\
&\quad -L_1(x)L_2(y)-L_2(x)L_1(y), \\
F_{1,3}(x, y) &= L_1(x)L_3(y-h_2)+L_2(x)L_2(y-h_2)-L_1(x)L_2(y-h_2), \\
F_{1,5}(x, y) &= L_1(x)L_3(y-h_3), \\
F_{3,3}(x, y) &= L_2(x-h_2)L_2(y-h_2).
\end{aligned}
$$

The graph of $F_{1,1}(x, y)$ is shown in Figure 4.

5. Remainder projectors

It is an important aspect of Boolean methods in multivariate interpolation that remainder formulas are obtained by duality techniques of Boolean algebra. For this reason we will use the procedure of Section 3 to construct a smallest Boolean algebra of commutative projectors containing a given set $\mathcal{K}$ of commutative projectors. We start with the description of the simplest cases.

We will denote by $\mathscr{C}^{k,l} = \mathscr{C}^{k,l}([a, b]\times[c, d])$ the space of continuously differentiable functions f with continuous mixed partial derivatives

$$D^{i,j}f = \frac{\partial^{i+j} f}{\partial x^i \, \partial y^j} \quad (i = 0, \ldots, k;\ j = 0, \ldots, l).$$

Let I be the identity operator and

$$P_m^c = I-P_m, \quad Q_n^c = I-Q_n$$

be the remainder projectors of P_m and Q_n respectively. We assume that P_m^c and Q_n^c have integral kernel representations with square integrable kernels $G_m(x, s)\in L_2([a, b]^2)$, $H_n(y, t)\in L_2([c, d]^2)$, i.e.,

$$f(x, y)-P_m(f)(x, y) = \int_a^b G_m(x, s)\,D^{\varkappa_m,0} f(s, y)\,ds,$$

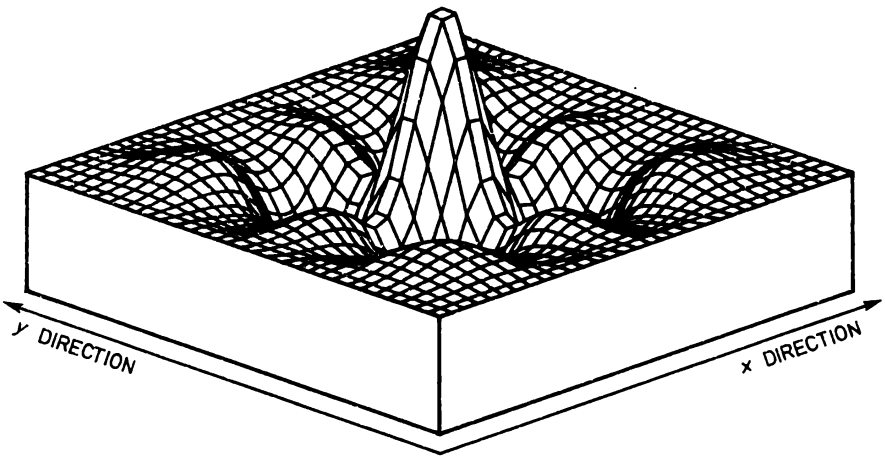

Fig. 4. The spline basis function $F_{1,1}(x, y)$ on $[-\pi, \pi]\times[-\pi, \pi]$

$$f(x, y)-Q_n(f)(x, y) = \int_c^d H_n(y, t)\, D^{0,\lambda_n} f(x, t)\, dt$$

with suitable numbers $\varkappa_m, \lambda_n \in \mathbb{N}$.

As an example we consider the polynomial case. We have

$$\langle f_{1,m}, \ldots, f_{k_m,m}\rangle = \Pi_{k_m-1},$$
$$\langle g_{1,n}, \ldots, g_{l_n,n}\rangle = \Pi_{l_n-1}$$

and $\varkappa_m = k_m$, $\lambda_n = l_n$. The kernel functions $G_m(x, s)$ and $H_n(y, t)$ are given by

$$G_m(x, s) = (x-x_1)\ldots(x-x_{k_m})\, M(x, x_1, \ldots, x_{k_m}, s),$$
$$M(x, x_1, \ldots, x_{k_m}, s) = [x, x_1, \ldots, x_{k_m}]\frac{(\bullet - s)_+^{k_m-1}}{(k_m-1)!},$$
$$H_n(y, t) = (y-y_1)\ldots(y-y_{l_n})\, M(y, y_1, \ldots, y_{l_n}, t).$$

Next we will derive the remainder formulas for the simplest Boolean interpolation projectors $P_m Q_n$ and $P_m \oplus Q_n$.

Note that for any two commuting projectors P and Q we have the *duality relations*

$$(PQ)^c = P^c \oplus Q^c,$$
$$(P \oplus Q)^c = P^c Q^c.$$

In particular, $(PQ)^c$ and $(P \oplus Q)^c$ are again commuting projectors.

For smooth functions $f \in \mathscr{C}^{k_m, l_n}$ we obtain the integral remainder formula of blending interpolation

$$f(x, y)-P_m \oplus Q_n(f)(x, y) = \int_a^b\int_c^d G_m(x, s)\, H_n(y, t)\, D^{k_m, l_n} f(s, t)\, dt\, ds.$$

Similarly the integral remainder formula of tensor product interpolation is given by

$$f(x, y)-P_m Q_n(f)(x, y) = \int_a^b G_m(x, s)\, D^{k_m,0} f(s, y)\, ds+\int_c^d H_n(y, t)\, D^{0,l_n} f(x, t)\, dt$$

$$-\int_a^b\int_c^d G_m(x, s)\, H_n(y, t)\, D^{k_m,l_n} f(s, t)\, dt\, ds.$$

The remainder formula for $C_r = P_1 Q_r \oplus \ldots \oplus P_r Q_1$ will be derived in a more general setting.

Let $\mathcal{L}$ be any set of commuting projectors. Then

$$\mathcal{K} = \{P\colon P \in \mathcal{L}\} \cup \{P^c\colon P \in \mathcal{L}\}$$

is again a set of commuting projectors. Moreover, $\mathcal{K}$ is closed with respect to complementation, i.e.

$$P \in \mathcal{K} \Rightarrow P^c = I-P \in \mathcal{K} \quad \text{for all } P.$$

Thus we may apply Theorem 1 to $\mathcal{K}$ and obtain a distributive lattice $\tilde{\mathcal{K}}$ of commuting projectors. Since the generating set $\mathcal{K}$ is closed with respect to complementation we obtain a refinement of Theorem 1.

THEOREM 3. *The distributive lattice $\tilde{\mathcal{K}}$ of commuting projectors generated from $\mathcal{K}$ is closed with respect to complementation, i.e. $\tilde{\mathcal{K}}$ is a Boolean algebra.*

The proof is a simple consequence of the following

LEMMA 1. *For every $j \in \mathcal{N}$ the relation*

$$A \in \mathcal{K}_j \Rightarrow A^c \in \mathcal{K}_j$$

is true.

Proof. We start with $j = 1$. Then $A \in \mathcal{K}_1$ implies

$$A = B_1 B_2 \oplus B_3 B_4 \quad (B_1, B_2, B_3, B_4 \in \mathcal{K}).$$

Then

$$A^c = (B_1^c \oplus B_2^c)(B_3^c \oplus B_4^c).$$

Since $B_1^c, B_2^c, B_3^c, B_4^c \in \mathcal{K}$ we obtain

$$B_1^c \oplus B_2^c,\ B_3^c \oplus B_4^c \in \mathcal{K}'' = \mathcal{K}_1$$

and thus $A^c \in \mathcal{K}_1$. Next assume $j > 1$ and

$$A = B_1 B_2 \oplus B_3 B_4 \in \mathcal{K}_j = \mathcal{K}''_{j-1}.$$

Then $B_1, B_2, B_3, B_4 \in \mathscr{K}_{j-1}$ and

$$A^c = (B_1^c \oplus B_2^c)(B_3^c \oplus B_4^c).$$

The induction hypothesis shows $B_1^c, B_2^c, B_3^c, B_4^c \in \mathscr{K}'_{j-1}$ and then

$$B_1^c \oplus B_2^c,\ B_3^c \oplus B_4^c \in \mathscr{K}''_{j-1} = \mathscr{K}_j.$$

This implies $A^c \in \mathscr{K}'_j$ which completes the proof of the lemma. ∎

Recall that $\tilde{\mathscr{K}} = \bigcup_{j=1}^{\infty} \mathscr{K}_j$ with

$$\mathscr{K}_j \subseteq \mathscr{K}'_j \subseteq \mathscr{K}''_j = \mathscr{K}_{j+1} \quad (j \geq 1), \quad \mathscr{K}_1 = \mathscr{K}''.$$

Now Theorem 3 follows immediately from the lemma.

Note that the finiteness of $\mathscr{K}$ again implies the finiteness of $\tilde{\mathscr{K}}$, the Boolean algebra generated from $\mathscr{K}$.

As an application we determine the remainder of Boolean interpolation defined by C_r.

THEOREM 4. *The remainder projector C_r^c is given by*

$$(P_1 Q_r \oplus P_2 Q_{r-1} \oplus \ldots \oplus P_r Q_1)^c = P_r^c I \oplus P_{r-1}^c Q_1^c \oplus \ldots \oplus P_1^c Q_{r-1}^c \oplus I Q_r^c.$$

Proof. The proof is carried out by induction. Taking into account the order relations

$$P_1^c \geq P_2^c \geq \ldots \geq P_r^c, \quad Q_1^c \geq Q_2^c \geq \ldots \geq Q_r^c$$

we can conclude

$$\begin{aligned}
&(P_1 Q_r \oplus \ldots \oplus P_r Q_1)^c \\
&\quad = (P_1 Q_r \oplus P_2 Q_{r-1} \oplus \ldots \oplus P_{r-1} Q_2)^c (P_r^c \oplus Q_1^c) \\
&\quad = (P_{r-1}^c I \oplus P_{r-2}^c Q_2^c \oplus \ldots \oplus P_1^c Q_{r-1}^c \oplus I Q_r^c)(P_r^c \oplus Q_1^c) \\
&\quad = (P_{r-1}^c P_r^c I \oplus P_{r-2}^c P_r^c Q_2^c \oplus \ldots \oplus P_1^c P_r^c Q_{r-1}^c \oplus P_r^c Q_r^c) \\
&\qquad \oplus (P_{r-1}^c Q_1^c \oplus P_{r-2}^c Q_2^c Q_1^c \oplus \ldots \oplus P_1^c Q_{r-1}^c Q_1^c \oplus I Q_r^c Q_1^c) \\
&\quad = (P_r^c I \oplus P_r^c Q_2^c \oplus \ldots \oplus P_r^c Q_{r-1}^c \oplus P_r^c Q_r^c) \\
&\qquad \oplus (P_{r-1}^c Q_1^c \oplus P_{r-2}^c Q_2^c \oplus \ldots \oplus P_1^c Q_{r-1}^c \oplus I Q_r^c) \\
&\quad = P_r^c I \oplus P_{r-1}^c Q_1^c \oplus \ldots \oplus P_1^c Q_{r-1}^c \oplus I Q_r^c
\end{aligned}$$

which completes the proof of Theorem 4. ∎

Combining Theorem 4 with Theorem 2 we obtain a *sum representation for the remainder of Boolean interpolation*:

$$\begin{aligned}
C_r^c &= I - C_r = I - P_1 Q_r \oplus \ldots \oplus P_r Q_1 \\
&= P_r^c + P_{r-1}^c Q_1^c + \ldots + P_1^c Q_{r-1}^c + Q_r^c - (P_1^c Q_r^c + \ldots + P_r^c Q_1^c).
\end{aligned}$$

Using the remainder kernels of polynomial interpolation we obtain a *remainder formula for polynomial Boolean interpolation* which holds for sufficiently smooth functions:

$$
\begin{aligned}
f(x, y) - C_r(f)(x, y)& \\
&= \int_a^b G_r(x, s)\, D^{k_r, 0} f(s, y)\, ds + \int_c^d H_r(y, t)\, D^{0, l_r} f(x, t)\, dt \\
&\quad + \sum_{j=1}^{r-1} \int_a^b \int_c^d G_{r-j}(x, s)\, H_j(y, t)\, D^{k_{r-j}, l_j} f(s, t)\, dt\, ds \\
&\quad - \sum_{j=1}^{r} \int_a^b \int_c^d G_{r+1-j}(x, s)\, H_j(y, t)\, D^{k_{r+1-j}, l_j} f(s, t)\, dt\, ds.
\end{aligned}
$$

We use the sum representation to derive an asymptotic *error estimate for spline Boolean interpolation.*

Let $k = 2q$ and let

$$D_k(s) = 2 \sum_{n=1}^{\infty} n^{-k} \cos(ns - k\pi/2)$$

denote the Bernoulli function of degree k. Following Korneichuk [34] we introduce the kernel

$$G_m(x, s) = D_k(x) - D_k(x-s) - \sum_{i=0}^{2^m - 1} L_m(x - ih_m)\big(D_k(ih_m) - D_k(ih_m - s)\big).$$

Then we have

$$f(x, y) - P_m(f)(x, y) = \frac{1}{2\pi} \int_0^{2\pi} G_m(x, s)\, D^{k, 0} f(s, y)\, ds$$

and

$$\sup_{0 \leqslant x \leqslant 2\pi} \frac{1}{2\pi} \int_0^{2\pi} |G_m(x, s)|\, ds \leqslant \gamma_k h_m^k$$

where γ_k is a constant independent of m. Using the sum representation of C_r we obtain

$$
\begin{aligned}
|f(x, y) - C_r(f)(x, y)| &\leqslant \gamma_r h_r^k (\|D^{k,0} f\|_\infty + \|D^{0,k} f\|_\infty) \\
&\quad + \sum_{j=1}^{r-1} \gamma_{r-j} \gamma_j h_{r-j}^k h_j^k \|D^{k,k} f\|_\infty \\
&\quad + \sum_{j=1}^{r} \gamma_{r+1-j} \gamma_j h_{r+1-j}^k h_j^k \|D^{k,k} f\|_\infty.
\end{aligned}
$$

Since $h_j = 2\pi/2^j$, it follows that there exists a constant γ independent of r such that the following *error estimate for spline Boolean interpolation* holds:

$$\|f - C_r(f)\|_\infty \leqslant \gamma(1+r)\, h_r^k = O\left(\frac{r+1}{2^{kr}}\right).$$

Using Theorem 2 the dimension of the space $\operatorname{ran}(C_r)$ of interpolants is given by

$$\dim\operatorname{ran}(C_r)$$

$$= \sum_{j=1}^{r-1} \left(\dim\operatorname{ran}(P_j Q_{r+1-j}) - \dim\operatorname{ran}(P_j Q_{r-j})\right) + \dim\operatorname{ran}(P_r Q_1).$$

For the example of spline Boolean interpolation this implies

$$\dim\operatorname{ran}(C_r) = (r+1)\, 2^r.$$

Note that the corresponding spline tensor product interpolation projector $P_r Q_r$ satisfies a similar error estimate as C_r:

$$\|f - P_r Q_r(f)\|_\infty = O(1/2^{kr}),$$

but $\dim\operatorname{ran}(P_r Q_r) = 2^{2r}$.

6. Extensions to higher dimensions

In the paper [16] we have extended bivariate Boolean interpolation to higher dimensions. We will briefly describe this method and will derive a representation formula for d-variate Boolean interpolation. Naturally, the notations and the combinatorical aspects are more cumbersome than in the bivariate case.

Let $D = [a_1, b_1] \times \ldots \times [a_d, b_d] \subset \mathscr{R}^d$ be a compact rectangle and $C(D)$ be the vector space of continuous functions on D. We consider the *parametrically extended interpolation projectors*:

$$P_u^{m_u}(f)(x_1, \ldots, x_d) = \sum_{i_u=1}^{a_u(m_u)} f(x_1, \ldots, x_{i_u,u}, \ldots, x_d)\, g_{i_u,u}^{m_u}(x_u)$$

$$(u = 1, \ldots, d).$$

Here we assume that $m_u \in \mathcal{N}$ and $a_u(m_u) \in \mathcal{N}$ such that

$$a_u(m_u) < a_u(m_u+1) \quad (u = 1, \ldots, d).$$

The sets of interpolation points are given by

$$\{x_{1,u}, \ldots, x_{a_u(m_u),u}\} \subseteq [a_u, b_u] \quad (u = 1, \ldots, d).$$

The elements are assumed to be distinct.

The functions

$$g_{i_u,u}^{m_u} \in \mathscr{C}[a_u, b_u] \quad (1 \leqslant i_u \leqslant a_u(m_u);\ u = 1, \ldots, d)$$

are the associated fundamental functions. For instance, $g_{1,u}^{m_u}, \ldots, g_{a_u(m_u),u}^{m_u}$ may be chosen as polynomial Lagrange functions in $\Pi_{a_u(m_u)-1}$.

The set

$$\mathscr{L} = \bigcup_{u=1}^{d} \{P_u^{m_u}:\ m_u \in \mathscr{N}\}$$

generates a distributive lattice $\mathscr{L}$ of commutative projectors (see Section 3). Taking $\mathscr{K} = \mathscr{L} \cup \mathscr{L}^c$ we obtain a smallest Boolean algebra $\tilde{\mathscr{K}}$ of commutative projectors containing $\mathscr{L}$. Special interpolation projectors of $\tilde{\mathscr{K}}$ are the tensor product interpolation projectors $P_1^{n_1} \ldots P_d^{n_d}$ which possess the representation

$$\begin{aligned} &P_1^{n_1} \ldots P_d^{n_d}(f)(x_1, \ldots, x_d) \\ &\quad = \sum_{i_1=1}^{a_1(n_1)} \cdots \sum_{i_d=1}^{a_d(n_d)} f(x_{i_1,1}, \ldots, x_{i_d,d})\, g_{i_1,1}^{n_1}(x_1) \ldots g_{i_d,d}^{n_d}(x_d). \end{aligned}$$

The interpolation properties of $P_1^{n_1} \ldots P_d^{n_d}$ are described by

$$\begin{aligned} &P_1^{n_1} \ldots P_d^{n_d}(f)(x_{i_1,1}, \ldots, x_{i_d,d}) = f(x_{i_1,1}, \ldots, x_{i_d,d}) \\ &\qquad (i_1 = 1, \ldots, a_1(n_1); \ldots; i_d = 1, \ldots, a_d(n_d)). \end{aligned}$$

The method of *d-variate Boolean interpolation* is defined by the projector

$$B_{q,d} = \bigoplus_{m_1 + \ldots + m_d = q} P_1^{m_1} \ldots P_d^{m_d}$$

[16]. The interpolation properties of $B_{q,d}(f)$ are given by

$$\begin{aligned} &B_{q,d}(f)(x_{i_1,1}, \ldots, x_{i_d,d}) = f(x_{i_1,1}, \ldots, x_{i_d,d}) \\ &\qquad (i_1 = 1, \ldots, a_1(m_1); \ldots; i_d = 1, \ldots, a_d(m_d);\ m_1 + \ldots + m_d = q). \end{aligned}$$

The projector of d-variate Boolean interpolation has an explicit *sum representation* [16]:

$$B_{q,d} = \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \sum_{m_1 + \ldots + m_d = q-j} P_1^{m_1} \ldots P_d^{m_d}.$$

(In this formula, empty sums are zero by definition.)

For $d = 2$ and $d = 3$ we obtain ([16]):

$$B_{q,2} = \sum_{m_1 + m_2 = q} P_1^{m_1} P_2^{m_2} - \sum_{m_1 + m_2 = q-1} P_1^{m_1} P_2^{m_2},$$

$$B_{q,3} = \sum_{m_1+m_2+m_3=q} P_1^{m_1} P_2^{m_2} P_3^{m_3}$$
$$-2 \sum_{m_1+m_2+m_3=q-1} P_1^{m_1} P_2^{m_2} P_3^{m_3} + \sum_{m_1+m_2+m_3=q-2} P_1^{m_1} P_2^{m_2} P_3^{m_3}$$

We have shown in Section 5, Theorem 4, that the remainder projector of $B_{q,2}$ is given by

$$(B_{q,2})^c = (P_1^{q-1})^c I \oplus (P_1^{q-2})^c (P_2^1)^c \oplus \ldots \oplus (P_1^1)^c (P_2^{q-2})^c \oplus I (P_2^{q-1})^c.$$

Our objective is to extend this formula for the remainder projector of d-variate Boolean interpolation $B_{q,d}$. In order to simplify the construction we introduce some new notation. Let

$$Q_1^u = P_u^1, \quad \ldots, \quad Q_s^u = P_u^s,$$
$$Q_i^u = 0 \quad \text{for } i \leqslant 0,$$
$$Q_i^u = I \quad \text{for } i > s$$
$$(u = 1, \ldots, d).$$

It is easily seen that

$$Q_{r,d} = \bigoplus_{\substack{i_1+\ldots+i_d=r \\ i_u \in \mathscr{Z}}} Q_{i_1}^1 \cdots Q_{i_d}^d = \bigoplus_{\substack{i_1+\ldots+i_d=r \\ 1 \leqslant i_u \leqslant r+1-d}} Q_{i_1}^1 \cdots Q_{i_d}^d.$$

Moreover, $Q_{r,d}$ is a chain satisfying

$$Q_{r,d} = 0 \qquad \text{for } r < d,$$
$$Q_{r,d} \leqslant Q_{r+1,d} \qquad \text{for } r \in \mathscr{Z},$$
$$Q_{r,d} = I \qquad \text{for } r \geqslant (s+1)d.$$

First, we determine the remainder projector of $Q_{r,2}$.

LEMMA 2. *The relation*

$$(Q_{r,2})^c = \bigoplus_{i+j=r-1} (Q_i^1)^c (Q_j^2)^c = \bigoplus_{i=0}^{r-1} (Q_i^1)^c (Q_{r-1-i}^2)^c$$

holds for $r \leqslant s$.

Proof. Note first that

$$(Q_i^1)^c = (Q_j^2)^c = 0 \quad \text{for } i, j > s$$

and

$$(Q_i^1)^c \geqslant (Q_{i+1}^1)^c, \quad (Q_j^2)^c \geqslant (Q_{j+1}^2)^c \quad \text{for } i, j \in \mathscr{Z}.$$

Then we obtain

$$\cdot \bigoplus_{i+j=r-1} (Q_i^1)^c (Q_j^2)^c = (Q_s^1)^c \oplus (Q_{s-1}^1)^c \oplus \ldots \oplus (Q_{r-1}^1)^c$$

$$\oplus (Q_{r-2}^1)^c (Q_1^2)^c \oplus \ldots \oplus (Q_1^1)^c (Q_{r-2}^2)^c \oplus (Q_{r-1}^2)^c \oplus \ldots \oplus (Q_s^2)^c$$

in view of $(Q_i^1)^c = (Q_j^2)^c = I$ for $i, j \leqslant 0$. Using

$$(Q_s^1)^c \leqslant \ldots \leqslant (Q_{r-1}^1)^c, \quad (Q_s^2)^c \leqslant \ldots \leqslant (Q_{r-1}^2)^c$$

we obtain

$$(\bigoplus_{i+j=r} Q_i^1 Q_j^2)^c = (Q_{r-1}^1)^c I \oplus (Q_{r-2}^1)^c (Q_1^2)^c \oplus \ldots \oplus (Q_1^1)^c (Q_{r-2}^2)^c \oplus I (Q_{r-1}^2)^c$$

which completes the proof of Lemma 2. ∎

Note that the formula

$$(P_1 Q_r \oplus P_2 Q_{r-1} \oplus \ldots \oplus P_r Q_1)^c = P_r^c I \oplus P_{r-1}^c Q_1^c \oplus \ldots \oplus P_1^c Q_{r-1}^c \oplus I Q_r^c$$

is a special case. The extension to higher dimensions is given in the following

THEOREM 5. *The equality*

$$(\bigoplus_{i_1+\ldots+i_d=r} Q_{i_1}^1 \cdots Q_{i_d}^d)^c = \bigoplus_{i_1+\ldots+i_d=r-d+1} (Q_{i_1}^1)^c \cdots (Q_{i_d}^d)^c$$

is valid.

We apply an induction argument. In view of Lemma 2 the theorem is true for $d = 2$. Note that

$$Q_{r,d} = \bigoplus_{i+i_d=r} Q_{i,d-1} Q_{i_d}^d$$

and

$$\begin{aligned} Q_{i,d-1} &\leqslant Q_{i+1,d-1} && (i \in \mathscr{Z}), \\ Q_{i,d-1} &= 0 && (i < 0), \\ Q_{i,d-1} &= I && (i \geqslant (s+1)(d-1)). \end{aligned}$$

We may apply Lemma 2 and obtain

$$\begin{aligned} (Q_{r,d})^c &= \bigoplus_{i+i_d=r-1} (Q_{i,d-1})^c (Q_{i_d}^d)^c \\ &= \bigoplus_{i+i_d=r-1} (\bigoplus_{i_1+\ldots+i_{d-1}=i+2-d} (Q_{i_1}^1)^c \cdots (Q_{i_{d-1}}^{d-1})^c)(Q_{i_d}^d)^c \\ &= \bigoplus_{i+i_d=r-1} (\bigoplus_{i_1+\ldots+i_{d-1}=i+2-d} (Q_{i_1}^1)^c \cdots (Q_{i_{d-1}}^{d-1})^c (Q_{i_d}^d)^c) \\ &= \bigoplus_{i_1+\ldots+i_d=r-d+1} (Q_{i_1}^1)^c \cdots (Q_{i_d}^d)^c. \ ∎ \end{aligned}$$

Taking into account that $(Q^u_{i_u})^c$ are chains it can be shown that

$$\Big(\bigoplus_{i_1+\ldots+i_d=r} Q^1_{i_1}\cdots Q^d_{i_d}\Big)^c = \bigoplus_{\substack{i_1+\ldots+i_d=r-d+1\\ 0\leq i_u\leq r+1-d}} (Q^1_{i_1})^c\cdots(Q^d_{i_d})^c$$

which indicates that $(B_{r,d})^c$ has a greater combinatorial complexity than $B_{r,d}$.

Using this formula it can be proved by induction on d that the remainder projector $I-B_{r,d}$ has the explicit sum representation

$$I-B_{r,d}$$
$$=\sum_{j=1}^{d}\sum_{h=j}^{d}(-1)^{j-1}\binom{h-1}{j-1}\sum_{1\leq\nu_1<\ldots<\nu_h\leq d}\ \sum_{\substack{i_{\nu_1}+\ldots+i_{\nu_h}=r-d+j\\ i_{\nu_1}\geq 1,\ldots,i_{\nu_h}\geq 1}}(P^{\nu_1}_{i_{\nu_1}})^c\ldots(P^{\nu_h}_{i_{\nu_h}})^c.$$

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