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THE CLASSIFICATION PROBLEM FOR THE CAPACITIES ASSOCIATED WITH THE BESOV AND TRIEBEL-LIZORKIN SPACES

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Introduction

This paper is a survey of the results of a study of the exceptional sets naturally associated with the two special function spaces, the Besov spaces $\Lambda_\alpha^{p,q}$ and the Triebel–Lizorkin spaces $F_\alpha^{p,q}$, $\alpha > 0$, p, $q > 0$, over n-dimensional Euclidean space $\boldsymbol{R}^n$. These exceptional sets (subsets of $\boldsymbol{R}^n$) arise as in [5] as the exceptional sets for a perfect functional completion, i.e. the sets up to which pointwise statements about the functions of the class can be made. In particular, it is the limit of the averages of these functions over balls (as the balls shrink to their common center) that exists in an almost everywhere sense which defines these classes; they are the sets where the limits may fail to exist. An equivalent method of defining these classes is as null sets of certain set functions on $\boldsymbol{R}^n$ – capacities. In this article, these capacities are referred to as Besov capacities and Triebel–Lizorkin capacities, respectively.

Results concerning these null sets have appeared over the years by many authors. Most are concerned with various special cases related to their particular problem. But because these spaces depend on three indices, it seems like a good idea to consider the general case and sort out the various relations among these classes of exceptional sets and especially to determine how they are related to the null sets for the more traditional Hausdorff measures. This we refer to as the classification problem.

Much of the work involving these classes, at least in the more exotic cases, comes from questions in differentiation theory and approximation theory. For example, the reader might consult [3], [16], [19], [21] or any of the sources referred to in these papers. Of course, the case $\alpha = 1$, $p = q = 2$ (in either the Besov or the Triebel–Lizorkin case!) is nothing more than the classical, sets of Newtonian capacity zero. These have found many important applications in the theory of partial differential equations of the second order, especially for linear elliptic equations.

In [3], the general case for the Besov spaces with three indices was considered. There the idea was to represent a Besov function as a trace of a Bessel potential. This is well known in the case $q = p$, but to use this idea when $q \neq p$, it was necessary to replace the ordinary Bessel potentials (of L^p functions on R^{2n}) by Bessel potentials of functions that belong to the mixed norm $L^p - L^q$ class, i.e. L^p on the first n-variables, and then L^q on the remaining n-variables. This approach was only partially successful. The present approach seems to be much better, though several open questions still remain – see III.11 for three of them.

Finally, I want to list three features of this article that the interested reader might find especially interesting:

(1) The null sets for the Triebel–Lizorkin capacities do not depend on q, at least for $p, q > 1$;

(2) The exceptional sets for $\Lambda_\alpha^{p,p}$ agree with those for $F_\alpha^{p,2}$, $p \geqslant 1$;

(3) In order to produce the results on $\Lambda_\alpha^{p,q}$ exceptional sets, it was found very useful to have an alternate (pointwise) description of a Besov function. Such a new representation is given in III.2, and as a consequence, the "nonlinear potentials" – in the sense of Hedberg–Wolff – are determined for the Besov spaces in III.4.

I. Capacity

The term *capacity* will be used for any non-negative set function that is monotone and finite on compact subsets of R^n. For a collection of capacities, there is a natural partial ordering:

$$\text{(1.1)} \qquad \operatorname{cap}_1(\cdot) \ll \operatorname{cap}_2(\cdot) \quad \text{iff } \operatorname{cap}_1(K) = 0 \text{ whenever } \operatorname{cap}_2(K) = 0.$$

These partial orderings are sometimes also referred to as "relations" among the capacities in the collection. Here and throughout, the symbol K is used to represent a compact subset of R^n. The equivalence relation corresponding to $\ll$ will be denoted by $\approx$. The classification problem (cp) for such a collection is to determine all such partial orderings (and hence all equivalences) within the collection. The extended classification problem is to do the same for the enlarged collection obtained by including the Hausdorff capacities H^h. Here, for $h = h(t)$ a monotone increasing function of $t \geqslant 0$ with $h(0) = 0$, we set

$$\text{(1.2)} \qquad H^h(K) = \inf \sum_j h(r_j)$$

where the infimum is over all countable coverings of K by balls; r_j denotes the radius of the jth ball of such a cover. Below we shall use $B(x, r)$ to denote a ball centered at $x \in \boldsymbol{R}^n$ of radius $r > 0$. It should be recalled that although Hausdorff h-measure (see eg. [8]) is not a capacity in our sense, it does have the same null sets as H^h.

The capacities considered in this note are those generated by the Besov spaces $\Lambda_\alpha^{p,q}$ on $\boldsymbol{R}^n$ and by the Triebel–Lizorkin spaces $F_\alpha^{p,q}$ on $\boldsymbol{R}^n$, $0 < \alpha < \infty$, $0 < p < \infty$, $0 < q \leqslant \infty$. Using the standard terminology (e.g. S, S' for the Schwartz spaces of C^∞ rapidly decreasing functions on $\boldsymbol{R}^n$ and its dual space, S_0 the subspace of S consisting of φ for which the Fourier transform $\hat{\varphi}$ has support disjoint from the origin, $S(\varphi) =$ support of φ), the spaces $\Lambda_\alpha^{p,q}$ and $F_\alpha^{p,q}$ can be defined as follows. Let $\Phi \in S$, with $S(\hat{\Phi}) \subset B(0, 2)$ and $\hat{\Phi}(\xi) = 1$ on $B(0, 1)$. Setting $\varphi_0 = \Phi$ and $\hat{\varphi}_k(\xi) = \hat{\Phi}(2^{-k}\xi) - \hat{\Phi}(2^{-k+1}\xi)$, for k an integer $\geqslant 1$, one has for $q < \infty$,

$$\Lambda_\alpha^{p,q} = \{u \in S' \colon |u|_{\alpha,p,q} \equiv \Big[\sum_{k=0}^{\infty} (2^{\alpha k} \|\varphi_k * u\|_p)^q\Big]^{1/q} < \infty\},$$

$$F_\alpha^{p,q} = \{u \in S' \colon \|u\|_{\alpha,p,q} \equiv \Big\|\Big[\sum_{k=0}^{\infty} (2^{\alpha k} |\varphi_k * u(\cdot)|_p)^q\Big]^{1/q}\Big\|_p < \infty\}.$$

Here $\|\cdot\|_p$ denotes the usual L^p norm/quasi-norm on $\boldsymbol{R}^n$, $0 < p < \infty$. The usual changes are made for $q = \infty$. Many of the properties of $\Lambda_\alpha^{p,q}$ and $F_\alpha^{p,q}$ needed in this note can be found in [22] or in [17]. For example, it is known that S is dense in both $\Lambda_\alpha^{p,q}$ and $F_\alpha^{p,q}$, and that the topologies are independent of the choice of Φ. Also $F_\alpha^{p,2}$ coincides with the class $\{G_\alpha * f \colon f \in h^p\}$, $0 < p < \infty$, and in particular with $\{G_\alpha * f \colon f \in h^p\}$, $1 < p < \infty$. Here h^p is the local Hardy space $\{f \in S' \colon \sup_{0<t<1} |\psi_t * f| \in L^p\}$, $\psi \in S$, $\psi_t(x) = t^{-n}\psi(x/t)$, $t > 0$, $\hat{G}_\alpha(\xi) = (1+|\xi|^2)^{-\alpha/2}$, $*$ denotes convolution over $\boldsymbol{R}^n$. Clearly, $F_\alpha^{p,p} = \Lambda_\alpha^{p,p}$. Perhaps a more familiar definition of $\Lambda_\alpha^{p,q}$, $1 \leqslant p, q < \infty$, is via differences: $u \in \Lambda_\alpha^{p,q}$ iff $u \in L^p$ and $\langle u \rangle_{\alpha,p,q} < \infty$;

$$\langle u \rangle_{\alpha,p,q} = \Big\{\int_{\boldsymbol{R}^n} \Big[\int_{\boldsymbol{R}^n} |\Delta_h^k u(x)|^p dx\Big]^{q/p} |h|^{-\alpha q - n} dh\Big\}^{1/q}, \tag{1.3}$$

for $0 < \alpha < k$, $k.=$ positive integer, $\Delta_h^1 u(x) = u(x+h) - u(x)$, $\Delta_h^k = \Delta_h^1 \Delta_h^{k-1}$. Also $|\cdot|_{\alpha,p,q}$ is comparable to $\|\cdot\|_p + \langle\cdot\rangle_{\alpha,p,q}$. Here and throughout, the term comparable (symbolically: $\sim$) is used between two quantities when their ratio is bounded above and below by positive finite quantities independent of the critical parameters. Finally, there are the dual spaces $(\Lambda_\alpha^{p,q})^* = \Lambda_{-\alpha}^{p',q'}$, especially for $\alpha > 0$; here $1 \leqslant p, q < \infty$, and $p' = p/(p-1)$. Similarly, $(F_\alpha^{p,q})^* = F_{-\alpha}^{p',q'}$.

Sometimes it is simpler, because of the homogeneity, to use the homogeneous spaces $\dot{\Lambda}_\alpha^{p,q}$ and $\dot{F}_\alpha^{p,q}$. These are defined by simply modifying the corresponding norms/quasi-norms by extending the sums in those norms/quasi-norms over all integers k rather than just over the non-negative integers; for this, we take $\hat{\varphi}_k(\xi) = \hat{\Phi}(2^{-k}\xi) - \hat{\Phi}(2^{-k+1}\xi)$ for all k, and identify functions/distributions that differ by a polynomial. The important point to keep in mind, however, is that elements in the spaces $\dot{F}$ and $\dot{\Lambda}$ have the same basic local behavior as those in F and Λ; they can be made to agree up to a C^∞ function – see [17] and [22]. In fact, an equivalent semi-norm on $\dot{\Lambda}_\alpha^{p,q}$ is $\langle\cdot\rangle_{\alpha,p,q}$, and $\Lambda_\alpha^{p,q} = \dot{\Lambda}_\alpha^{p,q} \cap L^p$.

Additional notation:

$$I_\alpha * f(x) = \int |x-y|^{\alpha-n} f(y)\, dy,$$

and

$$M_\alpha f(x) = \sup_{t>0} t^{\alpha-n} \int_{B(x,t)} f(y)\, dy; \qquad 0 < \alpha < n.$$

$M^+(K)$ = non-negative Radon measures with support in K; $\mu \in M^+(K)$, $\|\mu\|_1$ = total mass of μ. And generally the superscript "+" denotes the non-negative elements in a class (the "positive cone").

Now with $X = \Lambda_\alpha^{p,q}$ or $F_\alpha^{p,q}$, $0 < \alpha < \infty$, $0 < p, q < \infty$, and $\|\cdot\|_X$ the corresponding norm/quasi-norm, we set

$$\text{(1.4)} \qquad \operatorname{cap}(K; X) = \inf\{\|u\|_X^p\colon\ u \in S \ \&\ u \geqslant 1 \text{ on } K\}.$$

Note that because of the continuous embedding of X into the space of bounded continuous functions when $\alpha p > n$, we easily have for $x \in K$, $1 \leqslant |u(x)| \leqslant c\|u\|_X$, for some constant c. Hence $\operatorname{cap}(K; X) \neq 0$ unless $K = \emptyset$, in this case. (This also happens for $\operatorname{cap}(\cdot\,; F_\alpha^{p,q})$ when $\alpha p = n$, $0 < p < 1$, for example.) Hence we may immediately restrict our attention at least to those triples (α, p, q) for which $\alpha p \leqslant n$. Also, since the spaces X are locally convex when $1 \leqslant p, q < \infty$, the minimax theorem applies (see e.g. [14]) and yields the following alternate characterization:

$$\text{(1.5)} \qquad \operatorname{cap}(K; X)^{1/p} = \sup \|\mu\|_1,$$

where the supremum is over all $\mu \in M^+(K) \cap X^*$ for which $\|\mu\|_{X^*} \leqslant 1$.

In order to get an initial handle on the (cp) for these capacities, it is useful to compute the Hausdorff dimension of $\operatorname{cap}(\cdot\,; X)$. By this we shall mean the number d, $0 < d < n$, such that

$$\text{(1.6)} \qquad H^{d+\varepsilon} \ll \operatorname{cap}(\cdot\,; X) \ll H^{d-\varepsilon}$$

for all $0 < \varepsilon < d$. Here H^d now denotes H^h with $h(t) = t^d$. Setting $d = \dim\operatorname{cap}(\cdot\,; X)$ and $n-d = \operatorname{codim}\operatorname{cap}(\cdot\,; X)$, we easily have that (1.7) implies (1.8), where

$$\text{(1.7)} \qquad \operatorname{codim}\operatorname{cap}(\cdot\,; X_1) < \operatorname{codim}\operatorname{cap}(\cdot\,; X_2)$$

and

$$\text{(1.8)} \qquad \operatorname{cap}(\cdot\,; X_1) \ll \operatorname{cap}(\cdot\,; X_2).$$

We understand that $\dim \operatorname{cap}(\cdot\,; X) = 0$ when $\operatorname{cap}(\cdot\,; X) \gg H^{\varepsilon}$, for all $\varepsilon > 0$. We shall see that in most cases

$$\dim \operatorname{cap}(\cdot\,; X) = n - \alpha p \tag{1.9}$$

when $X = \Lambda_{\alpha}^{p,q}$ or $F_{\alpha}^{p,q}$. Also, from [14] this is well known for the spaces $F_{\alpha}^{p,2}$, $1 < p < \infty$. In fact, it is easy to see that $F_{\alpha+\varepsilon}^{p,2} \subset \Lambda_{\alpha}^{p,q} \subset F_{\alpha-\varepsilon}^{p,2}$, for $0 < \varepsilon < \alpha$. Hence (1.9) holds for the Besov spaces, for $1 < p, q < \infty$. Thus for these spaces, the (cp) is reduced to the study of capacities of the same dimensions. This then is where the subtle role of the exponent q comes into play.

Finally, we should mention that whenever there is a continuous embedding $X_1 \subset X_2$, then $\operatorname{cap}(\cdot\,; X_2) \preccurlyeq \operatorname{cap}(\cdot\,; X_1)$. However, this is a poor way to proceed since there are far more relations than there are embeddings.

II. Triebel-Lizorkin capacity

1. The Nilsson result

We begin with the rather remarkable fact that the null sets for the capacities $\operatorname{cap}(\cdot\,; F_{\alpha}^{p,q})$ are the same for all q, $1 < q < \infty$, at least for each $\alpha > 0$ and $1 < p < \infty$. This observation was made by P. Nilsson in the Fall of 1983. To do this, we show that the positive cone in $F_{-\alpha}^{p',q'}$ is the same as the positive cone in $F_{-\alpha}^{p',2}$. Consequently, (1.5) implies $\operatorname{cap}(\cdot\,; F_{\alpha}^{p,q}) \approx \operatorname{cap}(\cdot\,; F_{\alpha}^{p,2})$, $1 < q < \infty$, $1 < p < \infty$, $\alpha > 0$. Thus in this case, the study of the Triebel-Lizorkin capacities reduces to the study of the $F_{\alpha}^{p,2}$-capacities, i.e. to the Bessel capacities. These later capacities have been treated extensively in the literature – see [13], [14], [4].

Our assertion about the positive cones follows from

THEOREM 2.1. *If μ is a Borel measure on $\mathbf{R}^n$, then there are constants c_1 and c_2 independent of μ such that*

(a) $$\|\mu\|_{-\alpha,p',q'} \leqslant c_1 \|G_\alpha * \mu\|_{p'},$$

(b) $$\|M_\alpha^1 \mu\|_{p'} \leqslant c_2 \|\mu\|_{-\alpha,p',q'}.$$

Here $M_\alpha^1 \mu(x)$ is the maximal function $\sup_{0<t<1} t^{\alpha-n}\mu(B(x,t))$. The result follows from the theorem by observing the following non-homogeneous version of the Muckenhoupt–Wheeden result [15]: there is a constant c_3 independent of μ such that

$$\|G_\alpha * \mu\|_{p'} \leqslant c_3 \|M_\alpha^1 \mu\|_{p'}, \quad 1 < p < \infty. \tag{2.1}$$

From the well-known asymptotic behavior of G_α, it is clear that the reverse inequality to (2.1) trivially holds. Finally, notice that $\|G_\alpha * \mu\|_{p'}$ is equivalent to the $F_{-\alpha}^{p',2}$-norm of μ.

To see (a) of Theorem 2.1, write

$$\|\mu\|_{-\alpha,p',q'} \leqslant \|\sum_k (2^{-\alpha k}|\varphi_k * \mu(\cdot)|)\|_{p'}.$$

Then with $\hat{\psi}(\xi) = \hat{\Phi}(\xi) - \hat{\Phi}(2\xi)$, note that

$$\sum_{k \geqslant 0} 2^{(n-\alpha)k} \psi(2^k x) = J * G_\alpha(x)$$

where J is a bounded operator on $L^{p'}$. Consequently,

$$\|\mu\|_{-\alpha,p',q'} \leqslant \|J * G_\alpha * \mu\|_{p'} \leqslant c\|G_\alpha * \mu\|_{p'}. \tag{2.2}$$

For part (b), we use the characterizations of the F-spaces by "ball means of differences" as given in [22], p. 108. However, for simplicity we focus our attention here on the homogeneous case and note that there is a constant $c = c(\beta)$ such that if $\alpha \neq$ positive integer and $2\beta - 2 < \alpha < 2\beta$, for $\beta =$ positive integer, then

$$c\varrho^{\alpha-2\beta} \int \Delta^2_{\varrho h}(I_{2\beta} * \mu)(x) \cdot \Delta^\beta \Psi(-h)\, dh = \varrho^\alpha \{2^{2\beta} \Psi_{2\beta} * \mu(x) - 2\Psi_\varrho * \mu(x)\},$$

for $\Psi \in C_0^\infty(\boldsymbol{R}^n)$, $S(\Psi) \subset B(0, 2)$, $\Psi(x) = 1$, $|x| \leqslant 1$. Δ^β denotes the βth power of the n-dimensional Laplacians. Thus, one easily gets

$$\|M_\alpha \mu\|_{p'} \leqslant c\|I_{2\beta} * \mu\|_{\dot{F}} \tag{2.3}$$

where the $\dot{F}$-norm is for $\dot{F}^{p',q'}_{2\beta-\alpha}$. Finally, for integral α, one just uses the semigroup property of I_α.

2. The case $p = 1$, $q = 2$

Again for simplicity, we will just consider the homogeneous case. Now $F^{1,2}_\alpha = G_\alpha * H^1$, $H^1 =$ classical real Hardy space on $\boldsymbol{R}^n$. The dual of H^1 is BMO = space of functions of bounded mean oscillation. But $I_\alpha * \mu \in \mathrm{BMO}$ iff $M_\alpha \mu \in L^\infty$ (see [2]) and hence by Frostman's theorem (see [7], Chapter 2), we get

$$\mathrm{cap}(\cdot\,; \dot{F}^{1,2}_\alpha) \approx H^{n-\alpha}, \quad 0 < \alpha < n. \tag{2.4}$$

3. The case $0 < p < 1$, $q = 2$

Here the duality approach does not work, since the Hardy spaces H^p are not locally convex. The idea then is to use the atomic decomposition of H^p to work directly with the definition of $\mathrm{cap}(\cdot\,; F^{p,2}_\alpha)$, $0 < \alpha < n/p$, $0 < p < 1$. This yields the following result which gives, as a consequence, the lower bound

$$H^{n-\alpha p} \ll \mathrm{cap}(\cdot\,; F^{p,2}_\alpha). \tag{2.5}$$

THEOREM 2.2. *There is a constant c such that for all* $f \in H^p(\boldsymbol{R}^n)$,

$$\int [(G_\alpha * f)^*]^p \, dH^h \leqslant c \cdot \|f\|_{H^p}^p, \qquad 0 < \alpha < n/p, \; 0 < p \leqslant 1. \tag{2.6}$$

$h(r) = r^{\alpha p - n}$ *for* $0 < r \leqslant 1$, *and* r^n *for* $r \geqslant 1$.

Here F^* refers to the non-tangential maximal function $\sup |\varphi_\varepsilon * F(y)|$ where the supremum is over all $y \in \boldsymbol{R}^n$ and $\varepsilon > 0$ such that $|x-y| < \varepsilon$. Also, the integral in (2.6) should be understood in the sense of Choquet, i.e.

$$\int F^p \, dH = \int_0^\infty H\{F > \lambda\} \, d\lambda^p.$$

The proof of (2.6) follows closely the ideas contained in [12], at least in estimating

$$\int [(G_\alpha * f)^*]^p \, d\mu,$$

where μ is any Borel measure for which $\mu(B(x, r)) \leqslant (\text{constant}) \cdot h(r)$, for all $r > 0$ and all $x \in \boldsymbol{R}^n$. The passage from such measures to H^h follows from:

THEOREM 2.3. *The integral* $\int \varphi \, dH^h$, *for all lower semicontinuous* $\varphi \geqslant 0$, *is comparable to* : $\sup \int \varphi \, d\mu$, *where the supremum is over all Borel measures* μ *such that* $|||\mu|||_h \leqslant 1$, *where*

$$|||\mu|||_h = \sup_{x; r > 0} h(r)^{-1} \mu(B(x, r)). \tag{2.7}$$

This theorem can be viewed as an extension of the result of Frostman quoted earlier: $H^h(K) > 0$ iff there is a Borel measure μ with $S(\mu) \subset K$ and $|||\mu|||_h < \infty$. (See reference [23].)

An upper bound on $\mathrm{cap}(\cdot\,; F_\alpha^{p,2})$, $0 < \alpha < n/p$, $0 < p < 1$, is somewhat harder to obtain. Again the idea has been to exploit the atomic decomposition of H^p; however, no real satisfactory result is yet known. The obvious conjecture – and one that is satisfied for many examples – is that all of the capacities $\mathrm{cap}(\cdot\,; F_\alpha^{p,2})$, for $0 < \alpha < n/p$, $0 < p < 1$, are bounded above by a multiple of the $(n-\alpha p)$-dimensional Minkowski content, $M_{n-\alpha p}$, of Section III.9 below. (Some partial results of this type have recently been obtained by J. Orobitg at Universitat Autonoma, Barcelona.)

III. Besov capacity

1. The positive cone in the dual space

Using ideas due to J. Peetre [18] and J. Polking [20], one can prove the following lemma that leads to a useful characterization of the positive cone in $\Lambda_{-\alpha}^{p',q'}$, $\alpha > 0$, $1 < p < \infty$, $1 < q < \infty$.

LEMMA 3.1. *Let $\chi \in S$, $\alpha > 0$, $1 \leqslant p, q < \infty$, then there is a constant c independent of u such that*

$$\left\{\int_0^1 (\varepsilon^\alpha \|\chi_\varepsilon * u\|_p)^q \frac{d\varepsilon}{\varepsilon}\right\}^{1/q} \leqslant c\,|\chi|_{\alpha,1,1} \cdot |u|_{-\alpha,p,q}. \tag{3.1}$$

If, in addition, χ is nonnegative and radially decreasing, then the quantity on the left side of (3.1) *is a comparable norm for $(\Lambda^{p,q}_{-\alpha})^+$. The homogeneous version of* (3.1) *is*

$$\left\{\int_0^\infty (\varepsilon^\alpha \|\chi_\varepsilon * u\|_p)^q \frac{d\varepsilon}{\varepsilon}\right\}^{1/q} \leqslant c\,\langle\chi\rangle_{\alpha,1,1} \cdot \langle u\rangle_{-\alpha,p,q}. \tag{3.1$'$}$$

By taking an appropriate χ in Lemma 3.1, we have that

$$\left\{\int_0^1 (\varepsilon^\alpha \|\varepsilon^{-n} \mu(B(\cdot, \varepsilon))\|_{p'})^{q'} \frac{d\varepsilon}{\varepsilon}\right\}^{1/q'} \tag{3.2}$$

is an equivalent norm on $\mu \in (\Lambda^{p',q'}_{-\alpha})^+$, $\alpha > 0$, $1 < p < \infty$, $1 \leqslant q < \infty$. But upon noting that

$$\|\mu(B(\cdot, \varepsilon))\|_{p'}^{p'} \sim \varepsilon^n \int \mu(B(y, \delta\varepsilon))^{p'-1}\, d\mu(y),$$

where $\delta = 2$ for the upper bound and $\delta = \frac{1}{2}$ for the lower bound, we are led to

THEOREM 3.2. *For $\alpha > 0$, $1 < p < \infty$, $\mu \in (\Lambda^{p',q'}_{-\alpha})^+$ iff*

$$E_{\alpha,p,q}(\mu) \equiv \int_0^1 \left[\int \left(t^{\alpha p - n} \mu(B(x, t))\right)^{p'-1} d\mu(x)\right]^{q'/p'} \frac{dt}{t} \tag{3.3}$$

is finite, $1 < q < \infty$, and

$$E_{\alpha,p,1}(\mu) \equiv \sup_{0 < t \leqslant 1} \int \left(t^{\alpha p - n} \mu(B(x, t))\right)^{p'-1} d\mu(x) \tag{3.4}$$

is finite, $q = 1$. Furthermore, $E_{\alpha,p,q}(\cdot)^{1/q'}$ is an equivalent norm on the positive cone. (*Here E stands for "generalized energy."*)

2. A characterization of $\Lambda^{p,q}_\alpha$, $\alpha > 0$

Using Lemma 3.1, we can also give a new characterization of $\Lambda^{p,q}_\alpha$ as "almost" potentials. First consider $u \in \dot{\Lambda}^{p,q}_\alpha$. If we set

$$f(x, t) = |t|^{-\alpha} \Delta_t^k u(x), \quad 0 < \alpha < k \in \mathbf{Z}^+,\ x, t \in \mathbf{R}^n,$$

then clearly

$$\left\{\int \|f(\cdot, t)\|_p^q \frac{dt}{|t|^n}\right\}^{1/q} = \langle u\rangle_{\alpha,p,q}.$$

Now for $\varphi \in S$, set

$$\dot{S}_\alpha f(x) \equiv \int\int \varphi\left(\frac{x-y}{|t|}\right) f(y, t)\, |t|^{\alpha-2n}\, dy\, dt = \int \varphi_{|t|} * f^t(x)\, |t|^{\alpha-n}\, dt$$

where $f^t(y) = f(y, t)$. Then there is a constant c independent of u such that $\dot{S}_\alpha\hat{f}(\xi) = c \cdot \hat{u}(\xi)$. Hence $\dot{S}_\alpha f(x) = c \cdot u(x)$, for $u \in S_0$. The next result is the converse.

THEOREM 3.3. *If $f\colon \mathbf{R}^{2n} \to \mathbf{R}$ is such that*

$$\int \|f(\cdot, t)\|_p^q \frac{dt}{|t|^n} < \infty, \tag{3.5}$$

then $\dot{S}_\alpha f \in \Lambda_\alpha^{p,q}$ and there is a constant c such that

$$\langle \dot{S}_\alpha f \rangle_{\alpha,p,q} \leqslant c \left\{\int \|f(\cdot, t)\|_p^q \frac{dt}{|t|^n}\right\}^{1/q}, \quad 1 < p < \infty,\ 1 \leqslant q < \infty.$$

Thus $u \in \Lambda_\alpha^{p,q}$ iff $u = \dot{S}_\alpha f$ for some f satisfying (3.5).

The proof of this result follows by duality and Lemma 3.1. For $u \in \Lambda_\alpha^{p,q}$, we have $u = S_\alpha f$ where

$$S_\alpha f(x) \equiv \int_0^1 \varphi_t * f^t(x) \cdot t^\alpha \frac{dt}{t}, \tag{3.6}$$

and $f^t(y) = f(y, t)$ is now defined on $\mathbf{R}^n \times [0, 1]$ and satisfies

$$\int_0^1 \|f(\cdot, t)\|_p^q \frac{dt}{t} < \infty. \tag{3.7}$$

3. An equivalent Besov capacity

Using these S-potentials, we can easily formulate a definition of capacity which together with its dual will be equivalent to $\mathrm{cap}(\cdot\,; \Lambda_\alpha^{p,q})$. We take

$$\inf\left\{\left(\int_0^1 \|f(\cdot, t)\|_p^q \frac{dt}{t}\right)^{p/q} : f \geqslant 0\ \&\ S_\alpha f \geqslant 1 \text{ on } K\right\} \tag{3.8}$$

and a dual formulation

$$\sup\{\|\mu\|_1\colon \mu \in M^+(K)\ \&\ E_{\alpha,p',q'}(\mu) \leqslant 1\}. \tag{3.9}$$

The equality of (3.8) and the pth power of (3.9) follows from the minimax theorem in the standard way (see e.g. [14]), and because of Theorem 3.2, both are equivalent to $\mathrm{cap}(\cdot\,; \Lambda_\alpha^{p,q})$; $\alpha > 0$, $1 < p < \infty$, $1 \leqslant q < \infty$.

4. Nonlinear Besov potentials

We are thus motivated to use the following equivalent norms for $\Lambda_\alpha^{p,q}$ and $\Lambda_{-\alpha}^{p',q'}$ (but retaining the old notation for simplicity):

$$\|S_\alpha f\|_{\alpha,p,q} = \left(\int_0^1 \|f(\cdot,t)\|_p^q \frac{dt}{t}\right)^{1/q} \tag{3.10}$$

$$|\mu|_{-\alpha,p',q'} = \left(\int_0^1 [t^{\alpha-n}\|\mu(B(\cdot,t))\|_{p'}]^{q'} \frac{dt}{t}\right)^{1/q'}. \tag{3.11}$$

With this, we choose f_K and μ_K to be extremals for (3.8) and its dual capacity (the same as (3.9) except that (3.11) is used in place of the equivalent energy formulation). Then as in [14], these two elements are closely related and in our case, we get that $S_\alpha f_K(x)$ is comparable to

$$\int_0^1 [t^{\alpha p-n}\nu(B(x,t))]^{p'-1}\left(\int [t^{\alpha p-n}\nu(B(z,t))]^{p'-1}\,d\nu(z)\right)^{(q'/p')-1}\frac{dt}{t}. \tag{3.12}$$

Here ν is normalized μ_K, normalized so that $S_\alpha f_K$, which is bounded on $S(\mu_K)$, will have a bound there independent of the set K. Formula (3.12) – with upper endpoint of integration equal to one – defines the Besov space analogue of the nonlinear potentials associated with the Besov capacities. For the Bessel capacities $\operatorname{cap}(\cdot; F_\alpha^{p,2})$, the similar role is played by (3.12) with $q = p$; see [9]. In fact, it is just this observation – that both $(\Lambda_{-\alpha}^{p',p'})^+$ and $(F_{-\alpha}^{p',2})^+$ are characterized by (3.12) with $q = p$ – that implies the equivalence of $\operatorname{cap}(\cdot; \Lambda_\alpha^{p,p})$ and $\operatorname{cap}(\cdot; F_\alpha^{p,2})$; $\alpha > 0$, $1 < p < \infty$. Of course, we already know this from Theorem 2.1 since $\Lambda_\alpha^{p,p} = F_\alpha^{p,p}$.

5. Partial orderings via energy estimates

If we denote the nonlinear potential given in (3.12) by $W_{\alpha,p,q}^{\nu}(x)$, then the following energy estimates hold.

(i) Estimates valid for any codimension $\leqslant n$:

(a) if $\alpha p = \beta r$, $q'/p = s'/r$, $r \leqslant p$, then

$$E_{\alpha,p,q}(\mu) \leqslant E_{\beta,r,s}(\mu)\cdot\|\mu\|_1^{q'-s'}, \tag{3.13}$$

(b) if $\alpha p = \beta r$, $q/p = s/r$, $p \leqslant r$, then there is a constant c independent of μ such that

$$E_{\alpha,p,q}(\mu) \leqslant c\cdot E_{\beta,r,s}(\mu)\cdot[\sup_{x\in S(\mu)} W_{\beta,r,s}^{\mu}(x)]^{((r/p)-1)q'}, \tag{3.14}$$

(c) if $q \leqslant s$, then there is a c independent of μ such that

$$E_{\alpha,p,q}(\mu)^{1/q'} \leqslant c\cdot E_{\alpha,p,s}(\mu)^{1/s'}. \tag{3.15}$$

(ii) Estimates valid for codimension n only:

(d) if $p'/q' - r'/s' < \sigma$, $\sigma = (s-1)(r-p)/(p-1)(r-1)$, $s \leqslant r$, $p < r$, then there is a constant c such that

$$E_{n/p,p,q}(\mu)^{p'/q'} \leqslant c \cdot E_{n/r,r,s}(\mu)^{r'/s'} [\sup_{x \in S(\mu)} W^{\mu}_{n/r,r,s}(x)]^{\sigma}. \tag{3.16}$$

(In estimates (3.13)–(3.16), it may be necessary to interpret one or more of the energy functions appearing in an upper bound to involve integration over the interval (0, 2) rather than the customary (0, 1).)

Now to deduce partial ordering from these energy estimates, we need

THEOREM 3.4. *If* $\operatorname{cap}(K; \Lambda^{p,q}_{\alpha}) > 0$, $\alpha > 0$, $1 < p < \infty$, $1 \leqslant q < \infty$, *then there exists a Borel measure* μ *such that*

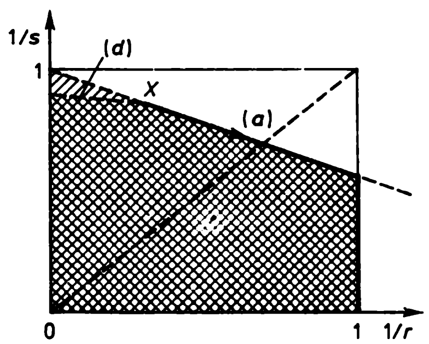
$$\mu \in M^{+}(K) \cap (\Lambda^{p',q'}_{-\alpha})^{+} \tag{3.17}$$

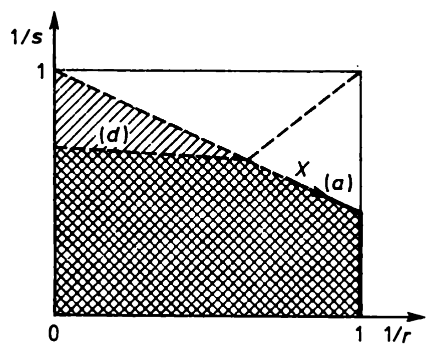
and

$$W^{\mu}_{\alpha,p,q}(x) \text{ is bounded on } S_{\mu}. \tag{3.18}$$

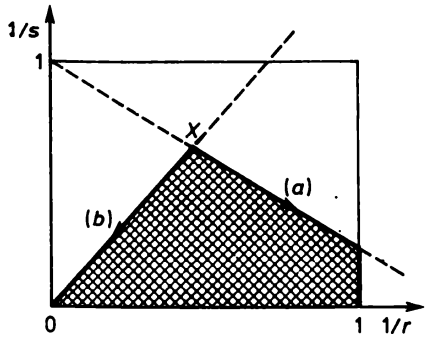
We summarize the results below with diagrams. The X represents the capacity $\operatorname{cap}(K; \Lambda^{p,q}_{\alpha})$ which we are assuming to be zero. The shaded region, including the solid lines with arrows represents possible capacities $\operatorname{cap}(K; \Lambda^{r,s}_{\beta})$ which must necessarily be zero as a consequence.

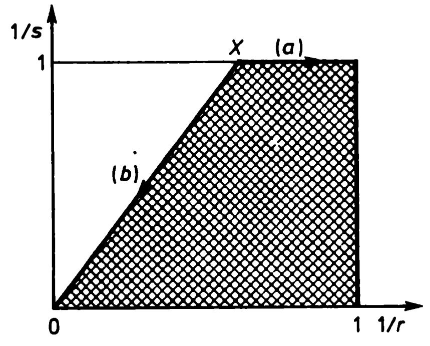
For $\alpha p = \beta r = n$:





For $\alpha p = \beta r = n-d$, $0 \leqslant d < n$:





Later, using estimates in terms of Hausdorff capacity, we will improve these diagrams, at least the codimension n diagrams, by showing that we may shade in the entire region below the line $r/s' = p/q'$.

6. Besov capacity of a ball

Here we discuss the calculation

$$\operatorname{cap}(B(x, t); \Lambda_\alpha^{p,q}), \quad \alpha > 0,\ 1 < p \leqslant n/\alpha,\ 1 \leqslant q < \infty.$$

To do this, we note that $\operatorname{cap}(K; \Lambda_\alpha^{p,q})^{q/p}$ is comparable to

$$\sup\{\|\mu\|_1\colon \mu \in M^+(K) \ \&\ W_{\alpha,p,q}^\mu \leqslant 1 \text{ on } S(\mu)\}. \tag{3.19}$$

Also there is a universal constant M such that $\operatorname{cap}(K; \Lambda_\alpha^{p,q})^{q/p}$ is comparable to

$$\inf\{\|\mu\|_1\colon \mu \in M^+(K) \ \&\ W_{\alpha,p,q}^\mu \geqslant 1 \text{ on } K \ \&\ W_{\alpha,p,q}^\mu \leqslant M \text{ on } S(\mu)\}. \tag{3.20}$$

With these we need only take μ to be a constant multiple of Lebesgue measure restricted to the ball to prove

THEOREM 3.5. *As* $t \to 0$,

$$\operatorname{cap}(B(x, t); \Lambda_\alpha^{p,q}) \sim \begin{cases} t^{n-\alpha p}, & \alpha p < n, \\ (\log 1/t)^{-p/q'}, & \alpha p = n. \end{cases} \tag{3.21}$$

7. Hausdorff capacity vs. Besov capacity

Now when the Besov capacity is countably subadditive, we can use (3.21) to obtain an upper bound in terms of H^h, where $h = h(t)$ is either of the two measure functions appearing in (3.21). However, countable subadditivity is only known in the case $p \leqslant q$. The present section is an attempt to give some Hausdorff capacity upper bound when $q < p$. The key is to investigate the relationship between the Besov capacity and the capacity corresponding to the class $D_{\alpha;\delta}^p$ given below. As in (3.6), form the operator ($\delta > 0$)

$$S_{\alpha;\delta} f(x) = \int_0^{1/2} \varphi_t * f^t(x) \cdot t^\alpha (\log 1/t)^{-\delta} \frac{dt}{t},$$

and then with $D_{\alpha;\delta}^p = \{S_{\alpha;\delta} f\colon f \in L^p(\mathbf{R}^n \times [0, \tfrac{1}{2}])\}$, the capacity $\operatorname{cap}(\cdot\,; D_{\alpha;\delta}^p)$ has the properties:

(3.22) $\operatorname{cap}(\cdot\,; \Lambda_\alpha^{p,q}) \leqslant \operatorname{cap}(\cdot\,; D_{\alpha;\delta}^p)$, for $1/q - 1/p < \delta$, $\alpha p \leqslant n$;

(3.23) $\operatorname{cap}(B(x, t); D_{\alpha;\delta}^p)$ is comparable to $t^{n-\alpha p}(\log 1/t)^{\delta p}$ as $t \to 0$, for $\alpha p < n$, and is comparable to $(\log 1/t)^{\delta p - p + 1}$ as $t \to 0$, for $\alpha p = n$;

(3.24) $\operatorname{cap}(\cdot\,; D_{\alpha;\delta}^p)$ is countably subadditive.

With these properties, we easily have

THEOREM 3.6. *For* $\alpha > 0$, $1 < p \leqslant n/\alpha$, $1 \leqslant q < \infty$,

$$\operatorname{cap}(\cdot\,; \Lambda_\alpha^{p,q}) \ll H^h \tag{3.25}$$

where (i) *if* $\alpha p < n$

$$\begin{aligned} h(t) &= t^{n-\alpha p}, \qquad p \leqslant q \\ &= t^{n-\alpha p}(\log 1/t)^{p/q-1+\varepsilon}, \qquad q < p,\ \varepsilon > 0, \end{aligned}$$

and (ii) *if* $\alpha p = n$

$$\begin{aligned} h(t) &= (\log 1/t)^{-p/q'}, \qquad p \leqslant q \\ &= (\log 1/t)^{-p/q'+\varepsilon}, \qquad q < p, \quad \varepsilon > 0. \end{aligned}$$

Notice that with this result, we can shade in the lower portion $p/q' < r/s'$ in our figures illustrating the codimension n case. This is because it is well known (see [4]) that $\operatorname{cap}(\cdot\,; B_{n/p}^{p,p}) \ll H^h$, for $h(t) = (\log 1/t)^{1-r}$, $r > p$. Thus we have:

THEOREM 3.7. $\operatorname{cap}(\cdot\,; \Lambda_{n/r}^{r,s}) \ll \operatorname{cap}(\cdot\,; \Lambda_{n/p}^{p,q})$ *whenever* (1) $p/q' < r/s'$ *or* (2) $p/q' = r/s'$ *and* $r \leqslant p$.

Also, there is the following lower bound:

THEOREM 3.8. *For* $1 < p < n/\alpha$, $H^{n-\alpha p} \ll \operatorname{cap}(\cdot\,; \Lambda_\alpha^{p,1})$.

This follows from (3.9) and (3.4), and the often-quoted theorem of Frostman.

8. Besov capacity of Cantor-type sets

To see that the region above the line $r/s' = p/q'$ in our diagrams cannot generally be shaded in as well, we compute the Besov capacity of a Cantor-type set. Let $\{l_k\}$ be a sequence of positive numbers that satisfy $l_k - 2l_{k-1} > 0$, $k = 1, 2, 3, \ldots$, $l_0 = 1$, and set E_k equal to the union of the 2^k intervals remaining in the interval $[0, 1]$ after open middle intervals of length l_k are removed from E_{k-1}. $E_0 = [0, 1]$. Set $E = \bigcap_k E_k^n$, the intersection of the products $E_k^n = E_k \times \ldots \times E_k$. Now following the ideas of Carleson [7] and Besicovitch–Taylor [6], we have

THEOREM 3.9.

(i) $$H^d(E) \sim \liminf_{k\to\infty} 2^{nk} l_k^d, \qquad 0 < d \leqslant n;$$

(ii) $$\operatorname{cap}(E; \Lambda_\alpha^{p,1}) \sim \liminf_{k\to\infty} 2^{nk} l_k^{n-\alpha p}, \qquad \alpha p < n;$$

(iii) $$\operatorname{cap}(E; \Lambda_\alpha^{p,q}) \sim \Big[\sum_k (2^{-kn} l_k^{\alpha p-n})^{q'/p}\Big]^{-p/q'}, \quad \alpha p < n,\ q > 1;$$

(iv) $$\operatorname{cap}(E; \Lambda_{n/p}^{p,q}) \sim \Big[\sum_k \{(\log(l_k/l_{k+1}))^{p/q'} 2^{-nk}\}^{q'/p}\Big]^{-p/q'}, \quad q > 1.$$

9. Minkowski content and Besov capacity

Here, we record a relationship between the Besov capacities and the finiteness of the set function

$$M_d(K) \equiv \liminf_{\varepsilon\to 0} \varepsilon^{d-n}|K_\varepsilon|, \tag{3.26}$$

the lower Minkowski content of K. Here $K_\varepsilon = \varepsilon$-neighborhood of K. An equivalent version of M_d is the set function

$$S_d(K) \equiv \liminf_{\varepsilon\to 0} N(\varepsilon; K)\varepsilon^d, \tag{3.27}$$

where $N(\varepsilon; K)$ is the minimum number of open balls of radius $\varepsilon > 0$ needed to cover K.

THEOREM 3.10. *For* $1 < p < n/\alpha$, $q > 1$, $M_{n-\alpha p}(K) < \infty$ *implies*

$$\operatorname{cap}(K; \Lambda_\alpha^{p,q}) = 0.$$

This result follows by considering the Besov capacity in mn-dimensions of the product set $K\times \ldots \times K$, $K \subset \boldsymbol{R}^n$. In fact, we can use the minimax theorem to show

$$[\operatorname{cap}(K; \Lambda_\alpha^{p,q}(\boldsymbol{R}^n))]^m \leqslant \operatorname{cap}(K\times \ldots \times K; \Lambda_{\alpha m}^{p,p}(\boldsymbol{R}^{mn})), \tag{3.28}$$

when $mp' = q'$. The capacity on the right side of (3.28) is an equivalent Bessel capacity which is zero when the standard Hausdorff measure of dimension $m(n-\alpha p)$ in $\boldsymbol{R}^{mn}$ of the product is finite (see [14]). Thus the conclusion follows by noting that $S_{n-\alpha p}(K)^m$ exceeds this Hausdorff measure.

Note that if we used the corresponding Minkowski capacity, $S^d(K) = \inf_{\varepsilon>0} N(\varepsilon; K)\varepsilon^d$, which is always finite, then one easily gets the estimate: $\operatorname{cap}(\cdot; \Lambda_\alpha^{p,q}) \ll S^{n-\alpha p}(\cdot)$, $\alpha p < n$, $q > 1$.

10. The case $p \leqslant 1$

Here we have

$$H^{n-\alpha p} \ll \operatorname{cap}(\cdot; \Lambda_\alpha^{p,p}) \tag{3.29}$$

for $0 < p < 1$, $n(1/p-1) < \alpha < n/p$. This follows from the trace theorems for the Besov spaces – see [10], [11]. In particular, on subsets of $\boldsymbol{R}^n$

$$\operatorname{cap}(\cdot; \Lambda_\alpha^{p,p}(\boldsymbol{R}^n)) \approx \operatorname{cap}(\cdot; F_{\alpha+n/p}^{p,2}(\boldsymbol{R}^{2n})) \ll H^{n-\alpha p},$$

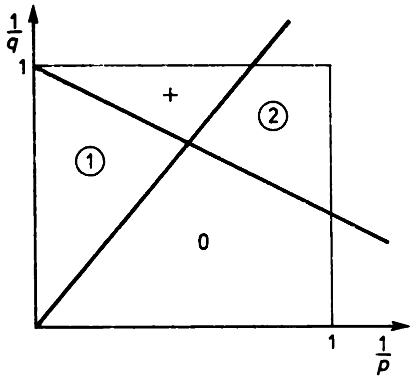
for $0 < p < 1$, $\alpha > n/p - n$. The last implication follows from (2.5). Also from (2.4) we have

$$\operatorname{cap}(\cdot\,; \Lambda_\alpha^{1,1}) \approx H^{n-\alpha}, \qquad 0 < \alpha < n.$$

11. Open questions

Several obvious questions remain.

(i) The dimension d-diagram, $d > 0$, appears to be far from complete – certainly less so that the $d = 0$ diagram. So what should the diagram look like, or rather what are the possibilities? For instance, if all of the capacities in the region marked with a $+$, in the diagram below, are positive on some given set while those in the region marked with a 0 are zero, then what are the possibilities for regions ① and ②? When $d = 0$ our theory plus the Cantor construction implies that the only possibility is that the capacities of region ① are zero and those of region ② are positive. Does this also remain true for $d > 0$?



(ii) Does the relation $\operatorname{cap}(\cdot\,; \Lambda_\alpha^{p,p}) \approx \operatorname{cap}(\cdot\,; F_\alpha^{p,p})$ persist for $0 < p < 1$?

(iii) What can be said about $\operatorname{cap}(\cdot\,; \Lambda_\alpha^{p,q})$ for $0 < p < 1$, $0 < q < \infty$?

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