COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 1

ON MAPS WHICH ARE PERFECT WITH RESPECT TO THE HEWITT REALCOMPACT EXTENSION

BY

A. BŁASZCZYK (KATOWICE)

The aim of this paper is to give a characterization of maps of a Tychonoff space which induce on the Hewitt realcompact extension maps carrying the remainder into the remainder. In the case of the Čech-Stone compactification and of the Katětov extension such characterizations are known. We give a generalization of a theorem of Dykes [1] on preimages of realcompact spaces.

All spaces are assumed to be Tychonoff and all maps are assumed to be continuous.

1. Preliminaries. A space X is *realcompact* iff it is a closed subspace of the product of copies of the real line.

The *Hewitt realcompact extension* ν_X: $X \subset \nu X$ is a dense embedding of X into a realcompact space, characterized (up to homeomorphism) by the following condition (Hewitt [5]):

(1) for each $f: X \to Y$, where Y is a realcompact space, there exists exactly one continuous map $\tilde{f}: \nu X \to Y$ completing the diagram

$$\begin{array}{ccc} X & \subset & \nu X \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \tilde{f}} & \\ Y & & \end{array}$$

From (1) it follows immediately that for each $f: X \to Y$ (Y need not be here realcompact) there exists exactly one continuous map νf: $\nu X \to \nu Y$ completing the diagram

(2)
$$\begin{array}{ccc} X & \subset & \nu X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \nu f} \\ Y & \subset & \nu Y \end{array}$$

A map $f: X \to Y$ will be said to be *v-perfect* provided the map $vf: vX \to vY$ carries the remainder into the remainder, i. e., $vf(vX \setminus X) \subset vY \setminus Y$ (in analogy to those maps in the diagram of the Čech-Stone compactification which behave in the same way and usually are called *perfect*; cf. Henriksen and Isbell [4]).

Clearly, if $f: X \to Y$ is a v-perfect map and Y is realcompact, then X is realcompact. Indeed, by condition (1), a space is realcompact iff the remainder of the Hewitt realcompact extension is empty.

For every X, the Hewitt realcompact extension vX can be constructed as a subspace of the Čech-Stone compactification βX. Namely, vX is equal to the intersection of the family of all sets $F^{-1}(E)$, where $F: \beta X \to \omega E$ (ωE denotes here the one-point Alexandroff compactification of the real line E) is the extension of a map $f: X \to E$ (cf. Hewitt [5] or Engelking [2], p. 156). By a slight modification of the argument, we infer that vX equals βX without all zero-sets, with respect to βX, contained in $\beta X \setminus X$.

A filter is a *z-filter* provided it consists of zero-sets; it is said to be a *z-ultrafilter* if it is maximal in the family of all z-filters. We say that a z-filter $\mathfrak{F}$ has the *countable intersection property* (*c. i. p.*) if each countable subfamily of $\mathfrak{F}$ has a non-empty intersection. It is known (cf. Hewitt [5] or Engelking [2], p. 155) that a space X is realcompact iff every z-ultrafilter in X with the c. i. p. has a non-empty intersection. Let $\mathfrak{F}$ be a z-ultrafilter in X. A point $x \in X$ is a *limit in X* of $\mathfrak{F}$ provided $\{x\} = \bigcap\{\mathrm{Cl}_X A: A \in \mathfrak{F}\}$. A point $y \in \beta X$ is a *limit in βX of a z-ultrafilter* $\mathfrak{F}$ provided $\{y\} = \bigcap\{\mathrm{Cl}_{\beta X} A: A \in \mathfrak{F}\}$.

The following theorem describes the Hewitt realcompact extension in terms of z-ultrafilters:

THEOREM 1 (Gillman and Jerison [3], p. 118). *Every point in vX is the limit in βX of a unique z-ultrafilter in X with the c. i. p. Conversely, the limit in βX of any z-ultrafilter in X with the c. i. p. belongs to vX.*

Proof. 1. It suffices to consider only points of the remainder of vX. Let $x_0 \in vX \setminus X$. The family

$$\mathfrak{F}(x_0) = \{Z \cap X: Z \text{ is a zero-set in } \beta X \text{ and } x_0 \in Z\} \tag{3}$$

is a z-ultrafilter in X. Indeed, by the remark on vX as βX without zero-sets, every zero-set which contains x_0 meets X. Since an intersection of a countable family of zero-sets is a zero-set, $\mathfrak{F}(x_0)$ has the c. i. p. Clearly, $x_0 \in \mathrm{Cl}_{\beta X}(Z \cap X)$ for every zero-set Z in βX such that $x_0 \in Z$. Thus x_0 is the limit of $\mathfrak{F}$ in βX. Uniqueness is obvious.

2. Clearly, every z-ultrafilter has the limit in βX. It remains to prove that it belongs to vX if it has the c. i. p. Let $\mathfrak{F}$ be such a z-ultrafilter and let $\{x_0\} = \bigcap\{\mathrm{Cl}_{\beta X} Z: Z \in \mathfrak{F}\}$. Suppose that $x_0 \notin vX$. Then there exists a function $f: \beta X \to I$ such that $f(x_0) = 0$ and $f(x) > 0$ for each $x \in vX$.

Clearly, $Z_n = f^{-1}([0, 1/n]) \cap X$ belongs to $\mathfrak{F}$ for every integer $n = 1, 2, \ldots$ But $\bigcap_{n=1}^{\infty} Z_n = \emptyset$; a contradiction.

2. ν-perfect maps. Let us note that each perfect map $f: X \to Y$ is ν-perfect. In fact, there exists, by (2), an extension νf: $\nu X \to \nu Y$. Since $X \subset \nu X \subset \beta X$, βX is equal (up to a homeomorphism) to the Čech-Stone compactification of the Hewitt realcompact extension νX. Then there exists a map βf: $\beta X \to \beta Y$ being an extension of νf. Since f is perfect, $\beta f(\beta X \setminus X) \subset \beta Y \setminus Y$. Thus $\nu f(\nu X \setminus X) \subset \nu Y \setminus Y$ which means that f is ν-perfect.

THEOREM 2. *A map* f: $X \xrightarrow{\text{onto}} Y$ *is ν-perfect iff*

(4) *for each $y \in Y$ and for each z-ultrafilter $\mathfrak{F}$ with the c. i. p. and empty intersection, there exists a $Z \in \mathfrak{F}$ such that the set $f(Z)$ is closed and* $Z \cap f^{-1}(y) = \emptyset$.

Proof. 1. Let us assume that condition (4) holds. Then there exists the map νf: $\nu X \to \nu Y$ completing diagram (2). It suffices to show that $\nu f(\nu X \setminus X) \subset \nu Y \setminus Y$. Suppose there exists a point $x_0 \in \nu X \setminus X$ such that $\nu f(x_0) = y \in Y$. By Theorem 1, the family $\mathfrak{F}(x_0)$, defined by (3), is a z--ultrafilter with the c. i. p. and empty intersection. Hence, by (4), there exists $Z \in \mathfrak{F}(x_0)$ such that $f(Z)$ is closed and $f^{-1}(y) \cap Z = \emptyset$. Consider the set $V = Y \setminus f(Z)$. Clearly, V is an open neighbourhood of y. Then there exists the set $W \subset \nu Y$ which is open in νY and such that $W \cap Y = V$. Since $(\nu f)^{-1}(W)$ is an open neighbourhood of x_0, there exists a continuous function g: $X \to I$ such that $g(x_0) = 0$ and $g(x) = 1$ for $x \notin (\nu f)^{-1}(W)$. Observe that $Z_0 = X \cap g^{-1}(0)$ belongs to $\mathfrak{F}(x_0)$ and $Z_0 \subset X \cap (\nu f)^{-1}(W)$. But we have

$$(\nu f)^{-1}(W) \cap X = f^{-1}(V) = f^{-1}\big(Y \setminus f(Z)\big) = X \setminus f^{-1}\big(f(Z)\big) \subset X \setminus Z,$$

whence $Z \cap Z_0 = \emptyset$; a contradiction.

2. Now assume that the map f: $X \to Y$ is ν-perfect. Let $\mathfrak{F}$ be an arbitrary z-ultrafilter with the c. i. p. and empty intersection, and let $y \in Y$. By Theorem 1, $\bigcap\{\mathrm{Cl}_{\beta X} Z\colon Z \in \mathfrak{F}\} = \{x_0\}$ and $x_0 \in \nu X \setminus X$. Since f is ν-perfect, $y \neq \nu f(x_0)$. Hence there exists a function h: $\nu Y \to I$ such that $h(y) = 1$ and $h\big(\nu f(x_0)\big) = 0$. Consider the composition $\varphi = h \circ f$ and the set $Z = X \cap \varphi^{-1}(0)$. Clearly, Z is a zero-set and, by Theorem 1, $Z \in \mathfrak{F}$. It is easy to see that $Z \cap f^{-1}(y) = \emptyset$ and $f(Z)$ is closed in Y. In fact,

$$f(Z) = f\big((\nu f)^{-1}(h^{-1}(0)) \cap X\big) = \nu f\big((\nu f)^{-1}(h^{-1}(0)) \cap X\big) = h^{-1}(0) \cap Y.$$

Thus condition (4) holds.

A map f: $X \to Y$ is said to be a *Z-map* if the image under f of each zero-set in X is closed in Y. A map f is a *WZ-map* if $(\beta f)^{-1}(y) = \mathrm{Cl}_{\beta X} f^{-1}(y)$

for every $y \in Y$, where $\beta f\colon \beta X \to \beta Y$ is an extension of f. Isiwata [6] has shown that all Z-maps are WZ-maps.

THEOREM 3. *If a map $f\colon X \xrightarrow{\text{onto}} Y$ is a WZ-map and $f^{-1}(y)$ is closed in νX for each $y \in Y$, then f is ν-perfect.*

Proof. Suppose, on the contrary, that there exists a point $x_0 \in \nu X \setminus X$ such that $\nu f(x_0) = y \in Y$. Clearly, $\beta f\colon \beta X \to \beta Y$ is an extension of νf. Then $\beta f(x_0) = y$. Since $f^{-1}(y)$ is closed in νX, $x_0 \notin \mathrm{Cl}_{\beta X} f^{-1}(y)$. But f is a WZ-map, hence $\beta f(x_0) \neq y$; a contradiction.

Note. The converse of Theorem 3 is not true. In fact, if X is realcompact, then each map $f\colon X \to Y$ is ν-perfect. Clearly, such a map is not, in general, a WZ-map. For example, if E is the real line, $A = \{x_0\} \cup (\beta E \setminus E)$, where $x_0 \in E$, and $\varphi\colon \beta E \to \beta E/A$ is the natural map, then the restriction $\varphi \mid E\colon E \to \beta E/A$ is ν-perfect but not a WZ-map.

COROLLARY (Dykes [1]). *If $f\colon X \xrightarrow{\text{onto}} Y$ is a WZ-map onto a realcompact space and $f^{-1}(y)$ is closed in νX for each $y \in Y$, then X is realcompact.*

Proof. The remainder of νY is empty. By Theorem 3, $\nu f(\nu X \setminus X) \subset \nu Y \setminus Y$. Hence the remainder of νX is empty.

Clearly, if $f\colon X \to Y$ is a map onto a realcompact space Y, then X is realcompact iff f is ν-perfect. Hence Theorem 2 can be considered as the final result on preimages of realcompact spaces. This result is actually a generalization of that of N. Dykes. The following theorem gives a criterion for a map to be ν-perfect.

THEOREM 4. *If a Z-map $f\colon X \to Y$ is onto and $f^{-1}(y)$ is Lindelöf for each $y \in Y$, then f is ν-perfect.*

Proof. We shall show that condition (4) holds. Suppose, on the contrary, that there exist $y \in Y$ and a z-ultrafilter $\mathfrak{F}$ with the c. i. p. and empty intersection such that, for each $Z \in \mathfrak{F}$, $Z \cap f^{-1}(y) \neq \emptyset$. Note that the family $\{Z \cap f^{-1}(y)\colon Z \in \mathfrak{F}\}$ is a z-filter with the c. i. p. in $f^{-1}(y)$. In fact, $\mathfrak{F}$ is closed with respect of countable intersections. Since $f^{-1}(y)$ is Lindelöf, $\bigcap\{Z \cap f^{-1}(y)\colon Z \in \mathfrak{F}\} \neq \emptyset$. Thus $\mathfrak{F}$ has a non-empty intersection; a contradiction.

Note. Easy examples show that the converse is not true.

REFERENCES

[1] N. Dykes, *Mappings and realcompact spaces*, Pacific Journal of Mathematics 31 (1969), p. 347-358.

[2] R. Engelking, *Outline of general topology*, Amsterdam 1968.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton 1960.

[4] M. Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Mathematical Journal 25 (1958), p. 83-106.

[5] E. Hewitt, *Rings of real-valued continuous functions, I*, Transactions of the American Mathematical Society 64 (1948), p. 54-99.
[6] T. Isiwata, *Mappings and spaces*, Pacific Journal of Mathematics 20 (1967), p. 455-480.

SILESIAN UNIVERSITY, KATOWICE

Reçu par la Rédaction le 23. 6. 1971;
en version modifiée le 25. 10. 1971