ANNALES
POLONICI MATHEMATICI
XXII (1969)

On a class of quasi-conformal mappings with invariant boundary points, I

The class E_Q and the general extremal problem

by J. Ławrynowicz (Łódź)

Introduction.

1. Let Δ and $\mathcal{E}$ denote the closed unit disc and the closed plane, respectively, and let the class S_Q be defined as follows:

Definition 1A. A function f is said to be of the class S_Q if it maps Δ onto itself Q-quasi-conformally with $f(0) = 0$ and $f(1) = 1$.

Clearly, this is equivalent to

Definition 1B. $f \in S_Q$ if $f(z) = f^*(z)$ identically for $z \in \Delta$, where f^* is defined in $\mathcal{E}$, maps it onto itself Q-quasi-conformally with $f^*(0) = 0$, $f^*(1) = 1$, $f^*(\infty) = \infty$, and satisfies $f^*(z) = 1/\overline{f^*(1/\bar{z})}$ for $z \in \mathcal{E}$, $z \neq 0, \infty$.

We have also

Definition 1C. $f \in S_Q$ if $f(z) = f^*(z)$ identically for $z \in \Delta$, where f^* is defined in $\mathcal{E}$ as follows:

(i) f^* is a function whose complex dilatation $\mu^* = \partial f^*/\partial\bar{z} : \partial f^*/\partial z$ satisfies

$$\mu^*(z) = e^{4i\arg z}\overline{\mu^*(1/\bar{z})} \quad \text{a.e. in } \mathcal{E}, \tag{1}$$

(ii) f^* maps $\mathcal{E}$ onto itself Q-quasi-conformally with $f^*(0) = 0, f^*(1) = 1$, $f^*(\infty) = \infty$.

Obviously, Definition 1B implies Definition 1C.

Conversely, suppose that $f \in S_Q$ according to Definition 1C. By the well-known theorem on existence and uniqueness (see e.g. [8], p. 204) there exists exactly one function f^* defined in $\mathcal{E}$ which maps it onto itself Q-quasi-conformally with $f^*(0) = 0$, $f^*(1) = 1$, $f^*(\infty) = \infty$, and has μ^* as its complex dilatation a.e. in $\mathcal{E}$. On the other hand, the function $f^{**}(z) = 1/\overline{f^*(1/\bar{z})}$ $(z \neq 0, \infty)$, $f^{**}(z) = z$ $(z = 0, \infty)$ is also defined in $\mathcal{E}$ and maps it onto itself Q-quasi-conformally with $f^{**}(0) = 0$, $f^{**}(1) = 1$, $f^{**}(\infty) = \infty$, and its complex dilatation μ^{**} satisfies $\mu^{**}(z) = e^{4i\arg z}\overline{\mu^*(1/\bar{z})}$, a.e. in $\mathcal{E}$. Since $\mu^{**}(z) = \mu^*(z)$ a.e. in $\mathcal{E}$, then $f^{**}(z) = f^*(z)$ identically in $\mathcal{E}$, and the proof of equivalence is completed.

2. Hence we see that the functional equation (1) for the complex dilatation μ^* of f^* together with $f^*(0) = 0$, $f^*(1) = 1$, $f^*(\infty) = \infty$ implies that $|f^*(z)| = 1$ for $|z| = 1$. This result is not trivial and it is rather difficult to obtain it from the Beltrami differential equation.

Naturally, this suggests looking for other functional equations concerned with μ^* which may imply some interesting geometric conditions for f^* and, consequently, for f. For instance, it would be interesting to derive a functional equation for μ^* which implies such a property as the invariance of the boundary points of f (cf. [17] and [5]). A functional equation with the properties described above would make it possible to apply the parametric method introduced by Shah Tao-shing [15] to the class under consideration, and then to obtain sharp estimates of some functionals.

The above facts were kindly pointed out to me by Professor F. W. Gehring during my stay at the Imperial College, London, where I held a scholarship of the Polish Academy of Sciences, under the guidance of Professor W. K. Hayman. I am also indebted to Professor J. Krzyż from Lublin for his helpful remarks during the preparation of this paper.

3. Let a be a real number such that a/π is irrational. The present paper is mainly concerned with a class E_Q, defined precisely in the next section, which is connected with the equation

$$\mu^*(z) = e^{2ia+4i\arg z}\overline{\mu^*(e^{ia}/\bar{z})} \quad \text{a.e. in } \mathcal{E} \tag{2}$$

in the same way as S_Q is connected with (1). The class is independent of the choice of a, and any $f \in E_Q$ satisfies $f(z) = z$ for $|z| = 1$. To the best of our knowledge functions of this class but under a stronger hypothesis $\arg f(z) = \arg z$ were first considered by Künzi [7] (pp. 25-26). On the other hand, this is a subclass of a class introduced in [11] (pp. 161-163).

We find a parametric representation for functions f of the class E_Q and determine the extremal functions for a wide class of functionals dependent on f. Our general results are then applied in order to derive the regions of variability of the functionals $f(z)/z$, $f(z)-z$ and $f(z_1)-f(z_2)$, z, z_1, z_2 being fixed, and f running over E_Q. The methods of proof are chosen so that they may also be applied to other classes of mappings, which are considered in Sections 17-24.

Next, analogous results are obtained for a class E_Q^*, defined precisely in Section 17, which is connected with the equation $\mu(z) = e^{-2ia}\mu(e^{ia}z)$, valid a.e. in $\mathcal{E}$; a/π being real and irrational. Finally, two families of classes of Q-quasi-conformal mappings with some analogous properties are introduced. They have a clear geometric interpretation. Two of those classes, E_Q^* and H_Q^*, the latter being defined in Section 22, may be interpreted as normalized "elliptic" and "hyperbolic" classes, and used in order to generate a wide subclass of the class of all normalized Q-quasi-conformal mappings of the closed plane onto itself in such a way that the methods given for E_Q can be transferred to these cases. This can be done by considering the classes $E_Q^{(1)} = E_Q^* \circ H_Q^*$, $E_Q^{(n)} = E_Q^{(1)} \circ E_Q^{(n-1)}$ $(n = 2, 3, \ldots)$ or $H_Q^{(1)} = H_Q^* \circ E_Q^*$, $H_Q^{(n)} = H_Q^{(1)} \circ H_Q^{(n-1)}$ $(n = 2, 3, \ldots)$, where $A \circ B$ denotes the class of all compositions $f \circ g$, i.e. $w = f(g(z))$; $f \in A$, $g \in B$. Another way of generalization is to consider various classes of quasi-conformal mappings which are solutions of Beltrami differential equations with separated variables, e.g. mappings with the complex dilatation of the form $\mu(z) = \mu_1(|z|)\mu_2(\arg z)$, $\mu(z) = \mu_1^*(\mathrm{re}\, z)\mu_2^*(\mathrm{im}\, z)$ or $\mu(z) = \hat{\mu}_1(z)\hat{\mu}_2(\bar{z})$, where $\mu_1, \mu_2, \mu_1^*, \mu_2^*, \hat{\mu}_1, \hat{\mu}_2$ depend on the shown variables only. The classes E_Q, E_Q^*, H_Q, H_Q^* may serve here as example. The final part of the paper is an announcement of the topics which will be investigated in subsequent papers.

The introduced classes of quasi-conformal mappings have a clear physical interpretation. The author hopes to apply the present and future results to the theory of physical phenomena in thin films of solids.

§ 1. The class E_Q.

4. DEFINITION 2A. A function f is said to be of class E_Q if it belongs to S_Q and if $f(z) = e^{i\arg z} f(|z|)$ for $z \in \Delta$, $z \neq 0$.

The definition implies

$$(3)\qquad \begin{aligned} |f(z)| &= |f(|z|)| \equiv R(|z|)\,, \\ \arg\big(f(z)/z\big) &= \arg f(|z|) \equiv \theta(|z|) \qquad (z \neq 0)\,. \end{aligned}$$

Now we obtain for E_Q analogues of Definitions 1B and 1C.

DEFINITION 2B. $f \in E_Q$ if $f(z) = f^*(z)$ identically for $z \in \Delta$, where f^* is defined in $\mathcal{E}$, maps it onto itself Q-quasi-conformally with $f^*(0) = 0$, $f^*(1) = 1$, $f^*(\infty) = \infty$, and satisfies $f^*(z) = e^{-i\alpha}/\overline{f^*(e^{i\alpha}/\bar{z})}$ for $z \in \mathcal{E}$, $z \neq 0, \infty$, where α is a real number such that α/π is irrational.

Definition 2A implies that f^*, defined by $f^*(z) = f(z)$ for $z \in \Delta$ and by $f^*(z) = 1/\overline{f(1/\bar{z})}$ for $z \notin \Delta$, $z \neq \infty$, $f^*(\infty) = \infty$, maps $\mathcal{E}$ onto itself Q-quasi-conformally with $f^*(0) = 0$, $f^*(1) = 1$, and satisfies in it $(z \neq 0, \infty)$

$$e^{-i\alpha}/\overline{f^*(e^{i\alpha}/\bar{z})} = e^{-i\alpha} f^*(z/e^{-i\alpha}) = e^{-i\alpha} e^{i\alpha + i\arg z} f^*(|z|) = f^*(z)$$

for any real α. Hence $f \in E_Q$ according to Definition 2B.

Conversely, suppose that $f \in E_Q$ according to Definition 2B. Clearly $f \in S_Q$ if we can prove that $|f(z)| = |z|$ for $|z| = 1$, and so we only have to verify that $f(z) = e^{i\arg z} f(|z|)$ for $z \in \Delta$, $z \neq 0$. To this end we notice first that setting $\zeta = e^{ia}/\bar{z}$ in the equation $f^*(\zeta) = e^{-ia}/\overline{f^*(e^{ia}/\bar{\zeta})}$ we get $f^*(e^{ia}/\bar{z}) = e^{-ia}/\overline{f^*(e^{2ia}z)}$, i.e. $1/\overline{f^*(e^{ia}/\bar{z})} = e^{-ia}f^*(e^{2ia}z)$. Hence $f^*(z) = e^{-ia}/\overline{f^*(e^{ia}/\bar{z})} = e^{-2ia}f^*(e^{2ia}z)$. Suppose now that we have $f^*(z) = e^{-2(n-1)ia}f^*(e^{2(n-1)ia}z)$ for a positive integer n. Taking $\zeta = e^{2(n-1)ia}z$ in $f^*(\zeta) = e^{-2ia}f^*(e^{2ia}\zeta)$, we obtain $f^*(e^{2(n-1)ia}z) = e^{-2ia}f^*(e^{2nia}z)$. Consequently

$$f^*(z) = e^{-2(n-1)ia}f^*(e^{2(n-1)ia}z) = e^{-2nia}f^*(e^{2nia}z) , \tag{4}$$

and so (4) holds for any positive integer n, and it can easily be seen that it holds for any integer n. Let us observe that since a/π is irrational, by a theorem of Kronecker, for any real ϑ there exists a sequence $\{2n_k a + 2m_k\pi\}$, where m_k $(k = 1, 2, \ldots)$ are integers, tending to ϑ as $k \to +\infty$. Setting $\vartheta = -\arg z$ we obtain $f^*(z) = e^{i\arg z}f^*(|z|)$, and since $f^*(z) = f(z)$ for $z \in \Delta$, the desired condition follows.

Definition 2C. $f \in E_Q$ if $f(z) = f^*(z)$ identically for $z \in \Delta$, where f^* is defined in $\mathcal{E}$ as follows:

(i) f^* is a function whose complex dilatation μ^* satisfies condition (2) where a is a real number such that a/π is irrational,

(ii) f^* maps $\mathcal{E}$ onto itself Q-quasi-conformally with $f^*(0) = 0, f^*(1) = 1$, $f^*(\infty) = \infty$.

Definition 2B implies that $f_z, f_{\bar{z}}$ exist a.e. in Δ (see e.g. [8], p. 172), and that

$$\begin{aligned} f^*_z(z) &= \frac{-e^{-ia}}{\{\overline{f^*(e^{ia}/\bar{z})}\}^2}\frac{\partial}{\partial z}\overline{f^*(e^{ia}/\bar{z})} = \frac{-e^{-ia}}{\{\overline{f^*(e^{ia}/\bar{z})}\}^2}\overline{f^*_{\bar{z}}(e^{ia}/\bar{z})} \\ &= \frac{-e^{-ia}}{\{\overline{f^*(e^{ia}/\bar{z})}\}^2}\overline{[f^*_{\bar{\zeta}}(\zeta)]}_{\zeta=e^{ia}/\bar{z}}\overline{\frac{\partial}{\partial\bar{z}}(e^{ia}/\bar{z})} \\ &= \frac{-e^{-2ia}}{z^2\{\overline{f^*(e^{ia}/\bar{z})}\}^2}\overline{[f^*_{\bar{\zeta}}(\zeta)]}_{\zeta=e^{ia}/\bar{z}} , \end{aligned}$$

and, similarly,

$$f^*_{\bar{z}}(z) = \frac{-e^{-ia}}{\{\overline{f^*(e^{ia}/\bar{z})}\}^2}\overline{[f^*_{\zeta}(\zeta)]}_{\zeta=e^{ia}/\bar{z}}\overline{\frac{\partial}{\partial z}(e^{-ia}/z)} = \frac{1}{\bar{z}^2\{\overline{f^*(e^{ia}/\bar{z})}\}^2}\overline{[f^*_{\zeta}(\zeta)]}_{\zeta=e^{ia}/\bar{z}} .$$

Hence, denoting by μ^* the complex dilatation of f^*, we get for μ^* the condition required in Definition 2C.

Conversely, suppose that $f \in E_Q$ according to Definition 2C. By the well-known theorem on existence and uniqueness (see e.g. [8], p. 204) the function f^* defined in that definition is determined uniquely. On the other hand, the function $f^{**}(z) = e^{-ia}/\overline{f^*(e^{ia}/\bar{z})}$ $(z \neq 0, \infty)$, $f^{**}(z) = z$ $(z = 0, \infty)$ is also defined in $\mathcal{E}$ and maps it onto itself Q-quasi-conformally with $f^{**}(0)=0$, $f^{**}(1)=1$, $f^{**}(\infty)=\infty$, and its complex dilatation μ^{**} satisfies $\mu^{**}(z)=e^{2ia+4i\arg z}\overline{\mu^{*}(e^{ia}/\bar{z})}$ a.e. in $\mathcal{E}$. Since $\mu^{**}(z)=\mu^{*}(z)$ a.e. in $\mathcal{E}$, then $f^{**}(z)=f^{*}(z)$ identically in $\mathcal{E}$, and the proof of equivalence is completed.

5. Definitions 2A, 2B and 2C are connected with certain functional equations for f, f^{*} and μ^{*}, respectively. It is natural to find for E_Q also a definition connected with a functional equation for μ. Clearly, there is no analogue of such a definition for S_Q.

DEFINITION 2D. A function belonging to S_Q is said to be of class E_Q if its complex dilatation μ satisfies $\mu(z)=e^{2i\arg z}\mu(|z|)$ a.e. in Δ.

Definition 2A implies that $f_z, f_{\bar{z}}$ exist a.e. in Δ (see e.g. [8], p. 172), and that, if $r=|z|$, $\varepsilon=e^{i\arg z}$, $\hat{f}(r)=[f(z)]_{z=r}$, we have

$$f_z(z)=\frac{\partial}{\partial r}\{\varepsilon\hat{f}(r)\}\frac{\partial}{\partial z}(z\bar{z})^{1/2}+\frac{\partial}{\partial\varepsilon}\{\varepsilon\hat{f}(r)\}\frac{\partial}{\partial z}(z/\bar{z})^{1/2}$$
$$=\tfrac{1}{2}(\bar{z}/z)^{1/2}\varepsilon\hat{f}'(r)+\tfrac{1}{2}(z\bar{z})^{-1/2}\hat{f}(r)\,.$$

Thus

(5) $$f_z(z)=\tfrac{1}{2}\{[\hat{f}'(r)]_{r=|z|}+(1/|z|)f(|z|)\}\,,$$

and, similarly,

(6) $$f_{\bar{z}}(z)=\tfrac{1}{2}e^{2i\arg z}\{[\hat{f}'(r)]_{r=|z|}-(1/|z|)f(|z|)\}\,.$$

Hence $\mu(z)=e^{2i\arg z}\mu(|z|)$ a.e. in Δ, i.e. $f\in E_Q$ according to Definition 2D.

Suppose now that $f\in E_Q$ according to Definition 2D. Setting $\mu^{*}(z)=\mu(z)$ for $z\in\Delta$ and $\mu^{*}(z)=e^{4i\arg z}\overline{\mu(1/\bar{z})}$ for $z\notin\Delta$, $z\neq\infty$, we see that

$$\exp(2ia+4i\arg z)\overline{\mu^{*}(e^{ia}/\bar{z})}$$
$$=\exp\big(-2ia+4i\arg(z/e^{-ia})\big)\overline{\mu^{*}(e^{ia}/\bar{z})}=\exp(-2ia)\mu^{*}(e^{ia}z)$$
$$=\exp(-2ia)\exp(2ia+2i\arg z)\mu^{*}(|z|)=\mu^{*}(z)$$

for any real a. Hence we conclude that $f\in E_Q$ according to Definition 2C, where f^{*} is defined by $f^{*}(z)=f(z)$ for $z\in\Delta$ and by $f^{*}(z)=1/\overline{f(1/\bar{z})}$ for $z\notin\Delta$, $z\neq\infty$, $f^{*}(\infty)=\infty$.

§ 2. Further properties of E_Q. Bounds for R and θ.

6. By relations (5) and (6) we can find two more definitions for the class E_Q. The second of them implies sharp estimates of R and θ (cf. (3)), given in Theorems 1 and 2, respectively.

DEFINITION 2E. $f\in E_Q$ if $f\in S_Q$ and $zf_z(z)-\bar{z}f_{\bar{z}}(z)=f(z)$ a.e. in Δ.

Definition 2A implies that $f_z, f_{\bar{z}}$ exist a.e. in Δ, and that relations (5) and (6) hold. Hence $zf_z(z)-\bar{z}f_{\bar{z}}(z)=e^{i\arg z}f(|z|)=f(z)$ a.e. in Δ, i.e. $f\in E_Q$ according to Definition 2E.

Conversely, suppose that $f \in E_Q$ according to Definition 2E. Hence, setting $r = |z|$, $\varepsilon = e^{i\arg z}$, we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left\{\frac{1}{\varepsilon}f(r\varepsilon)\right\} &= -\frac{1}{\varepsilon^2}f(r\varepsilon)+\frac{1}{\varepsilon}\left\{f_z(z)\frac{\partial}{\partial\varepsilon}(r\varepsilon)+f_{\bar z}(z)\frac{\partial}{\partial\varepsilon}(r/\varepsilon)\right\}\\ &= (1/\varepsilon^2)\{r\varepsilon f_z(z)-(r/\varepsilon)f_{\bar z}(z)-f(r\varepsilon)\}\\ &= e^{-2i\arg z}\{zf_z(z)-\bar z f_{\bar z}(z)-f(z)\} = 0 \qquad \text{a.e. in } \Delta.\end{aligned}$$

Consequently $e^{-i\arg z}f(z) = C(|z|)$, and choosing $z \geqslant 0$ we obtain $C(|z|) = f(|z|)$. Thus $f \in E_Q$ according to Definition 2A.

Definition 2F. $f \in E_Q$ if it is given by the formulae

$$(7)\qquad \begin{aligned}&f(z) = \exp\left(-\int_{|z|}^{1}\frac{1+\mu(r)}{1-\mu(r)}\frac{dr}{r}+i\arg z\right) \qquad \text{for } z \in \Delta,\ z \neq 0,\\ &f(z) = 0 \qquad \text{for} \qquad z = 0,\end{aligned}$$

where μ is measurable with $\sup\limits_{0<r\leqslant 1}|\mu(r)| < 1$ and $\operatorname{ess\,sup}\limits_{0<r\leqslant 1}|\mu(r)| \leqslant \dfrac{Q-1}{Q+1}$.

Remark 1. In general, μ is assumed to be complex-valued.

Definition 2A implies that $f_z, f_{\bar z}$ exist a.e. in Δ, and that relations (5) and (6) hold, where

$$(8)\qquad \hat f(r) = [f(z)]_{z=r} \qquad (r > 0).$$

Hence, by Definition 2D,

$$\mu(r) = e^{-2i\arg z}\frac{f_{\bar z}(z)}{f_z(z)} = \frac{\hat f'(r)-(1/r)\hat f(r)}{\hat f'(r)+(1/r)\hat f(r)} \qquad (r = |z|) \text{ a.e. in } \Delta.$$

Consequently,

$$\frac{\hat f'(r)}{\hat f(r)} = \frac{1+\mu(r)}{1-\mu(r)}(1/r) \qquad \text{a.e. in } \Delta,$$

and, by $\hat f(1) = f(1) = 1$,

$$(9)\qquad \hat f(|z|) = \exp\left(-\int_{|z|}^{1}\frac{1+\mu(r)}{1-\mu(r)}\frac{dr}{r}\right),$$

where the integral exists by Definition 2A. Since (8) and Definition 2A imply $f(z) = e^{i\arg z}\hat f(|z|)$ for $z \in \Delta$, $z \neq 0$, and $f(0) = 0$, we conclude that f satisfies (7) as desired.

Conversely, suppose that $f \in E_Q$ according to Definition 2F. Hence

$$(10)\qquad \frac{\hat f'(r)-(1/r)\hat f(r)}{\hat f'(r)+(1/r)\hat f(r)} = \mu(r) \qquad (r = |z|) \text{ a.e. in } \Delta,$$

where $\hat{f}$ is defined by (8). On the other hand, (7) implies $f(z) = e^{i\arg z}\hat{f}(|z|)$ for $z \in \Delta$, $z \neq 0$, whence (cf. (5) and (6))

$$\begin{aligned} f_z(z) &= \tfrac{1}{2}\{\hat{f}'(r) + (1/r)\hat{f}(r)\} \\ f_{\bar{z}}(z) &= \tfrac{1}{2}e^{2i\arg z}\{\hat{f}'(r) - (1/r)\hat{f}(r)\} \end{aligned} \qquad (r = |z|) \text{ a.e. in } \Delta . \tag{11}$$

From (10) and (11) we obtain $f_{\bar{z}}(z) = e^{2i\arg z}\mu(|z|)f_z(z)$ a.e. in Δ. Now, let us consider the Beltrami equation $w_{\bar{z}} = e^{2i\arg z}\mu(|z|)w_z$. By the theorem on existence and uniqueness quoted before there exists exactly one function f^* defined in Δ which maps it onto itself Q-quasi-conformally with $f^*(0) = 0$, $f^*(1) = 1$, and has $\mu^*(z) = e^{2i\arg z}\mu(|z|)$ as its complex dilatation a.e. in Δ. Hence $f^* \in E_Q$ according to Definition 2D. Since Definitions 2D and 2A are equivalent, we can repeat the previous considerations, which show that if $f \in E_Q$ according to Definition 2A, then $f \in E_Q$ according to Definition 2F, with f^* substituted for f. Hence $f^*(z) = f(z)$ identically in Δ. Since $f^* \in E_Q$ according to Definition 2A, we see that $f \in E_Q$ according to the same definition, and the proof is completed.

7. Before deriving the announced estimates of R and θ it is convenient to notice the following

LEMMA 1. *$f \in E_Q$ implies $f^{-1} \in E_Q$, and* (3) *implies*

$$\begin{aligned} |f^{-1}(w)| &= R^{-1}(|w|) , \\ \arg\big(f^{-1}(w)/w\big) &= \arg f^{-1}(|w|) = -\theta\big(R^{-1}(|w|)\big) \qquad (w \neq 0) , \end{aligned} \tag{12}$$

where $\arg\big(f^{-1}(w)/w\big) = -\arg\big(w/f^{-1}(w)\big)$.

This is an immediate consequence of Definition 2A.

8. Now we proceed to derive the bounds of R and θ (cf. (3)) when f ranges over E_Q.

THEOREM 1. *For any $f \in E_Q$ and $z \in \Delta$, $z \neq 0$, we have*

$$|z|^Q \leqslant |f(z)| \leqslant |z|^{1/Q} .$$

Both estimates are sharp for any $z \in \Delta$, $z \neq 0$, and $Q \in \langle 1, +\infty)$. The only extremal functions for every z are: $f(s) = |s|^{1/Q}e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$ for the upper bound, and $f(s) = |s|^{Q}e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$ for the lower bound.

Proof. Applying Definition 2F and setting $\varrho = |\mu|$, $\varphi = \arg\mu$, we get for $z \in \Delta$

$$\log\frac{1}{|f(z)|} = \int_{|z|}^{1} \operatorname{re}\frac{1+\varrho(r)e^{i\varphi(r)}}{1-\varrho(r)e^{i\varphi(r)}}\frac{dr}{r} = \int_{|z|}^{1} \frac{1-\varrho^2(r)}{1+\varrho^2(r)-2\varrho(r)\cos\varphi(r)}\frac{dr}{r}$$

$$\geqslant \int_{|z|}^{1}\frac{1-\varrho(r)}{1+\varrho(r)}\frac{dr}{r} \geqslant \int_{|z|}^{1}\frac{1}{Q}\frac{dr}{r} = -\frac{1}{Q}\log|z| = \log\frac{1}{|z|^{1/Q}} .$$

Equality holds only for $\mu(r) = -(Q-1)/(Q+1)$, which corresponds to $f(s) = |s|^{1/Q} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$. By Lemma 1 we may apply the result obtained to the inverse function, whence $|f(z)| \geqslant |z|^{Q}$. Here the only extremal function for every z is $f(s) = |s|^{Q} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$, which corresponds to $\mu(r) = (Q-1)/(Q+1)$.

Remark 2. Theorem 1 is also a consequence of a well-known inequality of Grötzsch [3].

THEOREM 2. *For any* $f \in E_Q$ *and* $z \in \Delta$, $z \neq 0$, *we have*

$$-\frac{1}{2}\left(Q-\frac{1}{Q}\right)\log\frac{1}{|z|} \leqslant \arg\frac{f(z)}{z} \leqslant \frac{1}{2}\left(Q-\frac{1}{Q}\right)\log\frac{1}{|z|},$$

where $\arg\big(f(z)/z\big) = 0$ *for* $z = 1$. *Both estimates are sharp for any* $z \in \Delta$, $z \neq 0$, *and* $Q \in \langle 1, +\infty)$. *The only extremal functions for every* z *are*: $f(s) = |s|^{\beta} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$ *for the upper bound, and* $f(s) = |s|^{\bar\beta} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$ *for the lower bound, where* $\beta = \frac{1}{2}(1-i)Q + \frac{1}{2}(1+i)(1/Q)$ *and the branch of* $\arg\big(f(s)/s\big)$ *is chosen in each case so that* $\arg f(1) = 0$.

Proof. Applying Definition 2F and setting $\varrho = |\mu|$, $\varphi = \arg\mu$, we get for $z \in \Delta$, $z \neq 0$

$$\arg\frac{f(z)}{z} = -\int_{|z|}^{1} \operatorname{im}\frac{1+\varrho(r)e^{i\varphi(r)}}{1-\varrho(r)e^{i\varphi(r)}}\frac{dr}{r} = \int_{|z|}^{1} \frac{-2\varrho(r)\sin\varphi(r)}{1+\varrho^2(r)-2\varrho(r)\cos\varphi(r)}\frac{dr}{r}$$

$$\leqslant \int_{|z|}^{1} \frac{2\varrho(r)\dfrac{1-\varrho^2(r)}{1+\varrho^2(r)}}{1+\varrho^2(r)-2\varrho(r)\dfrac{2\varrho(r)}{1+\varrho^2(r)}}\frac{dr}{r} = \int_{|z|}^{1} \frac{2\varrho(r)}{1-\varrho^2(r)}\frac{dr}{r}$$

$$\leqslant \int_{|z|}^{1} \frac{1}{2}\left(Q-\frac{1}{Q}\right)\frac{dr}{r} = \frac{1}{2}\left(Q-\frac{1}{Q}\right)\log\frac{1}{|z|}.$$

Equality holds only for $\mu(r) = q(1+q^2)^{-1}\{2q - i(1-q^2)\}$, where $q = (Q-1)/(Q+1)$, and this corresponds to $f(s) = |s|^{\beta} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$. The above result implies also the lower bound of $\arg\big(f(z)/z\big)$ given in Theorem 2, where the only extremal function for every z is $f(s) = |s|^{\bar\beta} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$, which corresponds to $\mu(r) = q(1+q^2)^{-1}\{2q + i(1-q^2)\}$.

§ 3. Parametric representation.

9. Here we give two theorems on parametric representation. First of them expresses the derivative g_t of a function $w = g(z, t)$, $g(z, 0) = z$, $g(z, 1) = f(z)$, $g(z, t) \in E_Q$, $0 \leqslant t \leqslant 1$, with respect to t in the form of an integral which depends only on the complex dilatation ν^* of the inverse function g^{-1}. This theorem is a consequence of the corresponding theorem for S_Q in the version due to Krushkal [4] (cf. also [10]), which seems to be the most convenient for our purposes. The second theorem expresses g_t in the form of an integral which depends only on the dilatation μ of f, and is proved directly.

THEOREM 3. *Suppose that $w = f(z)$ belongs to E_Q and has $\omega = \mu(z)$ as its complex dilatation. Moreover, suppose that the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, belong to S_Q and have complex dilatations*

$$\nu(z, t) = t\mu(z)\,. \tag{13}$$

Then $w = g(z, t)$, considered as a function of z and t, satisfies on $\Delta \times \{t\colon\ 0 \leqslant t \leqslant 1\}$ the equation

$$\frac{\partial w}{\partial t} = \frac{2w}{t}\int\limits_{|w|}^{1} \frac{(1/r)\nu^*(r, t)}{1-|\nu^*(r, t)|^2}\,dr \tag{14}$$

subject to the initial condition $g(z, 0) = z$, where ν^* is the complex dilatation of g^{-1}.

Remark 3. By Definition 2D the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, belong to E_Q.

Proof. By the theorem on parametrization for the class S_Q quoted above, $w = g(z, t)$ satisfies on $\Delta \times \{t\colon\ 0 \leqslant t \leqslant 1\}$ the equation

$$\frac{\partial w}{\partial t} = \frac{w(1-w)}{\pi}\iint\limits_{|\zeta|\leqslant 1}\left\{\frac{\psi(\zeta, t)}{\zeta(1-\zeta)(w-\zeta)} + \frac{\overline{\psi(\zeta, t)}}{\bar\zeta(1-\bar\zeta)(1-w\bar\zeta)}\right\}d\xi\,d\eta \tag{15}$$

$$(\zeta = \xi + i\eta)$$

subject to the initial condition $g(z, 0) = z$, where ψ is defined by

$$\psi(w, t) = \frac{\mu\big(g^{-1}(w, t)\big)}{1-t^2\big|\mu\big(g^{-1}(w, t)\big)\big|^2}\exp\big(-2i\arg g_w^{-1}(w, t)\big)\,. \tag{16}$$

Applying (13), we have

$$\psi(w, t) = \frac{(1/t)\,\nu\big(g^{-1}(w, t), t\big)}{1-\big|\nu\big(g^{-1}(w, t), t\big)\big|^2}\exp\big(-2i\arg g_w^{-1}(w, t)\big)\,.$$

On the other hand, it can easily be verified that

$$\nu\big(g^{-1}(w, t), t\big) = -\nu^*(w, t)\exp\big(2i\arg g_w^{-1}(w, t)\big)\,.$$

Consequently,

$$\psi(w, t) = -\frac{(1/t)\,\nu^*(w, t)}{1-|\nu^*(w, t)|^2}\,.$$

Substituting this result in (15) we obtain

$$\frac{\partial w}{\partial t}=\frac{w(1-w)}{\pi}\iint_{|\zeta|\leqslant 1}\frac{-1}{1-|\nu^*(\zeta,t)|^2}\left\{\frac{\nu^*(\zeta,t)}{\zeta(1-\zeta)(w-\zeta)}+\frac{\overline{\nu^*(\zeta,t)}}{\bar\zeta(1-\bar\zeta)(1-w\bar\zeta)}\right\}d\xi\,d\eta$$
$$=-\frac{w(1-w)}{\pi t}\iint_{|\zeta|<+\infty}\frac{\hat\nu(\zeta,t)}{1-|\hat\nu(\zeta,t)|^2}\,\frac{d\xi\,d\eta}{\zeta(1-\zeta)(w-\zeta)},$$

where $\hat\nu(w,t)=\nu^*(w,t)$ for $w\in\Delta$ and $\hat\nu(w,t)=e^{4i\arg w}\overline{\nu^*(1/\overline{w},t)}$ for $w\notin\Delta$.

Now we apply Definition 2D to g (cf. Remark 3). We get

$$\frac{\partial w}{\partial t}=-\frac{w(1-w)}{\pi t}\int_0^{+\infty}\int_{-\pi}^{\pi}\frac{e^{2i\vartheta}\hat\nu(r,t)}{1-|\hat\nu(r,t)|^2}\,\frac{r\,d\vartheta}{re^{i\vartheta}(1-re^{i\vartheta})(w-re^{i\vartheta})}\,dr$$
$$=-\frac{w(1-w)}{\pi i t}\int_0^{+\infty}\frac{(1/r)\hat\nu(r,t)}{1-|\hat\nu(r,t)|^2}\int_{|\zeta|=r}\frac{d\zeta}{(1-\zeta)(w-\zeta)}\,dr\,.$$

By the theorem on residues

$$\int_{|\zeta|=r}\frac{d\zeta}{(1-\zeta)(w-\zeta)}=\begin{cases}0 & \text{for } 0<r<|w|\,,\\ -2\pi i/(1-w) & \text{for } |w|<r<1\,,\\ 0 & \text{for } 1<r<+\infty\,.\end{cases}$$

Consequently,

$$\frac{\partial w}{\partial t}=\frac{2w}{t}\int_{|w|}^{1}\frac{(1/r)\hat\nu(r,t)}{1-|\hat\nu(r,t)|^2}\,dr\,.$$

Hence, since $\hat\nu(r,t)=\nu^*(r,t)$ for $0\leqslant r\leqslant 1$, the assertion of Theorem 3 follows.

Remark 4. Theorem 3 can also be proved directly by means of Definition 2F.

THEOREM 4. *Under the hypotheses of Theorem 3 the function $w=g(z,t)$, considered as a function of z and t, satisfies on $\Delta\times\{t\colon 0\leqslant t\leqslant 1\}$ the equation*

$$\frac{\partial w}{\partial t}=-2\exp\left(-\int_{|z|}^{1}\frac{1+t\mu(r)}{1-t\mu(r)}\,\frac{dr}{r}+i\arg z\right)\int_{|z|}^{1}\frac{(1/r)\mu(r)}{[1-t\mu(r)]^2}\,dr \tag{17}$$

subject to the initial condition $g(z,0)=z$.

Proof. By Definition 2F we have

$$g(z,t)=\exp\left(-\int_{|z|}^{1}\frac{1+t\mu(r)}{1-t\mu(r)}\,\frac{dr}{r}+i\arg z\right)\quad\text{for } z\in\Delta,\ z\neq 0\,,$$
$$g(z,t)=0\quad\text{for } z=0\,.$$

Hence the assertion of Theorem 4 follows.

§ 4. The general extremal problem in E_Q.

10. In Theorems 1 and 2 we have given sharp estimates for $|f(z)|$ and $\arg(f(z)/z)$ when f ranges over E_Q. Now we proceed to more general extremal problems. First we determine the extremal functions for any sufficiently regular real-valued functional $U = F(z_1, \dots, z_n; w_1, \dots, w_n)$ with fixed $z_1, \dots, z_n \in \Delta$, $w_1 = f(z_1), \dots, w_n = f(z_n)$, and f running over E_Q. Next we determine the extremal functions in an analogous problem with the additional condition that another real-valued functional G satisfying the same regularity conditions admits a given fixed value. In several cases this enables us to find the region of variability of the complex-valued functional $F + iG$.

THEOREM 5. *Let $U = F(\zeta_1, \dots, \zeta_n; \omega_1, \dots, \omega_n)$ be a real-valued function defined for $\zeta_k \in D_k$, $\omega_k \in D_{Q,k}$, where $D_k \subset \Delta$, $D_{Q,k} \supset \bigcup_g g(D_k)$ $(k = 1, \dots, n)$, $g \in E_Q$ and g ranges over E_Q. Suppose that $F \in C^1$ with respect to $\omega_1, \dots, \omega_n$. Then there exists a function $f \in E_Q$ for which the functional $U = F(z_1, \dots, z_n; g(z_1), \dots, g(z_n))$ attains its maximum when g ranges over E_Q; $z_k, z_k \in D_k$, $|z_k| \leqslant |z_{k-1}|$ $(k = 1, \dots, n)$, $z_0 = 1$, being fixed. The maximum is also attained for any function f_1 defined by $f_1(s) = f(s)$ if $|z_n| < |s| \leqslant 1$ and by $f_1(s) = f(z_n)f^*(s/z_n)$ if $|s| \leqslant |z_n|$, where $f^* \in E_Q$. Moreover, if f is not the identity function and if*

$$(18) \qquad \sum_{k=m+1}^{n} f(z_k) F_{\omega_k}\big(z_1, \dots, z_n; f(z_1), \dots, f(z_n)\big) \neq 0 \qquad (m = 0, \dots, n-1),$$

then we have

$$(19) \qquad f(s) = w_m |s/z_m|^{\beta_m(z_1, \dots, z_n; \varepsilon_m)} e^{i \arg(s/z_m)}$$
$$\text{for } |z_{m+1}| \leqslant |s| \leqslant |z_m| \quad (m = 0, \dots, n-1),$$

where

$$(20) \qquad \beta_m(z_1, \dots, z_n; \varepsilon_m)$$
$$= \frac{1}{2}\Big(Q + \frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_m\Big(Q - \frac{1}{Q}\Big)\exp\Big(-i \arg \sum_{k=m+1}^{n} w_k F_{\omega_k}(z_1, \dots, z_n; w_1, \dots, w_n)\Big),$$
$$\varepsilon_m = 1 \text{ or } -1, \qquad w_0 = 1,\ w_1 = f(z_1),\ \dots,\ w_n = f(z_n),$$

and the branch of $\arg(f(s)/s)$ is chosen for $|z_{m+1}| \leqslant |s| \leqslant |z_m|$ so that $f(s) \to w_m$ as $s \to z_m$. The theorem remains valid if "minimum" is substituted for "maximum".

Proof. It is well known (see [2], p. 324) that if F satisfies the analogous conditions with S_Q substituted for E_Q, there exists a function $f^* \in S_Q$ for which the functional $U = F(z_1, \dots, z_n; g(z_1), \dots, g(z_n))$ attains its maximum when g ranges over S_Q; $z_k, \bar{z}_k \in D_k$ $(k = 1, \ldots, n)$ being fixed. Furthermore, if f is not the identity function and if

$$A(s) = \sum_{k=1}^{n} \left\{ \frac{w_k(1-w_k)}{s(1-s)(w_k-s)} F_{w_k}(z_1, \ldots, z_n;\ w_1, \ldots, w_n) + \right.$$

$$\left. + \frac{\overline{w}_k(1-\overline{w}_k)}{s(1-s)(1-\overline{w}_k s)} \overline{F_{w_k}(z_1, \ldots, z_n;\ w_1, \ldots, w_n)} \right\} \neq 0 \quad \text{a.e. in } \Delta\,,$$

where $w_1 = f^*(z_1), \ldots, w_n = f^*(z_n)$, then the complex dilatation μ^* of f^{*-1} is given by the formulae

$$(21) \qquad |\mu^*(s)| = (Q-1)/(Q+1)\,, \qquad \arg \mu^*(s) = -\arg A(s) \quad \text{a.e. in } \Delta\,.$$

On the other hand, if μ denotes the complex dilatation of f, and if the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, belong to S_Q and have complex dilatations (13), respectively, then $w = g(z, t)$, considered as a function of z and t, satisfies on $\Delta \times \{t\colon 0 \leqslant t \leqslant 1\}$ equation (15) subject to the initial condition $g(z, 0) = z$, where ψ is defined by (16) (see [4]; cf. also [10]). Hence (21) can be written in the form

$$(22) \qquad \frac{d^{**}}{dt} F\big(z_1, \ldots, z_n;\ g(z_1, 1), \ldots, g(z_n, 1)\big) = 0\,,$$

where the asterisks denote that g_t is to be replaced by the integrand in the corresponding parametric equation (including the expression before the sign of integration), and that μ^* is to be replaced in it by $i\mu^*$. The restriction $A(s) \neq 0$ a.e. in Δ can be written in the form

$$(23) \qquad \frac{d^{*}}{dt} F\big(z_1, \ldots, z_n;\ g(z_1, 1), \ldots, g(z_n, 1)\big) \neq 0 \quad \text{a.e. in } \Delta\,,$$

where the asterisk denotes that g_t is to be replaced by the integrand in the corresponding parametric equation (including the expression before the sign of integration).

In our case, since E_Q is a normal family (cf. [8], pp. 76-77), there is a function $f \in E_Q$ for which the functional $U = F\big(z_1, \ldots, z_n;\ g(z_1), \ldots, g(z_n)\big)$ attains its maximum when g ranges over E_Q; z_k, $\bar{z}_k \in D_k$, $|z_k| \leqslant |z_{k-1}|$ $(k = 1, \ldots, n)$, $z_0 = 1$, being fixed. From Definition 2F it follows that the maximum is also attained for any function f_1 defined by $f_1(s) = f(s)$ if $|z_n| < |s| \leqslant 1$ and by $f_1(s) = f(z_n) f^*(s/z_n)$ if $|s| \leqslant |z_n|$, where $f^* \in E_Q$. Suppose that f is not the identity mapping. Formulae (21) do not hold, in general, in our case, but relation (22) and restriction (23) remain unchanged, and the argument given in [2], pp. 322-326, may be applied here. More generally, it should be remarked that this argument may be applied to any compact subclass of the class S_Q, and for any family of functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, which belong to S_Q and have complex dilatations $\omega = \nu(z, t)$, $\nu(z, 0) = 0$, $\nu(z, 1) = \mu(z)$, respectively, satisfying the regularity conditions described in [15], p. 403. Here μ denotes the complex dilatation of f. Clearly, we have to choose functions $\omega = \nu(z, t)$ so that every $w = g(z, t)$, $0 \leqslant t \leqslant 1$, belongs to the class under consideration. In each of these cases condition (22) and restriction (23) remain unchanged.

Applying (22) and (23) to our case, we get

$$|\mu^*(v)| = (Q-1)/(Q+1)$$

and

$$\arg \mu^*(v) = -\arg \sum_{k=m+1}^{n} f(z_k) F_{\omega_k}\big(z_1, \ldots, z_n;\ f(z_1), \ldots, f(z_n)\big) + \tfrac{1}{2}\pi(1-\varepsilon_m)$$

a.e. in $\{v\colon |f(z_{m+1})| \leqslant |v| \leqslant |f(z_m)|\}$, $m = 0, \ldots, n-1$, where μ^* is the complex dilatation of f^{-1}, and $\varepsilon_m = 1$ or -1. Here we assume that f is not the identity mapping and that restriction (23) is fulfilled. This restriction may be written in form (18). Now, since μ^* is a constant a.e. in $\{v\colon |f(z_{m+1})| \leqslant |v| \leqslant |f(z_m)|\}$ for $m = 0, \ldots, n-1$, we denote it there by γ_m. Hence, by Definition 2F, we have

$$f^{-1}(v) = z_m |v/f(z_m)|^{(1+\gamma_m)/(1-\gamma_m)} e^{i \arg(v/f(z_m))}$$

for $|f(z_{m+1})| \leqslant |v| < |f(z_m)|$, where the branch of $\arg\big(f^{-1}(v)/v\big)$ is chosen so that $f^{-1}\big(f(s)\big) \to z_m$ as $s \to z_m$. Now, applying Lemma 1 to f^{-1} (formulae (12)), we get formulae (19) with

$$\begin{aligned}\beta_m(z_1, \ldots, z_n;\ \varepsilon_m) &= \left(1 - i \operatorname{im} \frac{1+\gamma_m}{1-\gamma_m}\right) \Big/ \operatorname{re} \frac{1+\gamma_m}{1-\gamma_m} \\ &= (1 - 2\gamma_m + |\gamma_m|^2)/(1-|\gamma_m|^2) \qquad (m = 0, \ldots, n-1),\end{aligned}$$

whence, by the formulae for μ^* obtained before and $\mu^*(v) = \gamma_m$ a.e. in $\{v\colon |f(z_{m+1})| \leqslant |v| \leqslant |f(z_m)|\}$, $m = 0, \ldots, n-1$, relations (20) follow. Here the branch of $\arg\big(f(s)/s\big)$ is chosen for $|z_{m+1}| \leqslant |s| < |z_m|$ so that $f(s) \to$ $\to w_m$ as $s \to z_m$.

In case of the minimum of F the proof remains unchanged. We cannot decide a priori which system of ε_m $(m = 0, \ldots, n-1)$ corresponds to the case of the maximum, and which to the case of the minimum of the given functional F; $z_k, z_k \in D_k$ $(k = 1, \ldots, n)$ being fixed.

Remark 5. Theorem 5 can be proved with the help of Theorem 4 instead of Theorem 3, and also directly by means of Definition 2F.

11. In this section we determine the extremal functions in an analogous problem with the additional condition that another real-valued functional satisfying the same regularity conditions admits a given fixed value.

THEOREM 6. (i) *Let* $W = F(\zeta_1, \ldots, \zeta_n;\ \omega_1, \ldots, \omega_n)$ *be a complex-valued function defined for* $\zeta_k \in D_k$, $\omega_k \in D_{Q,k}$, *where* $D_k \subset \Delta$, $D_{Q,k} \supset \bigcup_g g(D_k)$ $(k = 1, \ldots, n)$, $g \in E_Q$ *and* g *ranges over* E_Q. *Suppose that* $F \in C^1$ *with respect to* $\omega_1, \ldots, \omega_n$. *Then there exists a function* $f \in E_Q$ *for which the functional* $U = \operatorname{re} F(z_1, \ldots, z_n;\ g(z_1), \ldots, g(z_n))$ *attains its maximum when* $\operatorname{im} F(z_1, \ldots, z_n;\ g(z_1), \ldots, g(z_n)) = \tau$, *and* g *ranges over* E_Q; $z_k, z_k \in D_k$, $|z_k| \leqslant |z_{k-1}|$ $(k = 1, \ldots, n)$, $z_0 = 1$, *and* τ,

$$\min_{g \in E_Q} \operatorname{im} F(z_1, \ldots, z_n;\ g(z_1), \ldots, g(z_n)) < \tau < \max_{g \in E_Q} \operatorname{im} F(z_1, \ldots, z_n;\ g(z_1), \ldots, g(z_n)),$$

being fixed. The maximum is also attained for any function f_1 *defined by* $f_1(s) = f(s)$ *if* $|z_n| < |s| \leqslant 1$ *and by* $f_1(s) = f(z_n) f^*(s/z_n)$ *if* $|s| \leqslant |z_n|$, *where* $f^* \in E_Q$.

(ii) *Suppose that all assumptions givens in* (i) *are fulfilled. Let* ε *denote a sequence* $\{\varepsilon_0, \ldots, \varepsilon_{n-1}\}$, *where* $\varepsilon_m = 1$ *or* -1 $(m = 0, \ldots, n-1)$, *and let* Λ_ε *denote the set of real numbers* λ *satisfying*

$$\sum_{k=m+1}^{n} f^{(\lambda,\varepsilon)}(z_k) F^{(\lambda)}_{\omega_k}(z_1, \ldots, z_n; f^{(\lambda,\varepsilon)}(z_1), \ldots, f^{(\lambda,\varepsilon)}(z_n)) \neq 0 \qquad (m = 0, \ldots, n-1).$$

Here $F^{(\lambda)} = \operatorname{re} F + \lambda \operatorname{im} F$, *the functions* $f^{(\lambda,\varepsilon)}$ *are defined by the formulae* $f^{(\lambda,\varepsilon)}(s) = s$ *for* $|s| = 1$ *and by*

$$f^{(\lambda,\varepsilon)}(s) = w_m^{(\lambda,\varepsilon)} |s/z_m|^{\beta_m(z_1, \ldots, z_n; \lambda, \varepsilon)} e^{i \arg(s/z_m)} \quad \text{for} \quad |z_{m+1}| \leqslant |s| < |z_m| \qquad (m = 0, \ldots, n-1),$$

where

$$\beta_m(z_1, \ldots, z_n;\ \lambda,\ \varepsilon) = \frac{1}{2}\left(Q + \frac{1}{Q}\right) - \frac{1}{2}\varepsilon_m\left(Q - \frac{1}{Q}\right) \exp\left(-i \arg \sum_{k=m+1}^{n} w_k^{(\lambda,\varepsilon)} F^{(\lambda)}_{\omega_k}(z_1, \ldots, z_n;\ w_1^{(\lambda,\varepsilon)}, \ldots, w_n^{(\lambda,\varepsilon)})\right),$$

$$w_0^{(\lambda,\varepsilon)} = 1, \quad w_1^{(\lambda,\varepsilon)} = f^{(\lambda,\varepsilon)}(z_1), \ldots, w_n^{(\lambda,\varepsilon)} = f^{(\lambda,\varepsilon)}(z_n),$$

and the branch of $\arg(f^{(\lambda,\varepsilon)}(s)/s)$ *is chosen for* $|z_{m+1}| \leqslant |s| < |z_m|$ *so that* $f^{(\lambda,\varepsilon)}(s) \to w_m^{(\lambda,\varepsilon)}$ *as* $s \to z_m$ *in the case of every* $\lambda \in \Lambda_\varepsilon$ *and each* ε. *Next let* Λ_ε^τ, $\Lambda_\varepsilon^\tau \subset \Lambda_\varepsilon$, *denote the set of numbers* $\lambda(\tau, \varepsilon)$ *such that*

$$\operatorname{im} F(z_1, \ldots, z_n;\ f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_1), \ldots, f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_n)) = \tau.$$

Finally let Λ_*^τ, $\Lambda_*^\tau \subset \bigcup_\varepsilon \Lambda_\varepsilon^\tau$, *denote the set of numbers* $\lambda_*(\tau)$ *for which* $\operatorname{re} F\big(z_1, \ldots, z_n; f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_1), \ldots, f^{(\lambda(\tau,\varepsilon),\varepsilon)}(z_n)\big)$ *attains its maximum when* $\lambda(\tau, \varepsilon)$ *ranges over* $\bigcup_\varepsilon \Lambda_\varepsilon^\tau$. *Suppose additionally that the extremal function f is not the identity function and that*

$$\sum_{k=m+1}^{n} f(z_k) F_{\omega_k}^{(\lambda_*(\tau))}\big(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)\big) \neq 0 \qquad (m = 0, \ldots, n-1)$$

for some $\lambda_*(\tau) \in \Lambda_*^\tau$. *Then there are: a sequence* $\varepsilon^0 = \{\varepsilon_0^0, \ldots, \varepsilon_{n-1}^0\}$, *where* $\varepsilon_m^0 = 1$ *or* -1 $(m = 0, \ldots, n-1)$, *and a number* $\lambda_0(\tau) = \lambda(\tau, \varepsilon^0)$, $\lambda_0(\tau) \in \Lambda_*^\tau$, *such that*

$$f(s) = f^{(\lambda_0(\tau),\varepsilon^0)}(s) \qquad \text{for} \qquad |z_n| \leqslant |s| \leqslant 1\,.$$

Moreover, each other function $f^{(\lambda_*(\tau),\varepsilon^*)}$, *where* $\lambda_*(\tau) \in \Lambda_*^\tau$, $\lambda_*(\tau) = \lambda(\tau, \varepsilon^*)$, $\varepsilon^* = \{\varepsilon_0^*, \ldots, \varepsilon_{n-1}^*\}$, $\varepsilon_m^* = 1$ *or* -1 $(m = 0, \ldots, n-1)$, *is also an extremal function for the problem under consideration in* E_Q *provided it is continued into the inner disc* $\{s\colon |s| \leqslant |z_n|\}$ *by any way described in* (i).

(iii) *The theorem remains valid if "minimum" is substituted for "maximum".*

Proof. We apply the well-known method of multipliers of Lagrange together with Theorem 5. Hence Theorem 6 follows immediately.

References

[1] L. V. Ahlfors, *Lectures on quasiconformal mappings*, Princeton-Toronto-New York-London 1966.

[2] P. P. Belinskii, *Solution of extremal problems in the theory of quasiconformal mappings by a variational method* (in Russian), Sibirsk. Mat. J. 1 (1960), pp. 303-330.

[3] H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes*, Ber. Verh. Sächs. Akad. Wiss. Leipzig 80 (1928), pp. 503-507.

[4] S. L. Krushkal, *A method of variation in the theory of quasiconformal mappings on closed Riemann surfaces* (in Russian), Dokl. A N SSSR 157 (1964), pp. 781-783.

[5] J. Krzyż, *On an extremal problem of F. W. Gehring*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 16 (1968), pp. 99-101.

[6] — and J. Ławrynowicz, *Quasiconformal mappings of the unit disc with two invariant points*, Michigan Math. Journ. 14 (1967), pp. 487-492; Errata, to appear.

[7] H. P. Künzi, *Quasikonforme Abbildungen*, Berlin-Göttingen-Heidelberg 1960.

[8] O. Lehto und K. I. Virtanen, *Quasikonforme Abbildungen*, Berlin-Heidelberg-New York 1965.

[9] J. Ławrynowicz, *On the parametrization of quasiconformal mappings in an annulus*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 18 (1964), pp. 23-52.

[10] — *Some parametrization theorems for quasiconformal mappings in an annulus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 15 (1967), pp. 319-323.

[11] — *On certain functional equations for quasiconformal mappings*, Ann. Polon. Math. 20 (1968), pp. 153-165 and p. 291.

[12] — *On a class of quasi-conformal mappings with invariant boundary points, II, Applications and generalizations*, this fascicle, pp. 325-347.

[13] A. Mori, *On an absolute constant in the theory of quasiconformal mappings*, Journ. Math. Soc. Japan 8 (1956), pp. 156-166.

[14] S. Saks and A. Zygmund, *Analytic functions*, Warszawa-Wrocław 1952.

[15] Shah Tao-shing, *Parametrical representation of quasiconformal mappings* (in Russian), Science Record N. S. 3 (1959), pp. 400-407.

[16] — and Fan Le-le, *On the parametric representation of quasiconformal mappings*, Scientia Sinica 2 (1962), pp. 149-162.

[17] O. Teichmüller, *Ein Verschiebungssatz der quasikonformen Abbildung*, Deutsche Mathematik 7 (1944), pp. 336-343.

[18] L. I. Volkovyskii, *Quasiconformal mappings* (in Russian), Lvov 1954.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
LABORATORY IN ŁÓDŹ

Reçu par la Rédaction le 2. 2. 1968