DISSERTATIONES MATHEMATICAE

ROZPRAWY MATEMATYCZNE

LIV

J. J. CHARATONIK

On fans

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CONTENTS

§ 1. Introduction

Investigating finite families of incomparable dendroids in [4], I studied especially simple dendroids, namely so-called *fans*. Afterwards I observed that the family of incomparable fans given in [4] can be constructed by simpler curves, namely by fans composed of countably many straight-line segments which have only one point, the top of the fan, in common. I call such fans *geometrical*. The notion of geometrical fans arises in a natural manner in many problems of the theory of curves, and leads to some other notions concerning fans, for instance that of *smoothness*. Further, I observed that smooth fans have interesting structural and mapping properties. The investigation of some of them is the purpose of the present paper.

After § 2, containing preliminaries, I introduce in § 3 the concept of smooth fans and study some conditions equivalent to this property. In § 4, I prove some connections between smoothness and uniform arcwise connectedness defined in my paper [4]. In § 5, I consider various properties of the end-point set of countable smooth fans which are invariant under continuity. Applying them, I construct in § 6 an arbitrary numerous finite family of geometrical fans, incomparable in the sense that none of them is a continuous image of another. The rest of the paper is concerned with the problem of geometrization of curvilinear fans, which is equivalent to the topological embedding of them in the geometrical fan whose end-point set is the Cantor discontinuum. I have not solved this problem definitively. In § 7, I show that the equivalent condition of this embeddability is a property which I call the *folding* property. In § 8, I prove, among other things, that every folding fan can be homotopically deformed in itself by means of retractions into a point. § 9 contains some theorems on the invariance of certain properties under confluent mappings (see, for this concept, my paper [5], p. 213). Finally, § 10 contains a summary of relations between the properties of fans which have been investigated.

I was encouraged to study fans by Professor B. Knaster, who contributed to my investigations his kind advice and valuable improvements. I express my gratitude to him.

§ 2. Preliminaries

The topological spaces under consideration will be assumed to be metric continua.

The following notation will be used: xy will denote an arbitrary arc with end-points x and y, $\delta(X)$ will be the diameter of X, $\varrho(x, a)$ the distance between x and a, $\varrho(x, A) = \inf\{\varrho(x, a): a \in A\}$, and $\lambda(A)$ the length of a rectifiable arc A. Further, $\mathscr{I}$ will denote the segment 01.

A point p of an arcwise connected continuum X is called a *point of order* $\mathfrak{r}$ *in the classical sense* if p is a common end-point of exactly $\mathfrak{r}$ arcs disjoint from one another beyond p and contained in X (writing $\text{Ord}_p X = \mathfrak{r}$, see [3], p. 230). In particular, a point p of X is called an *end-point of X in the classical sense* if $\text{Ord}_p X = 1$, and a *ramification point of X in the classical sense* if $\text{Ord}_p X \geqslant 3$. Henceforward the words "in the classical sense" will be omitted. The set of all end-points of X in this sense will be denoted by $E(X)$.

Recall that a *dendroid* is an arcwise connected and hereditarily unicoherent continuum (see e.g. [3], p. 239). A dendroid X is said to be a *fan* provided that there is in X only one ramification point, called the *top* of X (see [4], D11, p. 198).

Remark that every fan, as a dendroid, and thus a 1-dimensional continuum (see [3], (48), p. 239), can be homeomorphically embedded in a 3-dimensional Euclidean unit cube $\mathscr{I}^3$ by virtue of the Menger-Nöbeling theorem (see [9], § 40, VII, 1, p. 69). Thereby every fan will henceforth be considered as a subcontinuum of $\mathscr{I}^3$.

Let t be the top of a fan X, and e an end-point of it. Thus, by the definition of the fan,

$$X = \bigcup_{e \in E(X)} te.$$

The set $E(X)$ of all end-points (in the classical sense) of an arbitrary arcwise connected continuum X is not necessarily a G_δ-set, even if X is a dendroid. But if X is a fan, $E(X)$ is always a G_δ-set (see [10], 7.5, p. 311).

LEMMA 1. *If C_n is a sequence of continua in an hereditarily unicoherent continuum X, then $\operatorname*{Li}_{n \to \infty} C_n$ is a continuum.*

Proof. Since $\operatorname*{Li}_{n \to \infty} C_n$ is compact, it is sufficient to prove that $\operatorname*{Li}_{n \to \infty} C_n$ is connected. If $\operatorname*{Li}_{n \to \infty} C_n \neq 0$, suppose that A and B are non-empty compact sets such that

$$A \cup B = \operatorname*{Li}_{n \to \infty} C_n \quad \text{and} \quad A \cap B = 0.$$

and let $a \epsilon A$ and $b \epsilon B$. The set $\underset{n \to \infty}{\mathrm{Ls}\, C_n}$, being a continuum ([9], § 42, II, 6, p. 111), contains a continuum $I(a, b)$ irreducible between points a and b ([9], § 43, I, 1, p. 132). $\underset{n \to \infty}{\mathrm{Li}\, C_n}$ being supposed not to be connected, there exist in $\underset{n \to \infty}{\mathrm{Ls}\, C_n}$ a point $c \epsilon I(a, b) - \underset{n \to \infty}{\mathrm{Li}\, C_n}$, a neighbourhood U of c and a subsequence C_{n_m} such that none of C_{n_m} intersects U. Hence $L = \underset{n \to \infty}{\mathrm{Ls}\, C_{n_m}}$ $\neq 0$ and

$$(2.1) \qquad U \subset X - L,$$

where L is a continuum (see [9], loco cit.), and $(a) \cup (b) \subset L$. Denoting by $I'(a, b)$ a continuum irreducible in L between points a and b, we have $c \epsilon X - L$ by (2.1), whence $c \epsilon X - I'(a, b)$. Thus $I(a, b) \neq I'(a, b)$ contrary to the hereditary unicoherence of X (see [4], D1, p. 187).

COROLLARY 1. *If X is a dendroid and a_n and b_n are convergent sequences of points of X, then $\underset{n \to \infty}{\mathrm{Li}\, a_n b_n}$ is a continuum, and*

$$(2.2) \qquad a = \underset{n \to \infty}{\lim a_n} \quad and \quad b = \underset{n \to \infty}{\lim b_n}$$

imply

$$(2.3) \qquad ab \subset \underset{n \to \infty}{\mathrm{Li}\, a_n b_n}.$$

§ 3. Smoothness

A fan X with the top t is said to be *smooth* provided that if a sequence $\{a_n\}$ of points of X tends to a limit point a, then the sequence of arcs $\{t a_n\}$ is convergent, and

$$\underset{n \to \infty}{\mathrm{Lim}\, t a_n} = ta.$$

As an immediate consequence of this definition we have the following

COROLLARY 2. *If a fan X is smooth, then every subfan of X is also smooth* (the heredity of smoothness for fans).

THEOREM 1. *A fan $X = \underset{e \epsilon E(X)}{\bigcup} te$ is smooth if and only if for every two sequences $\{a_n\}$ and $\{b_n\}$ of points of X conditions (2.2) and*

$$(3.1) \qquad a_n b_n - (t) \ is \ connected \ for \ n = 1, 2, \ldots$$

imply

$$(3.2) \qquad ab = \underset{n \to \infty}{\mathrm{Lim}\, a_n b_n}.$$

Proof. If the above implication holds, the fan X is trivially smooth by putting $b_n = t$. Inversely, if this implication is not true, then there are in X two sequences of points $\{a_n\}$ and $\{b_n\}$ which satisfy (2.2), (3.1) and the negation of (3.2). By Corollary 1 we have (2.3). Without loss of generality we can assume in connection with (3.1) that

$$(3.3) \qquad\qquad a_n \,\epsilon\, tb_n \quad \text{for} \quad n = 1, 2, \dots.$$

Now consider two cases.

Case 1. $\underset{n\to\infty}{\mathrm{Li}}\, a_n b_n \neq \underset{n\to\infty}{\mathrm{Ls}}\, a_n b_n$. Thus there exist a point $c \,\epsilon\, \underset{n\to\infty}{\mathrm{Ls}}\, a_n b_n -$
$- \underset{n\to\infty}{\mathrm{Li}}\, a_n b_n$ and a sequence of points $\{c_{n_m}\}$ such that

$$(3.4) \qquad\qquad c_{n_m} \,\epsilon\, a_{n_m} b_{n_m} \quad \text{for} \quad m = 1, 2, \dots,$$

and

$$(3.5) \qquad\qquad c = \lim_{m\to\infty} c_{n_m}.$$

Hence $c_{n_m} \,\epsilon\, tb_{n_m}$ by (3.3) and (3.4), and

$$(3.6) \qquad\qquad c \,\epsilon\, \underset{n\to\infty}{\mathrm{Ls}}\, tb_n$$

by (3.5). If $c \,\epsilon\, X - tb$, then $tb \neq \underset{n\to\infty}{\mathrm{Ls}}\, tb_n$ by (3.6), and X is not smooth. If $c \,\epsilon\, tb$, we have $c \,\epsilon\, tb - ab$, because $ab \subset \underset{n\to\infty}{\mathrm{Li}}\, a_n b_n$ by Corollary 1. Thus

$$(3.7) \qquad\qquad c \,\epsilon\, ta - (a).$$

By substituting in the same corollary the point t for points b_n and b we conclude that $\underset{n\to\infty}{\mathrm{Li}}\, ta_n$ is a continuum and $ta \subset \underset{n\to\infty}{\mathrm{Li}}\, ta_n$, whence by (3.7) we have $c \,\epsilon\, \underset{n\to\infty}{\mathrm{Li}}\, ta_n$. This implies, in particular, that there exists a sequence of points c_n, where $n \neq n_m$, such that

$$(3.8) \qquad\qquad c_n \,\epsilon\, ta_n \quad \text{for} \quad n \neq n_m,$$

$$(3.9) \qquad\qquad c = \lim_{n\to\infty} c_n \quad \text{for} \quad n \neq n_m.$$

Thus, by (3.5) and (3.9), we have $c = \lim_{n\to\infty} c_n$, where $n = 1, 2, \dots$
Further, it follows from (3.3) and (3.4) that $a_{n_m} \,\epsilon\, tc_{n_m}$ for $m = 1, 2, \dots,$
whence $a \,\epsilon\, \underset{n\to\infty}{\mathrm{Ls}}\, tc_n$. But (3.7) implies $tc \subset ta - (a)$; therefore $tc \neq \underset{n\to\infty}{\mathrm{Ls}}\, tc_n$ and the fan X is not smooth.

Case 2. $\operatorname*{Li}_{n\to\infty} a_n b_n = \operatorname*{Ls}_{n\to\infty} a_n b_n$. Thus the sequence $\{a_n b_n\}$ is convergent, and we have $ab \subset \operatorname*{Lim}_{n\to\infty} a_n b_n$ by Corollary 1. Therefore the negation of (3.2) implies that there exists a point c such that

$$(3.10) \qquad\qquad c \,\epsilon\, \operatorname*{Lim}_{n\to\infty} a_n b_n - ab\,.$$

Thus there exists a sequence of points c_n with properties

$$(3.11) \qquad\qquad c_n \,\epsilon\, a_n b_n\,,$$

$$(3.12) \qquad\qquad c = \lim_{n\to\infty} c_n\,.$$

If X is smooth, then (2.2) and (3.12) imply

$$(3.13) \qquad ta = \operatorname*{Lim}_{n\to\infty} ta_n\,, \qquad tb = \operatorname*{Lim}_{n\to\infty} tb_n\,, \qquad \text{and} \qquad tc = \operatorname*{Lim}_{n\to\infty} tc_n\,.$$

But it follows from (3.3) and (3.11) that $ta_n \subset tc_n \subset tb_n$, whence $ta \subset tc \subset tb$ by (3.13). Thus $c \,\epsilon\, ab$ contrary to (3.10), which completes the proof.

It is easy to verify, e.g. by reduction ad absurdum, the following

COROLLARY 3. *If a fan* $X = \bigcup_{e \epsilon E(X)} te$ *is smooth and if for any two sequences* $\{a_n\}$ *and* $\{b_n\}$ *of points of* X *conditions* (2.2) *and* (3.1) *hold, then* $ab - (t)$ *is connected.*

The hypothesis of smoothness of the fan X is essential in Corollary 3. In fact, let X be the fan in the plane composed of the straight segment ab, where $a = (0, 1)$, $b = (0, -1)$, and of the sequence of polygonal lines of the form $a_n b_n c_n t$, where $a_n = (1/n, 1)$, $b_n = \big(1/n, -(n+1)/n\big)$, $c_n = \big(-1/n, -(n+1)/n\big)$, and $t = (0, 0)$. Then $ab = \operatorname*{Lim}_{n\to\infty} a_n b_n$, condition (3.1) holds, but t disconnects ab.

THEOREM 2. *A fan* $X = \bigcup_{e \epsilon E(X)} te$ *is smooth if and only if for every number* $\varepsilon > 0$ *there exists a number* $\eta > 0$ *such that for every point* $e \epsilon E(X)$ *the conditions*

$$(3.14) \qquad a \,\epsilon\, te\,, \qquad b \,\epsilon\, te\,, \qquad \text{and} \qquad \varrho(a, b) < \eta$$

imply

$$(3.15) \qquad\qquad \delta(ab) < \varepsilon\,.$$

Proof. Firstly, suppose that the fan X is smooth and that there is a number $\varepsilon > 0$ such that for every number $\eta > 0$ there exists a point $e \epsilon E(X)$ and points a and b which satisfy (3.14) with the negation of (3.15).

Thus taking $\eta = 1/n$, where $n = 1, 2, \ldots$, we see that there exist two sequences $\{a_n\}$ and $\{b_n\}$ of points of X such that

$$(3.16) \qquad a_n \, \epsilon \, t b_n,$$

$$(3.17) \qquad \varrho(a_n, b_n) < \frac{1}{n},$$

$$(3.18) \qquad \delta(a_n b_n) \geqslant \varepsilon.$$

By virtue of the compactness of X the sequence $\{a_n\}$ contains a convergent subsequence $\{a_{n_m}\}$. Put

$$(3.19) \qquad \lim_{m \to \infty} a_{n_m} = a.$$

Thus

$$(3.20) \qquad \lim_{m \to \infty} b_{n_m} = a$$

by (3.17). It follows from (3.18) that there exist points c_m such that

$$(3.21) \qquad c_m \, \epsilon \, a_{n_m} b_{n_m},$$

$$(3.22) \qquad \varrho(a_{n_m}, c_m) \geqslant \frac{\varepsilon}{2}.$$

The sequence $\{c_m\}$ contains a convergent subsequence $\{c_{m_k}\}$ by the compactness of X. Put

$$(3.23) \qquad \lim_{k \to \infty} c_{m_k} = c.$$

Therefore (3.19) and (3.22) imply that

$$(3.24) \qquad a \neq c.$$

The fan X being smooth, we conclude that

$$\operatorname{Lim}_{m \to \infty} a_{n_m} b_{n_m} = (a)$$

by (3.19) and (3.20) according Theorem 1, contrary to (3.24) by (3.21) and (3.23).

Secondly, suppose that the fan X is not smooth, i.e. that there exists a sequence $\{a_n\}$ of points of X which tends to a limit point a, i.e.

$$(3.25) \qquad a = \lim_{n \to \infty} a_n,$$

and is such that either $\{t a_n\}$ is not convergent, or it is convergent with

$$(3.26) \qquad \operatorname{Lim}_{n \to \infty} t a_n \neq t a.$$

By Corollary 1 we have

$$(3.27) \qquad\qquad ta \subset \underset{n\to\infty}{\mathrm{Li}}\ ta_n,$$

and obviously

$$(3.28) \qquad\qquad \underset{n\to\infty}{\mathrm{Li}}\ ta_n \subset \underset{n\to\infty}{\mathrm{Ls}}\ ta_n.$$

There exists a point c such that

$$(3.29) \qquad\qquad c \in \underset{n\to\infty}{\mathrm{Ls}}\, ta_n - ta,$$

because otherwise we would have $\underset{n\to\infty}{\mathrm{Ls}}\, ta_n \subset ta$ and, by (3.27) and (3.28), $ta = \underset{n\to\infty}{\mathrm{Li}}\ ta_n = \underset{n\to\infty}{\mathrm{Ls}}\ ta_n$ contrary to (3.26). By (3.29) there are a subsequence of arcs ta_{n_m} and a sequence of points c_m such that

$$(3.30) \qquad\qquad c_m \in ta_{n_m} \quad \text{for} \quad m = 1, 2, \ldots,$$

and

$$(3.31) \qquad\qquad c = \underset{m\to\infty}{\lim} c_m.$$

X being hereditarily unicoherent, $ta \frown tc$ is an arc or a point. If it is an arc, denote by b the end-point of that arc which is different from t; if it is a point, let $b = t$. In both cases we have

$$(3.32) \qquad\qquad tb = ta \frown tc$$

(in particular we can have $b = a$). It follows from (3.29) and (3.32) that

$$(3.33) \qquad\qquad b \neq c \neq a.$$

Further, (3.32) implies that

$$(3.34) \qquad\qquad b \in tc.$$

Corollary 1, the formula $\underset{m\to\infty}{\mathrm{Li}}\, tc_m \subset \underset{m\to\infty}{\mathrm{Ls}}\, tc_m$ and (3.31) imply $tc \subset \underset{m\to\infty}{\mathrm{Ls}}\, tc_m$, and we have $b \in \underset{m\to\infty}{\mathrm{Ls}}\, tc_m$ by (3.33). Thus there are a subsequence of arcs tc_{m_k} and a sequence of points b_k such that

$$(3.35) \qquad\qquad b_k \in tc_{m_k} \quad \text{for} \quad k = 1, 2, \ldots$$

and

$$(3.36) \qquad\qquad b = \underset{k\to\infty}{\lim} b_k.$$

Further, it follows from (3.32) that

$$(3.37) \qquad\qquad b \in ac.$$

Put

$$(3.38) \qquad\qquad A_k = a_{n_{m_k}} c_{m_k}.$$

We conclude by the same argument as above that

(3.39) $ac \subset \operatorname*{Ls}_{k \to \infty} A_k$

according to (3.25) and (3.31). Therefore there exist by (3.37) and (3.39) a subsequence of arcs A_{k_l} and a sequence of points b_l' such that

(3.40) $b_l' \epsilon A_{k_l}$ for $l = 1, 2, \ldots$

and

(3.41) $b = \lim_{l \to \infty} b_l'.$

Let ε be a positive number. For that ε there exists by hypothesis a positive number η such that for every arc te of X conditions (3.14) imply (3.15).

Remark that for any fixed l the points b_{k_l} and b_l' belong to the same arc te of the fan X, and

(3.42) $c_{m_{k_l}} \epsilon b_{k_l} b_l'$ for $l = 1, 2, \ldots$

by (3.35) and (3.40). By (3.36) and (3.41) the sequence

$$b_{k_1}, \; b_1', \; b_{k_2}, \; b_2', \; \ldots, \; b_{k_l}, \; b_l', \; \ldots$$

is convergent to the point b, and thus for sufficiently great indices l we have $\varrho(b_{k_l}, b_l') < \eta$, whence $\delta(b_{k_l} b_l') < \varepsilon$ by hypothesis. It implies by (3.42) that $\lim_{l \to \infty} c_{m_{k_l}} = \lim_{l \to \infty} b_{k_l}$, i.e. $c = b$ by (3.31) and (3.36) contrary to (3.33). This completes the proof.

§ 4. Uniform arcwise connectedness

Recall that a point set X is said to be *uniformly arcwise connected* (see [4], p. 193) if it is arcwise connected and if for every number $\varepsilon > 0$ there is a number k such that every arc $A \subset X$ contains points $a_0, a_1, \ldots, a_k$ such that

$$A = \bigcup_{=0}^{k-1} a_i a_{i+1},$$

$$\delta(a_i a_{i+1}) < \varepsilon \quad \text{for every } i = 0, 1, \ldots, k-1.$$

The following three theorems give sufficient and necessary conditions of uniform arcwise connectedness for fans.

THEOREM 3. *A fan* $X = \bigcup_{e \epsilon E(X)} te$ *is uniformly arcwise connected if and only if for every number* $\varepsilon > 0$ *there is a number* k *such that every arc* te

contains points $a_0, a_1, \ldots, a_k$ *such that*

$$te = \bigcup_{i=0}^{k-1} a_i a_{i+1},$$

(4.1) $\delta(a_i a_{i+1}) \leqslant \varepsilon$ *for every* $i = 0, 1, \ldots, k-1$.

P r o o f. If X is uniformly arcwise connected, then the condition in question is satisfied by the definition of uniform arcwise connectedness.

Inversely, remark that every arc $A \subset X$ is contained in the union of at most two arcs $te_1 \cup te_2$; thus putting $\varepsilon' = 2\varepsilon$ and $k' = 2k$ we have a decomposition of every arc A into at most k' arcs with diameters less than or equal to ε, i.e. less than ε', which proves the uniform arcwise connectedness of X.

COROLLARY 4. *A fan X is uniformly arcwise connected if and only if for every number $\varepsilon > 0$ there is a number k such that every arc te contains points* $a_0, a_1, \ldots, a_j$, *where* $j \leqslant k$ *such that*

$$te = \bigcup_{i=0}^{j-1} a_i a_{i+1}$$

and $\delta(a_i a_{i+1}) \leqslant \varepsilon$ *for every* $i = 0, 1, \ldots, j-1$.

In the same manner as in Theorem 3 one can prove the following

THEOREM 4. *A fan X is uniformly arcwise connected if and only if the condition of Theorem 3 is satisfied with* $\delta(a_i a_{i+1}) < \varepsilon$ *for every* $i = 0, 1,$ $\ldots, k-1$ *instead of* (4.1).

LEMMA 2. *Let a fan X be smooth and let ε be a positive number. If we take in every arc ab, such that $ab - (t)$ is connected, points*

(4.2) $a = x_0, x_1, \ldots, x_j, x_{j+1} = b$

such that

(4.3) $x_i \in x_{i-1} x_{i+1}$ *for* $i = 1, 2, \ldots, j,$

(4.4) $\varrho(x_i, x_{i+1}) = \varepsilon$ *for* $i = 0, 1, \ldots, j-1$ *and* $\varrho(x_j, x_{j+1}) \leqslant \varepsilon$,

then there exists a natural k with property $j \leqslant k$ for all arcs ab.

P r o o f ([1]). Suppose that it is not so. Thus

(4.5)   for every natural k there exists an arc $a_k b_k$ such that $a_k b_k - (t)$ is connected, and there exist points

$$a_k = x_0, x_1, \ldots, x_j, x_{j+1} = b_k$$

of this arc which satisfy (4.3), (4.4) and $j > k$.

([1]) I am indebted to Mrs. H. Patkowska for a simplified proof of this lemma and for some other improvements in this paper.

Let η be a positive number which corresponds to the number ε according to Theorem 2. The fan X being compact, we can take an $(\eta/4)$-net in X, i.e. a set of r points $y_1, y_2, \ldots, y_r$ of X (where the number r depends on η) with the property that for every point x of X there is an index i, $1 \leqslant i \leqslant r$, such that $\varrho(x, y_i) < \eta/4$. Let us consider an arc $a_r b_r$ which exists by (4.5). It contains $j + 2$ ($j > r$) points $a_r = x_0, x_1, \ldots, x_j, x_{j+1} = b_r$ which satisfy (4.3) and (4.4), and it is covered by at most r open balls S_i each of them has y_i as its centre and has a diameter less than $\eta/2$:

$$(4.6) \qquad\qquad \delta(S_i) < \eta/2.$$

Now we define a (finite) sequence $\{S_{i_n}\}$ of those balls. Put $a_r^0 = a_r$ and let S_{i_1} be that one of those balls S_i which contains the point a_r^0. If S_{i_m} is defined, let a_r^m be the last point of the arc $a_r b_r$ which belongs to $\overline{S_{i_m}}$, i.e. such a point of that arc that

$$\overline{a_r^m b_r} \cap \overline{S_{i_m}} = (a_r^m).$$

If $a_r^m \neq b_r$, we define $S_{i_{m+1}}$ to be that one of the balls S_i which contains the point a_r^m:

$$a_r^m \in S_{i_{m+1}}.$$

Otherwise, the sequence $\{S_{i_n}\}$ is composed of m elements only. So $\{S_{i_n}\}$ is well defined and contains at most r elements. Since the points a_r^{m-1} and a_r^m (with $m > 0$) belong to the same closed ball S_{i_m} by construction, we conclude from (4.6) and from Theorem 2 that

$$(4.7) \qquad\qquad \delta(a_r^{m-1} a_r^m) < \varepsilon.$$

Hence the arc $a_r b_r$ is covered by at most r arcs $a_r^{m-1} a_r^m$ of diameter less than ε. Thus the inequality $j > r$ implies that among points $x_0, x_1, \ldots, x_j$ there exist two of them, x_{i_0}, x_{i_0+1} which lie in the same arc $a_r^{m-1} a_r^m$. Therefore $\varrho(x_{i_0}, x_{i_0+1}) < \varepsilon$ by (4.7), contrary to (4.4).

COROLLARY 5. *Every smooth fan is uniformly arcwise connected.*

In fact, it suffices to put $a = t$ and $b = e$ in Lemma 2 and to apply Corollary 4 with $a_i = x_i$, where $k + 1$ from Lemma 2 is equal to k from Corollary 4.

Easy examples show that the theorem inverse to Corollary 5 is not true: there are uniformly arcwise connected and non-smooth fans, e.g. the Cantor hooked fan M_C^v described in [3], p. 240, E3.

§ 5. Countable smooth fans

A fan X with the top t is called *countable* or *uncountable* if $\mathrm{Ord}_t\, X = \aleph_0$ or $\mathrm{Ord}_t\, X = 2^{\aleph_0}$, respectively. In other words, X is a countable fan if and only if $E(X)$ is countable. E.g. the harmonic fan (see [3], p. 240) is count-

able, and the Cantor fan is uncountable. Let $\tau(X)$ be the degree of the non-local connectedness of X defined in [4], p. 190. If a fan X is countable, then $\tau(X)$ is a countable ordinal and inversely. Thus from Theorem 18 in [4], p. 192 we infer the following

COROLLARY 6. *If a fan Y is a continuous image of a fan X and if $E(X)$ is countable, then $E(Y)$ is at most countable.*

Now we prove the following

THEOREM 5. *If a smooth fan Y is a continuous image of a smooth fan X and if $\overline{E(X)}$ is countable, then $\overline{E(Y)}$ is at most countable.*

Proof. Put $X = \bigcup\limits_{e \in E(X)} te$ and $Y = \bigcup\limits_{e' \in E(Y)} t'e'$, where t and t' are the tops of the fans X and Y respectively, and let f be a continuous mapping of X onto Y. Suppose that $\overline{E(Y)}$ is uncountable. The set $E(Y)$ being countable by Corollary 6, there exists at least one arc $t'e_0' \subset Y$ such that $t'e_0' \cap \overline{E(Y)}$ is uncountable. Since $\overline{E(X)}$ is countable by hypothesis, the set

$$(5.1) \qquad B_0 = t'e_0' \cap \overline{E(Y)} - f\big(\overline{E(X)}\big)$$

is uncountable. Thus there exists a condensation point b_0 of B_0 such that $t' \neq b_0 \neq e_0'$. The fan Y being smooth, there is by Theorem 2 a number $\eta > 0$ such that

$$(5.2) \qquad \text{for every point } e' \epsilon E(Y) \text{ if } y_1 \epsilon t'e', y_2 \epsilon t'e' \text{ and } \varrho(y_1, y_2) < \eta, \text{ then}$$
$$\delta(y_1 y_2) < \tfrac{1}{2}\varrho(b_0, t').$$

Obviously we can assume

$$(5.3) \qquad \eta < \tfrac{1}{2}\min[\varrho(b_0, t'), \varrho(b_0, e_0')].$$

Let V be such a neighbourhood of the point b_0 that

$$(5.4) \qquad \delta(V) < \eta.$$

Thus

$$(5.5) \qquad t' \epsilon Y - V.$$

Putting

$$(5.6) \qquad B = V \cap B_0$$

we conclude that

$$(5.7) \qquad \text{the set } B \text{ is uncountable.}$$

Let $b \epsilon B$. The point b, being different from the end-point e_0' by (5.3) and (5.4), is a cluster point of the set $E(Y)$ by (5.6) and (5.1). Hence there exists a sequence of points e_i' such that

$$(5.8) \qquad b = \lim_{i \to \infty} e_i', \quad \text{where} \quad e_i' \epsilon E(Y) \cap V.$$

The set $E(X)$ being countable by hypothesis, we can arrange all points of this set in a sequence: $e_1, e_2, \ldots$ For every $i = 1, 2, \ldots$ the minimal index n_i of points of this sequence exists with property $e_i' \epsilon f(te_{n_i})$, and, $f^{-1}(e_i')$ being compact, for every $i = 1, 2, \ldots$ there is a first point in the arc te_{n_i} which is mapped on e_i' under f, i.e. such a point x_i that

(5.9) $x_i \epsilon te_{n_i}$ and $tx_i \cap f^{-1}(e_i') = (x_i)$.

Thus

(5.10) $f(x_i) = e_i'$.

X being compact, the sequence $\{x_i\}$ contains a convergent subsequence $\{x_{i_k}\}$. Putting

(5.11) $a = \lim_{k \to \infty} x_{i_k}$

we have

(5.12) $f(a) = b$

by (5.8) and (5.10), f being continuous. Therefore we have assigned to every point $b \epsilon B \subset Y$ exactly one point $a \epsilon X$ with properties (5.8)-(5.12). Let A_0 be the set of all such points a. Hence

(5.13) $f|A_0$ is one-to-one

and

(5.14) $f(A_0) = B$.

Thus it follows from (5.7) that A_0 is uncountable, whence, $E(X)$ being countable by hypothesis, there exists an arc $te_0 \subset X$ such that $te_0 \cap A_0$ is also uncountable. Consequently there is such a condensation point a_0 of the set $te_0 \cap A_0$ that

(5.15) $a_0 \epsilon te_0 \cap A_0$.

So by (5.14)

(5.16) $f(a_0) \epsilon B$.

X being compact and the mapping f being continuous, f is uniformly continuous. Thus, for the number $\eta > 0$ for which (5.2) holds, a number $\varepsilon' > 0$ exists such that

(5.17)   for arbitrary points x_1 and x_2 of X if $\varrho(x_1, x_2) < \varepsilon'$, then $\varrho(f(x_1), f(x_2)) < \eta$.

X being smooth, for this number ε' there is by Theorem 2 a number $\varepsilon'' > 0$ such that

(5.18)   for every point $e \epsilon E(X)$ if $x_1 \epsilon te, x_2 \epsilon te$ and $\varrho(x_1, x_2) < \varepsilon''$, then $\delta(x_1 x_2) < \varepsilon'$.

Since $f(a_0) \epsilon V$ by (5.16) and (5.6) and since f is continuous, there exists a neighbourhood U' of a_0 such that

(5.19) $$f(U') \subset V.$$

Obviously we can assume

(5.20) $$\delta(U') < \min(\varepsilon', \varepsilon'').$$

Thus (5.17) and (5.18) imply that

(5.21) for every point $e \epsilon E(X)$ if $x_1 \epsilon te \frown U'$ and $x_2 \epsilon te \frown U'$, then $\delta(f(x_1 x_2)) < \eta$.

The continuum $f(x_1 x_2)$ being locally connected, and thus arcwise connected, we have $f(x_1)f(x_2) \subset f(x_1 x_2)$, whence

(5.22) $$\delta(f(x_1)f(x_2)) < \eta$$

by (5.21). Further, (5.19) implies that if $x_1 \epsilon te \frown U'$ and $x_2 \epsilon te \frown U'$, then

(5.23) $$f(x_1) \epsilon V \quad \text{and} \quad f(x_2) \epsilon V.$$

Thus for every $y \epsilon f(x_1)f(x_2)$ we have

(5.24) $$\varrho(y, f(x_1)) < \eta$$

by (5.22). Since $b_0 \epsilon V$ by definition, (5.23) and (5.4) imply $\varrho(b_0, f(x_1)) < \eta$, and therefore $\varrho(y, b_0) < 2\eta$ by (5.24) and by the triangle axiom. Hence $\varrho(y, b_0) < \varrho(b_0, t')$ for $y \epsilon f(x_1)f(x_2)$ by (5.3), and we conclude that

(5.25) $$t' \epsilon Y - f(x_1)f(x_2)$$

for every end-point $e \epsilon E(X)$ and all points $x_1, x_2 \epsilon te \frown U'$.

Further, since $B \subset B_0$ by (5.6), we have $B \subset Y - f(\overline{E(X)})$ by (5.1), and thus (5.16) implies $f(a_0) \epsilon Y - f(\overline{E(X)})$. Consequently there exists a neighbourhood U'' of a_0 such that $U'' \frown E(X) = 0$. Let

$$U = U' \frown U''.$$

So U is a neighbourhood of a_0 and

(5.26) $$U \frown E(X) = 0,$$

(5.27) $$f(U) \subset V$$

by (5.19) and

(5.28) $$\delta(U) < \min(\varepsilon', \varepsilon'')$$

by (5.20). Putting

(5.29) $$A = U \frown te_0 \frown A_0$$

we see that

(5.30) $f|A$ is one-to-one

by (5.13) and (5.29).

Since all points of the set A_0 satisfy (5.8)-(5.12) by definition, we conclude from (5.29) that all points of the set A also satisfy them. Thus

(5.31)   to every point $a \epsilon A$ there exists a sequence of points x_i such that $a = \lim_{i\to\infty} x_i$, $x_i \epsilon te_{n_i} \frown U$, $f(x_i) = e'_i$, where $e'_i \epsilon E(Y) \frown V$, and $\lim_{i\to\infty} e'_i = b = f(a)$.

Further, a_0 being a condensation point of $te_0 \frown A_0$ by definition, we conclude from (5.29) that A is uncountable.

Let a^1 and a^2 be points of A such that

(5.32) $a^1 \neq a^2$,

(5.33) $a^1 \epsilon ta^2$,

belonging to the same component of the set $U \frown te_0$. So

(5.34) $a^1 a^2 \subset U$.

Putting

(5.35) $b^1 = f(a^1)$ and $b^2 = f(a^2)$

we conclude

(5.36) $b^1 \neq b^2$

by (5.32) and (5.30). According to statement (5.31) there exists a sequence of points x_i^1 with properties listed in that statement, in particular such that

(5.37) $a^1 = \lim_{i\to\infty} x_i^1$,

(5.38) $x_i^1 \epsilon te_{n_i}^1$.

X being compact, the sequence of end-points $\{e_{n_i}^1\}$ contains a convergent subsequence $\{e_{n_{ij}}^1\}$. Putting

(5.39) $c = \lim_{j\to\infty} e_{n_{ij}}^1$

we conclude that $tc = \operatorname{Lim}_{j\to\infty} te_{n_{ij}}^1$ by the smoothness of X, whence

(5.40) $a^1 \epsilon tc$

by (5.37) and (5.38). Since $a^2 \epsilon A$ by definition, we have

(5.41) $a^2 \epsilon U$

by (5.29). But $c \epsilon X - U$ by (5.39) and (5.26); thus (5.40), (5.34) and (5.33) imply

(5.42) $a^2 \epsilon a^1 c$.

It follows from Theorem 1 by (5.37) and (5.39) that

$$a^1 c = \operatorname*{Lim}_{j \to \infty} x^1_{i_j} e^1_{n_{i_j}};$$

thus, by (5.41) and (5.42), there exists a convergent sequence of points p_j such that

(5.43) $$p_j \,\epsilon\, x^1_{i_j} e^1_{n_{i_j}} \frown U$$

and

(5.44) $$\lim_{j \to \infty} p_j = a^2.$$

Thus since $p_j \,\epsilon\, t e^1_{n_{i_j}} \frown U$ by (5.43), and since $x^1_{i_j} \,\epsilon\, t e^1_{n_{i_j}} \frown U$ by (5.31), we have

(5.45) $$t' \,\epsilon\, Y - f(p_j) f(x^1_{i_j})$$

by (5.25). The sequence x^1_i having properties listed in (5.31), in particular $f(x^1_{i_j}) = e'^1_{i_j}$, we can write (5.45) in the form

$$t' \,\epsilon\, Y - f(p_j) e'^1_{i_j}.$$

Therefore

(5.46) $$f(p_j) \,\epsilon\, t' e'^1_{i_j}.$$

Further, $b^1 = \lim_{j \to \infty} e'^1_{i_j}$ by (5.31); thus, Y being smooth,

(5.47) $$t' b^1 = \operatorname*{Lim}_{j \to \infty} t' e'^1_{i_j}.$$

It follows from (5.44) and (5.35) that $\lim_{j \to \infty} f(p_j) = b^2$. So

(5.48) $$b^2 \,\epsilon\, t' b^1$$

by (5.46) and (5.47).

Now let us consider a sequence $\{x^2_i\}$ with the properties listed in (5.31), in particular such that $a^2 = \lim_{i \to \infty} x^2_i$ and $x^2_i \,\epsilon\, t e^2_{n_i}$. Since X is smooth, we have $t a^2 = \operatorname*{Lim}_{i \to \infty} t x^2_i$. Thus it follows from (5.33) that there exists a convergent sequence of points q_i such that $q_i \,\epsilon\, t x^2_i \frown U$ for $i = 1, 2, \ldots$ and

(5.49) $$\lim_{i \to \infty} q_i = a^1.$$

So we have $t' \,\epsilon\, Y - f(q_i) f(x^2_i)$ by (5.25), i.e. $t' \,\epsilon\, Y - f(q_i) e'^2_i$ because $f(x^2_i) = e'^2_i$ according to (5.31). Thus

(5.50) $$f(q_i) \,\epsilon\, t' e'^2_i.$$

Further, $b^2 = \lim_{i \to \infty} e_i'^2$ by (5.31), thus, Y being smooth, we have

(5.51) $t'b^2 = \lim_{i \to \infty} t' e_i'^2$.

It follows from (5.49) and (5.35) that $\lim_{i \to \infty} f(q_i) = b^1$; so we have $b^1 \epsilon t' b_2$ by (5.50) and (5.51), which implies $b^1 = b^2$ by (5.48), contrary to (5.36). This completes the proof.

The problem arises whether the hypothesis of the smoothness of Y is essential in Theorem 5; in other words, whether we can generalize Theorem 5 to all fans Y, without the hypothesis of their smoothness.

COROLLARY 7. *If the set* $\overline{E(X)} - E(X)$ *of a countable smooth fan* X *is countable and if a mapping* f *of* X *onto a smooth fan* Y *is continuous, then* Y *is a countable fan and* $\overline{E(Y)} - E(Y)$ *is countable.*

The hypothesis of the countability of the fan X is essential in Corollary 7 because e.g. the dendroid D constructed by Lelek in [10], § 9, p. 314 is a geometrical fan with the property $\overline{E(D)} = D$ and, by Corollary 17 in § 7 of this paper (see p. 29 below), it is a continuous image of the Cantor fan, which is manifestly uncountable and the set of its end-points is closed.

A set which does not contain any non-empty subset which is dense in itself is called *scattered*.

COROLLARY 8. *If the set* $\overline{E(X)}$ *of a smooth fan* X *is scattered and if a mapping* f *of* X *onto a smooth fan* Y *is continuous, then* $\overline{E(Y)}$ *is scattered.*

In fact, if $\overline{E(X)}$ is scattered, it is countable (see [8], § 18, V, p. 141); thus X must be a countable fan, whence $\overline{E(Y)}$ is countable by Theorem 5. Suppose that $\overline{E(Y)}$ contains a non-empty subset A which is dense in itself. Consequently $\bar{A}$ is a perfect, and thereby an uncountable subset of $\overline{E(Y)}$ (see [11], Corollary (5.51), p. 54).

In the same way we can prove

COROLLARY 9. *If the set* $\overline{E(X)} - E(X)$ *of a countable smooth fan* X *is scattered and if a mapping* f *of* X *onto a smooth fan* Y *is continuous, then* Y *is a countable fan and* $\overline{E(Y)} - E(Y)$ *is scattered.*

The hypothesis of the countability of the fan X is essential in this corollary. It can be seen from the same example as for Corollary 7.

Remark that the property $\dim \overline{E(X)} = 0$ is not invariant under continuous mappings even for countable geometrical fans. In fact, arrange in a sequence all rational numbers of the Cantor set $\mathscr{C}$:

$$r_1, r_2, \ldots, r_n, \ldots,$$

and take points p_n of the Euclidean plane with polar coordinates $\varrho = r_n$, $\varphi = 1/n$. Further, let the point p_0 have coordinates $\varrho = 1, \varphi = 0$. Join every point p_i for $i = 0, 1, \ldots$ with the origin O by a straight line segment. The set

$$X = \bigcup_{i=0}^{\infty} \overline{Op_i}$$

is a countable geometrical fan. Thus $E(X)$ consists of points p_i $(i = 0, 1, \ldots)$ only, and $\overline{E(X)} - E(X)$ is the Cantor set of points of the segment $\overline{Op_0}$. Let $g \colon \mathscr{C} \to \mathscr{I}$ be the Cantor continuous function (see [8], § 24a, VIa, p. 236), and let g^* be the extension of g by assigning to g on every interval of $\mathscr{I} - \mathscr{C}$ the same value as that which g takes at the endpoints of these intervals ("fonction scalariforme de Cantor"). Thus $g^* \colon \mathscr{I} \to \mathscr{I}$ is continuous. Put

$$f(p) = \left(g^*(\varrho), \varphi\right) \quad \text{for every } p = (\varrho, \varphi) \epsilon X.$$

Hence f maps continuously the fan X onto a countable geometrical fan $f(X)$ which has the property $\overline{E(f(X))} = \overline{Op_0}$, whence $\dim \overline{E(f(X))} = 1$.

§ 6. Applications. Families of n incomparable countable geometrical fans for each $n = 2, 3, \ldots$

In [4] families of n incomparable plane fans for each $n = 2, 3, \ldots$, have been constructed, but those fans were non-smooth. They were even non-uniformly arcwise connected. Now we shall construct arbitrarily large finite families of plane fans D_i^0 with the property that each of those fans will be geometrical, and thus smooth and thereby uniformly arcwise connected (see Corollary 5).

Let F_{Hk} be the fan defined in [4], (43), p. 200. We now define another fan F_{Sk} which will be contained in F_{Hk} (the notation is taken from [4], pp. 199 and 200).

Arrange in a sequence all rational numbers of the segment 01. Let

$$r_1, r_2, \ldots, r_s, \ldots$$

be this sequence. Take the points $p_{k;s}$ of the set $A_k - A_{k-1}$ and assign to each of them a point $q_{k;s} \epsilon \overline{Op_{k;s}}$ having the polar coordinate $\varrho = r_s$. Put

(6.1) $$F_{Sk} = F_{Hk-1} \cup \bigcup_{s=1}^{\infty} \overline{Oq_{k;s}}.$$

F_{Sk} defined in this way is a geometrical fan with the top O. $E(F_{Sk})$ consist of the points $q_{k;s}$ and of the end-points of F_{Hk-1}, i.e.

$$E(F_{Sk}) = E_k^0 \cup A_{k-1},$$

where

(6.2) $$E_k^0 = \bigcup_{s=1}^{\infty} q_{k;s}.$$

Obviously the set $E(F_{Sk})$ is countable and we have

(6.3) $$\overline{E_k^0} = F_{Hk-1} \cup E_k^0$$

by (6.2) and by the definition of points $q_{k;s}$ determining the lengths of the straight segments $\overline{Oq_{k;s}}$ which approach the segment $\overline{Op_{k-1;s}}$. According to (46) in [4], p. 200 and to (6.1) we have

(6.4) $$\tau(F_{Sk}) = k.$$

Denote by F_{Sk}^* a fan symmetric to F_{Sk} with respect to point O. Thus by (6.4)

(6.5) $$\tau(F_{Sk}^*) = k.$$

Put for an arbitrary natural number $n > 1$ and for every $i = 0, 1, \ldots$
$\ldots, n-1$

(6.6) $$D_i^0 = F_{Hn+i} \cup F_{Sn-i}^*.$$

Hence by T29 in [4], p. 198

(6.7) $$\tau(D_i^0) = n + i.$$

Let $F^0 \subset F_C$ be an arbitrary subcontinuum of the fan F_C which contains the top O and is such that

(6.8) $$\overline{E(F^0)} \text{ is countable.}$$

Hence $F^0 \cup F_{Sk}^*$ is a fan. We now prove the following property of continuous mappings of $F^0 \cup F_{Sk_1}^*$ onto F_{Sk_2}.

P. *If F_{Sk_2} is a continuous image of $F^0 \cup F_{Sk_1}^*$, then $k_2 \leqslant k_1$.*

Proof. Let f be such a mapping, i.e. let

(6.9) $$f(F^0 \cup F_{Sk_1}^*) = F_{Sk_2}.$$

Putting

(6.10) $$A = E_{k_2}^0 \cap f(F^0),$$

(6.11) $$B = E_{k_2}^0 - A,$$

we have identically

(6.12) $$E^0_{k_2} = A \cup B.$$

It follows from (6.10) that $A \subset E(f(F^0))$, whence by (6.8)

(6.13) $$\bar{A} \text{ is countable}$$

according to Theorem 6. Since $A \cap B = 0$ by (6.11) and $\bar{A} \cup \bar{B} = F_{Hk_2-1} \cup$
$\cup A \cup B$ by (6.3) with $k = k_2$, it follows from (6.13) that $F_{Hk_2-1} \subset \bar{B}$;
thus

(6.14) $$\bar{B} = B \cup F_{Hk_2-1}.$$

The set $I(\bar{B}) = F_{Hk_2-1} \cup \bigcup_{q \in B} \overline{Oq}$ being by definition a subfan of the
fan F_{Sk_2}, we conclude from (6.14) that

(6.15) $$\tau(I(\bar{B})) = k_2.$$

Further, $B \cap f(F^0) = 0$ by (6.10) and (6.11), and thus $B \subset f(F^*_{Sk_1})$
by (6.9), whence $I(\bar{B}) \subset f(F^*_{Sk_1})$; therefore

(6.16) $$\tau(I(\bar{B})) \leqslant \tau(f(F^*_{Sk_1})).$$

Since $\tau(f(F^*_{Sk_1})) \leqslant \tau(F^*_{Sk_1})$ by Theorem T18 in [4], p. 192, $k_2 \leqslant k_1$
follows from (6.16) by (6.15) and (6.5).

Let n be an arbitrary natural number, D^0_i the fans defined by (6.6)
and f a continuous mapping of D^0_i into D^0_j, where $i \neq j$. It ought to be
proved that

(6.17) $$f(D^0_i) \neq D^0_j.$$

Assume first that $i < j$, i.e. that

(6.18) $$n+i < n+j.$$

We have, according to (6.7), $\tau(D^0_i) = n+i$ and $\tau(D^0_j) = n+j$; con-
sequently (6.17) follows by (6.18) and T18 in [4], p. 192.

Assume next that $j < i$, i.e. that

(6.19) $$n-i < n-j.$$

Consider the retraction

$$r(p) = \begin{cases} p, & \text{when} \quad p \in F^*_{Sn-j}, \\ 0, & \text{when} \quad p \in F_{Hn+j}. \end{cases}$$

Then, if we suppose that (6.17) does not hold, the continuous mapping
$g = rf$ maps D^0_i onto F^*_{Sn-j}:

(6.20) $$g(D^0_i) = F^*_{Sn-j}.$$

Let h be a homeomorphism which maps F_{Hn+i} into F_C in such a manner that $h(A_{n+i}) \subset C$, and let $F^0 = h(F_{Hn+i})$. Putting $k_1 = n-i$, i.e. $F^*_{Sk_1} = F^*_{Sn-i}$ and $k_2 = n-j$, i.e. $F^*_{Sk_2} = F^*_{Sn-j}$, we have identically

$$(6.21) \qquad F^0 \cup F^*_{Sk_1} = h(F_{Hn+i}) \cup F^*_{Sn-i}.$$

Further, let h_1 be a homeomorphism of $F_{Hn+i} \cup F^*_{Sn-i}$ defined as follows:

$$h_1(p) = \begin{cases} h(p), & \text{when} \quad p \in F_{Hn+i}, \\ p, & \text{when} \quad p \in F^*_{Sn-i}. \end{cases}$$

Therefore, by (6.21), we have $F^0 \cup F^*_{Sk_1} = h_1(F_{Hn+i} \cup F^*_{Sn-i})$, whence $F^0 \cup F^*_{Sk_1} = h_1(D_i^0)$. It follows $D_i^0 = h_1^{-1}(F^0 \cup F^*_{Sk_1})$ and, by (6.20), $gh_1^{-1}(F^0 \cup F^*_{Sk_1}) = F_{Sk_2}$. In consequence of property P recently proved we have $k_2 \leqslant k_1$, i.e. $n-j \leqslant n-i$, contrary to (6.19).

§ 7. Folding fans

A fan X with the top t is said to be *folding* provided that a real-valued continuous function $\vartheta: X \to \mathscr{I}$ exists such that

$$(7.1) \qquad\qquad\qquad \vartheta(t) = 0,$$

(7.2) for every end-point $e \in E(X)$ the partial function $\vartheta|te$ is a homeomorphism of te into $\mathscr{I}$.

An immediate consequence of the above definition is the heredity of the folding property for fans:

COROLLARY 10. *If a fan X is folding, then every subfan of X is also folding.*

THEOREM 6. *If a fan X has the following property*:

(7.3) *for every two different points x_1 and x_2 of X $x_1 \epsilon tx_2$ implies $\varrho(t, x_1) < \varrho(t, x_2)$,*

then X is folding.

Proof. In fact, putting $d = \max\{\delta(te): e \in E(X)\}$, we have

$$\vartheta(x) = \frac{1}{d} \cdot \varrho(t, x) \quad \text{for every} \quad x \epsilon X.$$

THEOREM 7. *If all arcs te of a fan X are rectifiable and if*

$$(7.4) \qquad x = \lim_{n \to \infty} x_n \quad \text{implies} \quad \lambda(tx) = \lim_{n \to \infty} \lambda(tx_n),$$

then X is folding.

Proof. In fact, putting $l = \max\{\lambda(te): e \epsilon E(X)\}$ we have

$$\vartheta(x) = \frac{1}{l} \cdot \lambda(tx) \quad \text{for every } x \epsilon X.$$

An arc A contained in the 3-dimensional Euclidean space is said to be a *polygonal line* if it is the union of a finite number of straight line segments. A fan X with the top t is said to be *polygonal* if every arc te is a polygonal line and if the numbers of straight line segments consisting the arcs te are bounded in common. In particular if all arcs te in X are straight line segments, then X is said to be *geometrical*. It is quite obvious that every geometrical fan is smooth.

THEOREM 8. *If a polygonal fan is smooth, then it is folding.*

Proof. Let a fan X be polygonal. Every arc te of X, being a polygonal line, is rectifiable. All the numbers of straight line segments of arcs te being bounded in common and X being smooth, (7.4) holds. Thus X is folding by Theorem 7.

In particular, every geometrical fan being obviously polygonal and smooth, we infer from Theorem 8 the following

COROLLARY 11. *Every geometrical fan is folding.*

The hypothesis of the smoothness of the fan is essential in Theorem 8, because e.g. the Cantor hooked fan M_C^v described in [3], E3, p. 240 is polygonal, non-smooth and non-folding.

Now recall a well-known notion. A *quasicomponent* of a point p in a set A is the common part of all closed-open subsets of A containing p. In other words, it is a set of all points $q \epsilon A$ such that the set A is connected between p and q (see e.g. [9], § 41, V, p. 92).

LEMMA 3. *For every folding fan X the quasicomponents of the set $X - (t)$ have the form $te - (t)$, i.e. they coincide with the components of that set.*

Proof. Since every component of a set is contained in a quasicomponent of that set, and since every component of the set $X - (t)$ has the form $te - (t)$, it suffices to show that for any two different end-points e', e'' of X there exists a closed-open subset S of the set $X - (t)$ which contains e' but does not contain e''. To construct such an S let us take an arbitrary sequence of real numbers $\{\varepsilon_n\}$ such that

(7.5) $$\min[\vartheta(e'), \vartheta(e'')] > \varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > \ldots > 0,$$

(7.6) $$\lim_{n \to \infty} \varepsilon_n = 0,$$

and put

(7.7) $$X_n = \vartheta^{-1}([\varepsilon_n, 1]),$$

where $[a, b]$ denote the set of real numbers x such that $a \leqslant x \leqslant b$. It follows from (7.5) and (7.7) that $e', e'' \epsilon X_n$ for every n,

(7.8) $X_n \subset \text{Int}(X_{n+1})$ for every n,

and, by virtue of (7.6), that

(7.9) $$X - (t) = \bigcup_{n=1}^{\infty} X_n.$$

Let us remark that, the mapping ϑ being continuous by definition, $\vartheta^{-1}((\varepsilon_n, 1])$ is open in X_n as well as in $X - (t)$, and so

(7.10)   if $U \subset \vartheta^{-1}((\varepsilon_n, 1])$ is an open set in X_n, then it is an open set in $X - (t)$.

Further, the mapping ϑ being continuous, X_n are compact by (7.7) and, $\vartheta|te$ being a homeomorphism for every $e \epsilon E(X)$, the components of X_n are arcs of the form xe, where $x \epsilon te$ and $\vartheta(x) = \varepsilon_n$ (in the particular case of $\vartheta(e) = \varepsilon_n$ the arc can be reduced to a single point). It is well known that, X_n being compact, there exists for every n and for any two given end-points e' and e'' a set S_n which is closed-open in X_n and such that

(7.11) $e' \epsilon S_n$ and $e'' \epsilon X_n - S_n$ for $n = 1, 2, \ldots$

It is not difficult to define (e.g. by induction) a sequence of closed-open sets S_n which satisfy (7.11) and are such that

(7.12) $S_1 \subset S_2 \subset \ldots \subset S_n \subset S_{n+1} \subset \ldots,$

(7.13) $S_{n+1} - S_n \subset X_{n+1} - X_n$ for $n = 1, 2, \ldots$

Finally put

(7.14) $$S = \bigcup_{n=1}^{\infty} S_n.$$

So $e' \epsilon S$ and $e'' \epsilon X - S$ by (7.11). Let us remark that

(7.15) $X_n \cap S = S_n$ for $n = 1, 2, \ldots$

by (7.13).

We now prove that S is open in $X - (t)$. Let $p \epsilon S$. There exists by (7.14) a natural n such that

(7.16) $p \epsilon S_n.$

The set S_n being a subset of X_n, we conclude that $p \epsilon \vartheta^{-1}([\varepsilon_n, 1])$ by (7.7), whence

(7.17) $p \epsilon \vartheta^{-1}((\varepsilon_{n+1}, 1])$

by (7.5). It follows from (7.16) and (7.12) that $p \, \epsilon \, S_{n+1}$. This set being open in X_{n+1}, there exists a neighbourhood U of p in X_{n+1} such that

(7.18) $$U \subset S_{n+1},$$

and, by (7.17), $U \subset \vartheta^{-1}((\varepsilon_{n+1}, 1])$. So U is open in $X - (t)$ by (7.10), and $U \subset S$ by (7.18) and (7.14). Thus S is open in $X - (t)$.

Finally we prove that S is closed in $X - (t)$. Let us consider a convergent sequence $\{p_k\}$ of points of $X - (t)$ with the limit point p:

(7.19) $$p = \lim_{k \to \infty} p_k,$$

and let

(7.20) $$p \, \epsilon \, X - (t).$$

It suffices to prove that if

(7.21) $$p_k \, \epsilon \, S \quad \text{for} \quad k = 1, 2, \ldots,$$

then

(7.22) $$p \, \epsilon \, S.$$

It follows from (7.20) and (7.9) that there is a natural n such that $p \, \epsilon \, X_n$; thus $p \, \epsilon \, \text{Int}(X_{n+1})$ by (7.8). Therefore we can admit by (7.19) that

(7.23) $$p_k \, \epsilon \, X_{n+1} \quad \text{for} \quad k = 1, 2, \ldots$$

Assuming (7.21) we infer from (7.14) and (7.23) that $p_k \, \epsilon \, X_{n+1} \cap S$ for $k = 1, 2, \ldots$, i.e. $p_k \, \epsilon \, S_{n+1}$ by (7.15). Since S_{n+1} is closed in X_{n+1} by definition, and X_{n+1} is closed in $X - (t)$, we have $p \, \epsilon \, S_{n+1}$ by (7.19) and (7.20); thus (7.22) follows by (7.14). This finishes the proof of the lemma.

THEOREM 9. *Every folding fan can be homeomorphically embedded into the Cantor fan.*

Proof. Let a fan

$$X = \bigcup_{e \epsilon E(X)} te$$

be folding. By virtue of Lemma 3 the quasicomponents of the set $X - (t)$ have the form $te - (t)$. It is known (see [9], § 41, V, 3, p. 93) that there exists a continuous mapping of $X - (t)$ into the Cantor set $\mathscr{C}$ such that the inverse image of every point under that mapping is a quasicomponent of the set $X - (t)$ ("quasikomponententreue Abbildung"). Denote this mapping by ψ. Hence

$$\psi \colon X - (t) \to \mathscr{C},$$

$$\psi^{-1}(c) = te - (t) \quad \text{for every } c \, \epsilon \, \psi(X - (t)) \text{ and for an } e \, \epsilon \, E(X).$$

Suppose we have in the Euclidean plane E^2 a system of polar co-ordinates ϱ, φ with the pole at the point O. Consider the Cantor discontinuum C in the arc $0 \leqslant \varphi \leqslant 1$ of the circumference $\varrho = 1$, i.e. the set of points $p = (1, \varphi)$, where $\varphi = \sum\limits_{i=0}^{\infty} 2c_i/3^i$ and $c_i = 0$ or 1. The union

$$(7.24) \qquad\qquad F_C = \bigcup_{p \epsilon C} \overline{Op}$$

is a fan homeomorphic with the Cantor fan. We assign to every point x of X a point $h(x) \epsilon F_C$ in the following way:

$$h(t) = O;$$

if $x \epsilon te - (t)$, then $h(x) = (\varrho, \varphi)$, where $\varrho = \vartheta(x)$ and $\varphi = \psi(x)$.

Thus $h\colon X \to F_C$. The mappings ϑ and ψ being continuous, it is easy to see that so is h. Further, $\vartheta | te$ being a homeomorphism by definition and ψ being quasicomponentwise, h is one-to-one, and thus it is a homeomorphism.

The following corollaries follow from Theorem 9.

COROLLARY 12. *To every folding fan X there exists a geometrical fan which is homeomorphic with X.*

Conversely, every geometrical fan being folding by Corollary 11, we infer

COROLLARY 13. *The class of all folding fans is identical with that of all geometrical fans, i.e. with the class of all subfans of the Cantor fan.*

The Cantor fan lying in the Euclidean plane E^2 by definition, we have

COROLLARY 14. *Every folding fan is homeomorphic with a plane fan.*

The hypothesis of the folding property is essential in Corollary 14 as is shown by Borsuk's example of a non-plane fan (see [2], p. 233).

Every folding fan being homeomorphical with a geometrical fan by Theorem 9, and every geometrical fan being obviously smooth, we have

COROLLARY 15. *Every folding fan is smooth.*

The problem arises whether the inverse theorem is true, i.e. whether a smooth fan is folding.

Corollaries 15 and 5 give

COROLLARY 16. *Every folding fan is uniformly arcwise connected.*

The same example as for Corollary 5 shows that the inverse theorem to Corollary 16 is not true.

THEOREM 10. *Every folding fan is a continuous image of the Cantor fan.*

Proof. Let C and F_C be, as before, the Cantor discontinuum and the Cantor fan, respectively, described in the proof of Theorem 9 (see for-

mula (7.24) above), and let $F \subset F_C$ be the image of a folding fan X under a homeomorphism h according to Theorem 9. The point O being the top of F_C, we can write

$$F = \bigcup_{e \epsilon E(F)} \overline{Oe}.$$

Denote by f an arbitrary continuous mapping of C onto F:

$$f(C) = F,$$

and let f^* be a mapping defined as follows: for every $p \epsilon C$ the mapping $f^*|Op$ transforms linearly the arc Op onto the arc $Of(p)$. So f^* is defined for every $x \epsilon F_C$, and we have

$$f^*(F_C) = F.$$

It is easy to prove, by the continuity of f and by the smoothness o F, that f^* is continuous. So $h^{-1}f^*$ maps continuously F_C onto X.

Every geometrical fan being folding by Corollary 11, Theorem 10 implies the following

COROLLARY 17. *Every geometrical fan is a continuous image of the Cantor fan.*

Every subcontinuum of the Cantor fan being an arc or a geometrical fan, we have

COROLLARY 18. *Every subcontinuum of the Cantor fan is a continuous image of it.*

There exist other continua K which have the property pointed out in Corollary 18, namely the property that every subcontinuum of K is a continuous image of K, e.g. hereditarily locally connected continua or the closure of the set of points (x, y) with $y = \sin 1/x$, $0 < x \leqslant 1$. Obviously there are continua which do not have this property, e.g. every locally connected but not hereditarily locally connected continuum. It seems an interesting problem to characterize topologically all continua having this property.

Corollary 17 says that for all geometrical fans there exists a common model under continuous mappings, namely the Cantor fan. The problems arise whether common models exist for all fans, for all uniformly arcwise connected fans, and for all smooth fans.

Similarly, Theorem 9 says that in the class of all folding fans (i.e., by Corollary 12, fans homeomorphic with geometrical) there exists a universal element, i.e. such an element U of this class that every other element can be homeomorphically embedded in U. The problems arise whether universal elements exist in the classes of all fans, of all uniformly arcwise connected fans, and of all smooth fans.

Remark that there exists neither a common model nor a universal element in the class of all geometrical countable fans. The former assertion follows from the obvious existence of geometrical countable fans with arbitrarily great degree of non-local connectedness, which is a countable ordinal, and from a theorem (see [4], T18, p. 192) according to which the degree of the model cannot be smaller than that of the image. The latter assertion, i.e. that no universal element exists in the class of all geometrical countable fans, follows beside the above premises from a theorem (see [4], T13, p. 191) according to which the degree of non-local connectedness of the containing fan cannot be smaller than that of the contained fan.

§ 8. Contractibility

Recall that a continuum X is said to be *contractible* if every mapping $f\colon X \to X$ is homotopic to a constant. In particular for fans this definition is equivalent to the following one: a fan X with the top t is *contractible* if there exists a continuous mapping $h\colon X \times \mathcal{J} \to X$ (called a *homotopy*) such that

$$(8.1) \qquad h(x, 0) = x \quad \text{and} \quad h(x, 1) = t \quad \text{for every } x \in X.$$

Let us observe that

THEOREM 11. *Every folding fan is contractible.*

In fact, it follows from Theorem 9 that instead of an arbitrary folding fan we can consider the homeomorphic image X of a fan which is contained in the Cantor fan F_C defined by (7.24). It suffices to put for $x = (\varrho, \varphi)$ and $s \in \mathcal{J}$

$$h(x, s) = \left(\varrho \cdot (1 - s), \varphi\right).$$

Remark that we can define the homotopy in another way, namely we can put

$$(8.2) \qquad h(x, s) = \begin{cases} x & \text{for} \quad s \leqslant 1 - \varrho, \\ (1 - s, \varphi) & \text{for} \quad 1 - \varrho < s. \end{cases}$$

It is easy to state that h is continuous and satisfies (8.1).

Recall that if a continuous mapping f of X onto $f(X)$ has properties $f(X) \subset X$ and $f(x) = x$ for each $x \in f(X)$, then f is said to be a *retraction* (see [1], 3, p. 154).

Denote by i_s, for any fixed number $s \in \mathcal{J}$, a mapping of X onto $X \times (s)$ such that

$$(8.3) \qquad i_s(x) = (x, s) \quad \text{for every } x \in X.$$

It is remarkable that, by definition, if the mapping h is given by (8.2), then for every fixed number $s \in \mathcal{S}$ the superposition $h i_s$ is a retraction. Call a mapping $h: X \times \mathcal{S} \to X$ to be a *retracting homotopy* if it is a homotopy and if $h i_s$ is a retraction for every $s \in \mathcal{S}$, where $i_s: X \to X \times (s)$ is the mapping defined by (8.3). Thus we get the following specification of Theorem 11:

COROLLARY 19. *Every folding fan is contractible even in such a manner that the homotopy in question is retracting.*

The inverse of Theorem 11 is not true as is shown by the following example of a harmonic hooked fan F_H^v which is contractible and non-smooth, and thus non-folding by Corollary 15.

Put in the polar coordinates (ϱ, φ)

$$p_0 = (1, 0), \qquad p_n = (1, 2^{1-n}), \qquad q_n = (\tfrac{1}{2}, \tfrac{3}{4} \cdot 2^{1-n}) \qquad \text{for} \qquad n = 1, 2, \ldots,$$

and let

$$F_H^v = \overline{Op_0} \cup \bigcup_{n=1}^{\infty} (\overline{Op_n} \cup \overline{p_n q_n}).$$

So F_H^v is a polygonal fan with the top O and with end-points p_0 and q_n for $n = 1, 2, \ldots$ It is not smooth by construction. To show that F_H^v is contractible it suffices to define $h(x, s)$ as follows. $h(O, s) = O$ for every $s \in \mathcal{S}$. If $x \neq O$, then $h(x, s)$ belongs to the same component of the set $F_H^v - (O)$ as the point x for every $s \in \mathcal{S}$. Further, if ϱ_1 and ϱ are the radii, i.e. the first polar coordinates of points $h(x, s)$ and x respectively, we put

$$\varrho_1 = \begin{cases} (1 + 2s)\varrho & \text{if} & \varrho < \tfrac{1}{2} \text{ and } 0 \leqslant s < \tfrac{1}{2}, \\ (1 - 2s)\varrho + 2s & \text{if} & \tfrac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } 0 \leqslant s < \tfrac{1}{2}, \\ 4(1 - s)\varrho & \text{if} & \varrho < \tfrac{1}{2} \text{ and } \tfrac{1}{2} \leqslant s \leqslant 1, \\ 2(1 - s)[(1 - 2s)\varrho + 2s] & \text{if} & \tfrac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } \tfrac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

It is easy to verify that the $h(x, s)$ thus defined maps continuously $F_H^v \times \mathcal{S}$ onto F_H^v and that (8.1) holds with $t = O$.

Let us remark that the homotopy $h(x, s)$ just defined is not retracting. So the following problem arises concerning the theorem inverse to Corollary 19: is every contractible fan with a retracting homotopy a folding one? Or is it perhaps a smooth fan only?

As regards the generalization of Theorem 11 to smooth fans, the following problem is open: is every smooth fan contractible? Moreover, we have a further unsolved question: is every smooth and contractible fan folding? It is a partial case of the problem on p. 28.

§ 9. Confluent mappings

A continuous mapping f of a topological space X onto a topological space Y is *confluent* if for every subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [5], p. 213, where also some theorems were proved concerning confluent mappings of continua).

If f is a confluent mapping of a locally connected continuum, then f is quasi-monotone (see [5], IX, p. 215). Consequently, all invariants of locally connected continua under quasi-monotone mappings are also invariants under confluent mappings (see [5], loco cit.). The property of being an arc is one of those invariants. In fact, the property of being an arc is invariant under monotone mappings (see [11], (1.1), p. 165) and under interior mappings (see [11], (1.3), p. 184); thus it is invariant under quasi-monotone mappings by Theorem (8.5) in [11], p. 153. So we have

COROLLARY 20. *A confluent image of an arc is an arc.*

Now we prove that the property of being a fan is invariant under confluent mappings.

THEOREM 12. *A confluent image of a fan is a fan (or an arc), and the top of the model is mapped on the top of the image.*

Proof. Let X be a fan with the top t, and let f be a confluent mapping of X. X being a dendroid by definition, $f(X)$ is a dendroid by virtue of Corollary 1 in [5], p. 219. Suppose that there are in $f(X)$ two different ramification points a and b. So one of them, say a, is different from $f(t)$. Thus there is a subcontinuum Q of $f(X)$ such that $a \epsilon Q \subset f(X) - \{f(t)\}$ and that

(9.1) Q is not an arc.

So $f^{-1}(Q) \subset X - (t)$, whence every component of $f^{-1}(Q)$ is an arc. Let A be any of those components. The mapping f, being confluent, maps every component of $f^{-1}(Q)$ onto Q, whence $f(A) = Q$. But $f|A$ is also confluent (see [5], I, p. 213); consequently Q is an arc by Corollary 20, contrary to (9.1).

LEMMA 4. *If a fan Y is an image of a fan X under a confluent mapping f and if Q is a subcontinuum of Y containing its top, then $f^{-1}(Q)$ is connected.*

Proof. Let t be the top of X, and t' the top of Y. Thus $t' \epsilon Q$ by hypothesis, and since $t' = f(t)$ by Theorem 12, we have $t \epsilon f^{-1}(Q)$.

Assume firstly that Q is neither an arc nor a point, and suppose that $f^{-1}(Q)$ is not connected. So there exists a component A of $f^{-1}(Q)$ such that $A \subset X - (t)$. Thus A is an arc or a point. The mapping f, being con-

fluent, maps every component of $f^{-1}(Q)$ onto Q, whence $f(A) = Q$. But $f|A$ is also confluent (see [5], loco cit.); consequently Q is either an arc by Corollary 20 or a point, which is a contradiction. Hence

(9.2)   if Q is neither an arc nor a point, then $f^{-1}(Q)$ is connected.

Assume secondly that Q either is an arc or reduces to a point, namely to the point t'. Since the ramification point t' belongs to Q by hypothesis, there is in Y such a sequence of continua Q_n that

$$\ldots \subset Q_n \subset \ldots \subset Q_2 \subset Q_1,$$

$$Q = \bigcap_{n=1}^{\infty} Q_n,$$

and

$$\text{no } Q_n \text{ is an arc or a point.}$$

Thus $f^{-1}(Q_n)$ is connected by (9.2), and, being obviously compact it is a continuum. So $f^{-1}(Q) = \bigcap_{n=1}^{\infty} f^{-1}(Q_n)$ is the common part of the decreasing sequence of continua $f^{-1}(Q_n)$; therefore it is a continuum (see [9], § 42, II, 5, p. 110). The proof is finished.

THEOREM 13. *A confluent image of a smooth fan is a smooth fan (or an arc).*

Proof. Let X be a smooth fan with the top t, and let f be a confluent mapping of X. It follows from Theorem 12 that $f(X)$ is a fan, and that $t' = f(t)$ is the top of $f(X)$.

Suppose that $f(X)$ is not smooth. Thus there exists in $f(X)$ a sequence of points y_n which is convergent to a point y_0, and is such that there exists a point c with the property

(9.3) $c \in \operatorname*{Ls}_{n\to\infty} t'y_n - t'y_0.$

Hence there is a subsequence of arcs $t'y_{n_m}$ and there are points c_m such that

(9.4) $c_m \in t'y_{n_m},$

and

(9.5) $c = \lim_{m\to\infty} c_m.$

It follows from Lemma 4 that

'(9.6) $f^{-1}(t'y_0)$ is connected,

and from (9.3) that

(9.7) $f^{-1}(c) \cap f^{-1}(t'y_0) = 0$.

Now consider a sequence of points $\{x_{n_m}\}$ such that $f(x_{n_m}) = y_{n_m}$. This sequence contains a convergent subsequence $\{x_{n_{m_k}}\}$. Put

(9.8) $x_0 = \lim_{k\to\infty} x_{n_{m_k}}$.

The sequence of points $y_{n_{m_k}}$ being convergent to the point y_0, we have

(9.9) $f(x_0) = y_0$

by the continuity of f. The fan X being smooth by hypothesis, (9.8) implies

$$tx_0 = \operatorname*{Lim}_{k\to\infty} tx_{n_{m_k}},$$

whence obviously

(9.10) $f(tx_0) = f(\operatorname*{Lim}_{k\to\infty} tx_{n_{m_k}}) = f(\operatorname*{Ls}_{k\to\infty} tx_{n_{m_k}})$.

Further, we have

(9.11) $f(\operatorname*{Ls}_{k\to\infty} tx_{n_{m_k}}) = \operatorname*{Ls}_{k\to\infty} f(tx_{n_{m_k}})$

(see [6], Lemma 8.4, p. 23), and since

$$f(tx_{n_{m_k}}) \supset t'y_{n_{m_k}},$$

we conclude from (9.10) and (9.11) that

(9.12) $f(tx_0) \supset \operatorname*{Ls}_{k\to\infty} t'y_{n_{m_k}}$.

It follows from (9.5) and (9.4) that

$$c = \lim_{k\to\infty} c_{m_k}, \quad \text{where} \quad c_{m_k} \epsilon t'y_{n_{m_k}};$$

thus $c \epsilon \operatorname*{Ls}_{k\to\infty} t'y_{n_{m_k}}$, and by (9.12) we have $c \epsilon f(tx_0)$, whence

(9.13) $f^{-1}(c) \cap tx_0 \neq 0$.

The mapping f being continuous and $t'y_0$ being compact, $f^{-1}(t'y_0)$ is compact, thus by (9.6) it is a continuum. It follows that $f^{-1}(t'y_0)$ is a dendroid as a subcontinuum of a dendroid X (see [3], (49), p. 240). Since $t \epsilon f^{-1}(t'y_0)$ by the definition of t', and since $x_0 \epsilon f^{-1}(t'y_0)$ by (9.9), we have $tx_0 \subset f^{-1}(t'y_0)$, which gives

$$f^{-1}(c) \cap f^{-1}(t'y_0) \neq 0$$

by (9.13), contrary to (9.7). Thus the proof is finished.

It is easy to see that the hypothesis of confluency of the mapping is essential in Theorem 13: there are continuous but non-confluent mappings of smooth fans onto non-smooth ones.

Let us call each fan for which there exists a confluent mapping onto an arc — a *confluent fan*. Remark that in particular no fan F_{S_k} described in § 6 of this paper is a confluent fan. The following problem (due to B. Knaster), which is a partial case of the problem asked after Corollary 15, seems to be interesting: is any smooth and confluent fan folding?

§ 10. The summarized table

The following seven properties of fans have been considered in the present paper:

S — smooth (p. 7),

UAC — uniform arcwise connected (p. 12),

G — geometrical (p. 25),

F — folding (p. 24),

E — embeddable into the Cantor fan (p. 27).

CR — contractible by a retracting homotopy (p. 31)
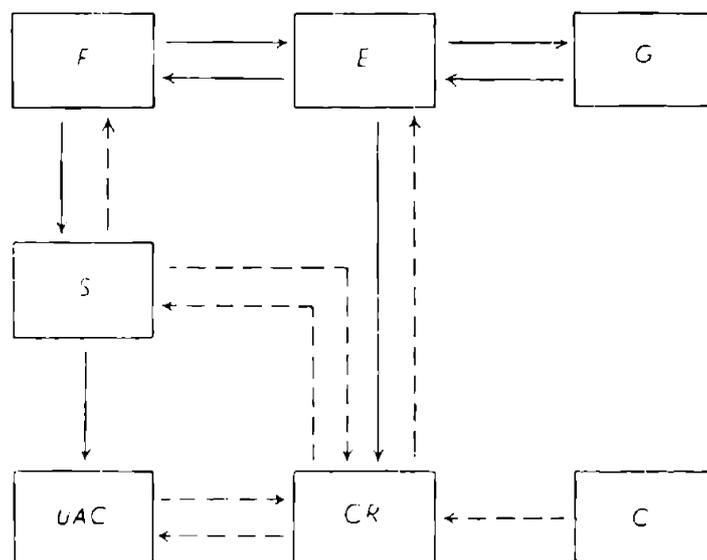
C — confluent (p. 35).

Table 1

The theorems proved on relations between these properties have exclusively the form of implications or of equivalences between a single notion and an other single notion. It gives us a possibility to represent the results in the form of the following table 1, in which continuous arrows denote implications, dotted arrows denote question marks (open prob-

lems), and the absence of arrows denote the lack of implications confirmed by examples (except the cases where an arrow is a direct consequence of other arrows).

It can be seen that we have 16 implications (we include those implications which are direct consequences of other implications and we regard an equivalence as two implications), 15 negations of implications; hence the number of open problems is 11. This number reduces to 7 by the equivalences between E, F and G. Moreover, the following problems remain open.
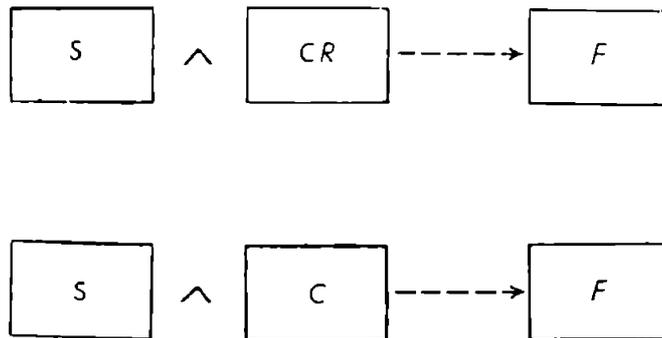

Table 2

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INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS OF THE WROCLAW UNIVERSITY