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# COLLECTIONWISE HAUSDORFF PROPERTY IN PRODUCT SPACES

BY

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**1. Introduction.** A $T_1$-topological space is called *collectionwise Hausdorff* if for every discrete family $\{z_t\}_{t\in T}$ of its points there exists a disjoint family $\{V_t\}_{t\in T}$ of open sets such that $z_t \in V_t$. Clearly, every collectionwise normal space is collectionwise Hausdorff.

Assuming Gödel's Axiom of Constructibility $V = L$, Fleissner proved in [5] that every normal first countable space is collectionwise Hausdorff. Fleissner's theorem and a classical result of Bing [2], stating that every collectionwise normal Moore space is metrizable, raised the necessity of a more thorough investigation of the relation between collectionwise Hausdorff property, normality and collectionwise normality.

In this paper we study the relation between these properties in products of paracompact spaces. We show that in appropriate models of set theory there exist examples of paracompact first countable spaces $X$ such that $X^2$ is collectionwise Hausdorff but not collectionwise normal and that one may additionally assume either that the space $X^2$ is normal or that it is non-normal.

The construction of our examples depends heavily on the beautiful technique developed by Fleissner in [6] and a recent result of Pol [13].

We adopt the notation and terminology from [4]. By $B = B(\omega_2)$ we denote the *Baire space of weight* $\omega_2$, i.e. $B = D(\omega_2)^\omega$, where $D(\omega_2)$ is a discrete space of cardinality $\omega_2$. A space is called *perfect* if its every open subset is an $F_\sigma$-set.

A subset $E$ of an ordinal number $\lambda$ is *stationary in* $\lambda$ if it intersects every closed unbounded subset of $\lambda$. Let $E(\omega_2)$ denote the following statement:

$E(\omega_2)$ There exists a set $E \subset \omega_2$ of ordinals of cofinality $\omega$, which is stationary in $\omega_2$, but for no $\lambda < \omega_2$ the set $E\cap\lambda$ is stationary in $\lambda$.

It is known (cf. Fleissner [6]) that the conjunction $E(\omega_2)+MA+$ $+\neg CH$ of $E(\omega_2)$, Martin's Axiom (see, e.g., [8] or [9]) and the negation of the Continuum Hypothesis and also the conjunction $E(\omega_2)+CH$ of

$E(\omega_2)$ and the Continuum Hypothesis are consistent with the ZFC axioms for set theory. Moreover, Gödel's Axiom of Constructibility $V = L$ (see, e.g., [3]) implies $E(\omega_2)+CH$.

The aim of this paper is to describe the following examples:

EXAMPLE 1. $(E(\omega_2)+MA+\neg CH)$ A first countable paracompact space $X$ such that $X^2$ is collectionwise Hausdorff and perfectly normal, but not collectionwise normal.

EXAMPLE 2. $(E(\omega_2)+CH)$ A first countable paracompact space $X$ such that $X^2$ is collectionwise Hausdorff and perfect, but not normal.

EXAMPLE 3. $(E(\omega_2))$ A perfectly normal collectionwise Hausdorff space which is not collectionwise normal (¹).

The paper is organized as follows. Section 2 is devoted to some preliminary notions and lemmas. In Section 3 we describe a basic construction, which is used in Section 4 to obtain Examples 1 and 2. In Section 5 Example 3 is constructed.

**2. Preliminaries.** A subset $A$ of a space $Y$ is called a *selector* for a family $\mathscr{D} = \{D_s\colon s \in S\}$ of subsets of $Y$ if $|A \cap D_s| = 1$ for every $s \in S$.

A covering $\mathscr{D} = \{D_s\colon s \in S\}$ of a space $Y$, consisting of pairwise disjoint non-empty subsets, is called a *regular decomposition* of $Y$ if the following two conditions are satisfied:

(i) every selector for $\mathscr{D}$ is $\sigma$-discrete in $Y$;

(ii) no family $\{U_s\}_{s\in S}$ of open subsets of $Y$ such that $D_s \subset U_s$ for $s \in S$ is point-countable.

The following theorem has been recently proved by Pol [13]:

THEOREM. $(E(\omega_2))$ *There exists a regular decomposition of $B$ of cardinality $\omega_2$.*

If $\mathscr{D} = \{D_s\colon s \in S\}$ is a family of sets, then by $\mathscr{D}^{(2)}$ we denote the family

$$\mathscr{D}^{(2)} = \{D_s \times D_t\colon (s, t) \in S^2\}.$$

In the sequel we shall need

LEMMA 1. *If $\mathscr{D} = \{D_s\colon s \in S\}$ is a regular decomposition of a metric space $Y$, then the family $\mathscr{D}^{(2)}$ is a regular decomposition of $Y^2$.*

Proof (communicated to the author by R. Pol). Let $A$ be a selector for $\mathscr{D}^{(2)}$. We shall show first that $A$ is $\sigma$-discrete in $Y^2$. For $s \in S$ put $A_s = A \cap (D_s \times Y)$. Since the projection of $A_s$ onto the second axis is a selector for $\mathscr{D}$, we deduce that

(1) the set $A_s$ is $\sigma$-discrete in $Y^2$.

(¹) The space $F$ constructed by Fleissner in [6] and the space from our Example 1 have also the properties required in Example 3, so that the only advantage of Example 3 lies in a relatively weak set-theoretic assumption.

Let $T$ be a selector for the family $\mathscr{A} = \{A_s: s \in S\}$. Since the projection of $T$ onto the first axis is a selector for $\mathscr{D}$, we deduce that

(2) every selector for $\mathscr{A}$ is $\sigma$-discrete in $Y^2$.

By (1) the set $A_s$ can be written in the form

$$A_s = \bigcup_{m,n<\omega} A_s(m, n),$$

where for every distinct $x, y \in A_s(m, n)$ we have $\varrho(x, y) \geqslant 1/n$ ($\varrho$ denotes a metric in $Y^2$).

Put $A_{m,n} = \bigcup\{A_s(m, n): s \in S\}$. If $U$ is an open subset of $Y^2$ of diameter $< 1/n$, then clearly $|U \cap A_s(m, n)| \leqslant 1$. Therefore the set $U \cap A_{m,n}$ is a part of a selector for $\mathscr{A}$ and hence, by (2), $U \cap A_{m,n}$ is $\sigma$-discrete in $Y^2$. From the local $\sigma$-discreteness of $A_{m,n}$ it follows that the sets $A_{m,n}$ and, consequently, also the set $A = \bigcup_{m,n<\omega} A_{m,n}$ are $\sigma$-discrete in $Y^2$.

To complete the prбof of Lemma 1 it suffices to note the following fact:

(3) If $\mathscr{D} = \{D_s: s \in S\}$ is a regular decomposition of a space $Y$, then no family $\{H_s\}_{s \in S}$ of open subsets of $Y^2$ such that $D_s \times D_s \subset H_s$ for $s \in S$ is point-countable.

To prove (3) assume, on the contrary, that the family $\{H_s\}_{s \in S}$ is point-countable. Let $\Delta = \{(y, y): y \in Y\}$ be the diagonal of $Y^2$ and put $E_s = (D_s \times D_s) \cap \Delta$ and $W_s = H_s \cap \Delta$. The mapping $h: \Delta \to Y$, defined by $h(y, y) = y$, is clearly a homeomorphism and we have $D_s = h(E_s) \subset h(W_s) = U_s$, which contradicts the regularity of $\mathscr{D}$, since the family $\{U_s\}_{s \in S}$ of open subsets of $Y$ is point-countable.

In the sequel we shall also need the following easy lemma:

LEMMA 2. *If $\{z_t\}_{t \in T}$ is a family of points of $Z$ and if $\mathscr{U} = \{U_t\}_{t \in T}$ is a $\sigma$-locally finite family of open subsets of $Z$ such that $z_t \in U_t$ for $t \in T$ and $z_t \notin \overline{U}_r$ for $t \neq r$, then there exists a disjoint family $\{V_t\}_{t \in T}$ of open subsets of $Z$ such that $z_t \in V_t$ for $t \in T$.*

Proof. Let $\mathscr{U} = \bigcup_{n<\omega} \mathscr{U}_n$, where the families $\mathscr{U}_n$ are locally finite. For every $t \in T$ find an $n(t) < \omega$ such that $U_t \in \mathscr{U}_{n(t)}$. It suffices to put

$$V_t = U_t \setminus \bigcup_{i=1}^{n(t)} \bigcup \{\overline{U}_r:\ U_r \in \mathscr{U}_i,\ r \neq t\}.$$

**3. Basic construction.** Assume $E(\omega_2)$ and let $S$ be an arbitrary subset of the unit interval $I = [0, 1]$, which has cardinality $\omega_1$, contains the set $Q$ of rational numbers of $I$, and satisfies the condition: $1 - s \in S$ if $s \in S$. In this section we shall use $S$ to construct a first countable, paracompact space $X$ such that $X^2$ is collectionwise Hausdorff, but not collectionwise normal.

Our construction depends heavily on the methods developed by Fleissner in [6] and on the approach of R. Pol, who first used regular decompositions to obtain examples of collectionwise Hausdorff non-collectionwise normal spaces (unpublished).

By Pol's theorem, there exists a regular decomposition $\mathscr{D} = \{D_s: s \in S\}$ of $B$ indexed by $S$. Define a family $\mathscr{E} = \{E_s: s \in S\}$ of subsets of $B$ by putting

$$E_s = \begin{cases} B & \text{if } s \in Q, \\ D_s \cup D_{1-s} & \text{otherwise}. \end{cases}$$

Clearly, we have

(4) $$E_s = E_{1-s} \text{ and } D_s \subset E_s \quad \text{for } s \in S,$$

and it follows easily from Lemma 1 and the countability of $Q$ that

(5) If $f: \mathscr{E}^{(2)} \to B^2$ is a function of choice, i.e. if $f(E_s \times E_r) \in E_s \times E_r$ for $(s, r) \in S^2$, then the set $f(\mathscr{E}^{(2)}) = \{f(E_s \times E_r) \mid (s, r) \in S^2\}$ is $\sigma$-discrete in $B^2$.

Let us consider the set $S$ with the topology of the subspace of the Sorgenfrey line (see [4], Example 1.2.1). Since $S$ is first countable, perfectly normal and paracompact, and $B$ is metrizable, the product space $S \times B$ is also first countable, perfectly normal and paracompact (Michael [10]). Let

$$X = X_S = \bigcup_{s \in S} (\{s\} \times E_s)$$

be the subspace of $S \times B$. Naturally, the space $X$ is dense in $S \times B$ and $Z = X^2$ is contained in $(S \times B)^2$ which is homeomorphic to $S^2 \times B^2$. We shall denote by $\pi$ the projection of $(S \times B)^2$ onto $B^2$.

I. *The space $X^2$ is collectionwise Hausdorff.*

Let $Z_0 = \{z_t\}_{t \in T}$ be a discrete subset of $Z = X^2$ and let $\mathscr{G}$ be a $\sigma$-locally finite base in $B$. For every $t \in T$ the point $z_t$ has the form $z_t = (s_t, b_t, s_t^*, b_t^*)$, where $(s_t, s_t^*) \in S^2$ and $(b_t, b_t^*) \in E_{s_t} \times E_{s_t^*} \subset B^2$. As $Z_0$ is discrete, for every $t \in T$ there exists an open subset $H_t$ of $(S \times B)^2$ such that

(6) $$z_t \in H_t \text{ and } z_t \notin \bar{H}_r \quad \text{for } t \neq r.$$

Find elements $G_t, G_t^* \in \mathscr{G}$ such that

(7) $$b_t \in G_t, \quad b_t^* \in G_t^* \quad \text{and} \quad \{s_t\} \times G_t \times \{s_t^*\} \times G_t^* \subset H_t.$$

Let us fix a pair $(G, G^*) \in \mathscr{G}^2$ and put $T(G, G^*) = \{t \in T: G_t = G \text{ and } G_t^* = G^*\}$. Clearly,

(8) $$T = \bigcup_{(G,G^*) \in \mathscr{G}^2} T(G, G^*).$$

It follows from (6) and (7) that if $t, r \in T(G, G^*)$ and $t \neq r$, then $(s_t, s_t^*) \neq (s_r, s_r^*)$.

Hence the function $f = f_{(G,G^*)}$ defined for $t \in T(G, G^*)$ by

$$f(E_{s_t} \times E_{s_t^*}) = (b_t, b_t^*) \in E_{s_t} \times E_{s_t^*}$$

is a part of a function of choice on $\mathscr{E}^{(2)}$ and hence — by (5) and (7) — the set $\{(b_t, b_{t*})\}_{t \in T(G,G^*)}$ is $\sigma$-discrete in $B^2$ and is contained in $G \times G^*$.

Since $B^2$ is metrizable, there exists a $\sigma$-discrete in $B^2$ family $\{W_t(G, G^*)\}_{t \in T(G,G^*)}$ of open subsets of $B^2$ such that

(9) $\quad (b_t, b_t^*) \in W_t(G, G^*) = W_t(G_t, G_t^*\} \subset G \times G^* \quad$ for $t \in T(G, G^*)$.

It follows from (8), (9) and the $\sigma$-local finiteness of the family $\mathscr{G}^{(2)} = \{G \times G^*)_{(G,G^*) \in \mathscr{G}^2}$ that

(10) $\quad$ the family $\{\pi^{-1}(W_t(G, G^*))\}_{(G,G^*) \in \mathscr{G}^2, t \in T(G,G^*)} = \{\pi^{-1}(W_t(G_t, G_t^*))\}_{t \in T}$ is $\sigma$-locally finite in $(S \times B)^2$.

For $t \in T$ define open subsets $U_t$ of $X^2$ by

$$U_t = H_t \cap \pi^{-1}(W_t(G_t, G_t^*)) \cap X^2.$$

By (10) the family $\{U_t\}_{t \in T}$ is $\sigma$-locally finite in $Z = X^2$ and from (6) and (9) we infer that $z_t \in U_t$ and that if $t \neq r$, then $z_t \notin \overline{U}_r$. By Lemma 2 there exists a disjoint family $\{V_t\}_{t \in T}$ of open subsets of $X^2$ such that $z_t \in V_t$, which proves that $X^2$ is collectionwise Hausdorff.

Remark 1. Making use of an adequate generalization of Lemma 1, one can easily prove that in fact $X^\omega$ is collectionwise Hausdorff.

II. *$X^2$ is not collectionwise normal.*

Since the set $\{(s, 1-s)\}_{s \in S}$ is discrete and closed in $S^2$, the family $\{F_s\}_{s \in S}$ of closed subsets of $X^2$, where $F_s = \{s\} \times E_s \times \{1-s\} \times E_{1-s}$, is discrete in $X^2$. If $X^2$ were collectionwise normal, there would exist a disjoint family $\{U_s\}_{s \in S}$ of open subsets of $X^2$ such that $F_s \subset U_s$. Let $V_s$, $s \in S$, be open subsets of $(S \times B)^2$ such that $V_s \cap X^2 = U_s$. From the density of $X^2$ in $(S \times B)^2$ we deduce that the family $\{V_s\}_{s \in S}$ is also disjoint.

The family $\{H_s\}_{s \in S}$ of open subsets of $B^2$, where $H_s = \pi(V_s)$, satisfies the following conditions:

(11) $$D_s \times D_s \subset E_s \times E_s = E_s \times E_{1-s} \subset H_s,$$

(12) $$\{H_s\}_{s \in S} \text{ is point-countable.}$$

Condition (11) follows immediately from (4) and the inclusion $F_s \subset V_s$. To prove (12) it suffices to observe that the space $S^2$ is separable and that for every $y \in B^2$ the space $\pi^{-1}(y)$ is homeomorphic to $S^2$ and, therefore, it can intersect only countably many elements of a disjoint family $\{V_s\}_{s \in S}$.

The existence of the family $\{H_s\}_{s \in S}$ contradicts (3).

**4. Construction of Examples 1 and 2.**

*Construction of Example* 1. Assume $E(\omega_2)$, Martin's Axiom and the negation of the Continuum Hypothesis. It is known that Martin's Axiom and the negation of the Continuum Hypothesis imply the existence of a subspace $A$ of $I$ of cardinality $\omega_1$ such that its every subset is a relative $F_\sigma$ (see Tall [15]). By putting $S = A \cup \{1-a: a \in A\} \cup Q$ we can find a subspace $S$ of $I$ with the above-mentioned properties, containing $Q$ and satisfying the condition: $1-s \in S$ if $s \in S$.

Let us use $S$ to construct the space $X = X_S$, as shown in Section 3. We already know that $X$ is first countable and paracompact and that $X^2$ is collectionwise Hausdorff but not collectionwise normal. It suffices to show that $X^2$ is perfectly normal. From Przymusiński [14] we infer that the space $S^2$ is perfectly normal, assuming that $S$ is considered with the topology of the subspace of the Sorgenfrey line. Therefore the space $S^2 \times B^2$, containing $X^2$, is also perfectly normal (Morita [12]), which completes the proof.

Remark 2. In fact, as it follows from Alster and Przymusiński [1], the space $X^\omega$ is perfectly normal. Hence, by Remark 1, the space $X^\omega$ is collectionwise Hausdorff and perfectly normal, but not collectionwise normal.

*Construction of Example* 2. Assume $E(\omega_2)$ and the Continuum Hypothesis. By Michael [11], Continuum Hypothesis implies the existence of an uncountable subspace $A$ of $I$ such that $A \supset Q$ and if $U$ is an open subset of $A$ containing $Q$, then $|A \setminus U| \leqslant \omega_0$. Putting $S = A \cup \{1-a: a \in A\}$ we can find a subspace $S$ of $I$ with the above-mentioned properties satisfying additionally the condition: $1-s \in S$ if $s \in S$.

Let us use $S$ to construct the space $X = X_S$ as shown in Section 3. It suffices to show that $X^2$ is perfect and non-normal. The perfectness of $X^2$ follows from the fact that the space $S^2$, where $S$ is considered with the topology of the subspace of the Sorgenfrey line, is perfect (Heath and Michael [7]), which implies that the space $S^2 \times B^2$, containing $X^2$, is also perfect (Michael [10]).

We shall prove the non-normality of $X^2$. Assume on the contrary that $X^2$ is a normal space. The sets $C = \{(s, 1-s): s \in Q\}$ and $D = \{(s, 1-s): s \in S \setminus Q\}$ are closed and disjoint in $S^2$, hence the subsets

$$K = \bigcup_{s \in Q} (\{s\} \times E_s \times \{1-s\} \times E_{1-s})$$

and

$$L = \bigcup_{s \in S \setminus Q} (\{s\} \times E_s \times \{1-s\} \times E_{1-s})$$

of $X^2$ are also closed and disjoint. Therefore there exist open subsets $U_0, V_0$ of $X^2$ such that $K \subset U_0$, $L \subset V_0$ and $U_0 \cap V_0 = \emptyset$. Let $U$ and $V$ be open in $(S \times B)^2$ and such that $U \cap X^2 = U_0$ and $V \cap X^2 = V_0$. From the density of $X^2$ in $(S \times B)^2$ we deduce that $U \cap V = \emptyset$. For $s \in S$ and $n = 1, 2, \ldots$ let $U_n(s) = [s, s+1/n) \cap S$ be basic neighbourhoods of the point $s$ in $S$.

Put

$$H_s(n) = \bigcup \{G \times G^*\colon G, G^* \text{ are open in } B \text{ and } U_n(s) \times G \times U_n(1-s) \times G^* \subset V\}.$$

Clearly, $H_s(n)$ are open subsets of $B^2$ and

$$(13) \qquad H_s = \bigcup_{n=1}^{\infty} H_s(n) \supset E_s \times E_{1-s} = E_s \times E_s \supset D_s \times D_s \quad \text{for } s \in S \setminus Q.$$

For $s \in Q$ put $H_s = B^2$. By the countability of $Q$, to obtain a contradiction with (3) and hence to complete the proof, it is enough to show that for every $n = 1, 2, \ldots$ the family $\{H_s(n)\}_{s \in S\setminus Q}$ is point-countable, since this implies that the family $\{H_s\}_{s\in S}$ is also point-countable.

Fix an $n = 1, 2, \ldots$, choose a point $(b, b^*) \in B^2$ and let $S_0 = \{s \in S\setminus Q\colon (b, b^*) \in H_s(n)\}$. If $S_0$ were uncountable, then by the definition of $S$ there would exist a $q \in \bar{S}_0 \cap Q$, where $\bar{S}_0$ denotes the closure of $S_0$ in the euclidean topology of $S$. Therefore we could choose a sequence $\{s_m\}_{m=1}^{\infty}$ of points of $S_0$ converging in a usual sense to $q$. In such a case we would have

$$V \supset \bigcup_{m=1}^{\infty} \big(U_n(s_m) \times \{b\} \times U_n(1-s_m) \times \{b^*\}\big)$$

and, consequently, the point $(q, b, 1-q, b^*) \in K$ would belong to the closure of $V$, which is impossible.

**5. Construction of Example 3.** Assume $E(\omega_2)$ and let $\mathscr{D} = \{D_\alpha\colon \alpha < \omega_1\}$ be a regular decomposition of $B$. Take an arbitrary space $Y$ with the following properties:

(i) $Y$ is perfectly normal;

(ii) $Y$ contains a closed, discrete subspace $Z = \{z_\alpha\colon \alpha < \omega_1\}$;

(iii) the space $Y \setminus Z$ is discrete;

(iv) every disjoint family of open subsets of $Y$ intersecting $Z$ is countable.

For instance, the well-known Bing's example [2] of a perfectly normal, non-collectionwise normal space has the above-mentioned properties (continuum should be replaced by $\omega_1$).

Let $X$ be a subspace of a perfectly normal space $Y \times B$ (Morita [12]) defined by

$$X = \big((Y\setminus Z) \times B\big) \cup \bigcup_{\alpha<\omega_1} (\{z_\alpha\} \times D_\alpha).$$

The proof that the space $X$ is collectionwise Hausdorff but not collectionwise normal is similar (though simpler) to the proof of the analogous properties of the space $X = X_S$ considered in Section 3 and, therefore, is omitted (cf. also Fleissner [6], Example F).

Remark 3. Note that in all our examples it sufficed to assume the existence of a metric space $M$ having a regular decomposition of cardinality $\omega_1$ instead of the more restrictive condition $E(\omega_2)$.

PROBLEM 1. Does there exist a (separable?) paracompact space $X$ such that $X^2$ is collectionwise normal but not paracompact? (**P 984**)

PROBLEM 2. Does there exist a "real" example of a perfectly normal and collectionwise Hausdorff space, which is not collectionwise normal? (**P 985**)

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