COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

LATTICE ORDERED ALGEBRAS GENERATED BY A SYSTEM OF IDEALS

BY

J. JAKUBÍK (KOŠICE)

Fuchs [3] has shown that to each universal algebra $\mathfrak{A} = (A; F)$ with a given system B of subsets satisfying certain conditions ("system of r-ideals") there corresponds a lattice ordered algebra $\mathfrak{B} = (B; F, \wedge, \vee)$ such that all operations $f \in F$ are isotone. He proved that the construction of the lattice ordered algebra $\mathfrak{B}$ enables one to generalize the classical theorems on ideals in commutative rings to the case of universal algebras ([3], part 2).

In this note we consider the following problems (raised in [3]; for exact formulation cf. section 1.2): how can algebras $\mathfrak{B}$, constructed in this way, be intrinsically characterized, and under which condition $\mathfrak{A}$ is uniquely determined by $\mathfrak{B}$? Analogical problems for ideals in semilattices and for ideals in Boolean algebras were studied by Nachbin [5].

1. BASIC NOTIONS

1.1. For the partially ordered algebras we shall use the terminology of [1] and [2]. The symbols $\cap$, $\cup$ and $\wedge$, $\vee$ denote the set-theoretical and lattice operations, respectively; $A \subset B$ means that A is a subset of B (the equality being not excluded). The cardinality of a set M is denoted by card M.

Let $\mathfrak{A} = (A, F)$ be any algebra. Suppose that to each subset $X \subset A$ there corresponds a subset $X_r \subset A$ such that the following conditions are satisfied:

1° $X \subset X_r$.

2° $X \subset Y_r \Rightarrow X_r \subset Y_r$.

3° $X_r = \bigcup Y_r$, where Y runs over all finite subsets of X (i.e operation $X \to X_r$ is a generalized closure operator of finite character).

Then the system $B = \{X_r\}_{X \subset A}$ is a *system of r-ideals of* $\mathfrak{A}$.

B is partially ordered under set inclusion. It is known (cf. [3]) that B is a complete lattice satisfying

$$\bigwedge_{i\in I} X_r^i = \bigcap_{i\in I} X_r^i, \tag{1}$$

$$\bigvee_{i\in I} X_r^i = (\bigcup_{i\in I} X^i)_r = (\bigcup_{i\in I} X_r^i)_r. \tag{2}$$

If $f \in F$ is an n-ary operation and if $X_r^1, \ldots, X_r^n \in B$, then we put

$$f(X_r^1, \ldots, X_r^n) = \{f(x^1, \ldots, x^n) : x^i \in X_r^i\}_r. \tag{3}$$

The operation f defined on B is isotone; by this we mean that from $X_r^i, Y_r^i \in B, X_r^i \subset Y_r^i (i = 1, \ldots, n)$ it follows that $f(X_r^1, \ldots, X_r^n) \subset f(Y_r^1, \ldots$ $\ldots, Y_r^n)$. We get a lattice ordered algebra $\mathfrak{B} = (B; F, \wedge, \vee) = \mathfrak{J}(\mathfrak{A}, B)$ If B is fixed, then we write also $\mathfrak{J}(\mathfrak{A})$ instead of $\mathfrak{J}(\mathfrak{A}, B)$.

Remark. In the definition of an r-system given above X is any subset of A; if we consider only non-empty subsets $X \subset A$ and if for these subsets conditions 1°, 2° and 3° are satisfied, then the system $S = \{X_r : X \subset A, X \neq \emptyset\}$ need not be a lattice. For example, if A is any set with card $A > 1$ and if for any $X \subset A, X \neq \emptyset$, we put $X_r = X$, then the system S is not directed by relation $\supset$.

According to 2°, $\emptyset_r \subset X_r$ for any $X \subset A$. A system $\{X_r\}_{X\subset A}$ of r-ideals is said to be *regular* if $\emptyset_r = \emptyset$. (If we put $X_r = A$ for any $X \subset A$, then the system of r-ideals $\{X_r\}_{X\subset A} = \{A\}$ is not regular.)

1.2. The problems formulated by Fuchs in [3] are as follows:

(a) Under what conditions an algebra $(B; G, \wedge, \vee)$ can be obtained from an algebra $(A; F)$ with $G \subset F$ as an r-ideal system?

(b) Under what conditions does the isomorphy of r-ideal systems imply the isomorphy of the algebras from which they were constructed?

It is remarked in [3] that if $(B; G, \wedge, \vee)$ satisfies the condition given in (a), then each element of the lattice $(B; \wedge, \vee)$ is a join of compact elements (cf. also Nachbin [6]). (An element c of a lattice L is *compact* if $c \leqslant \bigvee_{i\in I} x_i$ implies the existence of a finite subset $I_1 \subset I$ satisfying $c \leqslant \bigvee_{i\in I_1} x_i$; L is compact, if each element of L is compact.)

Čornayová [4] has solved problem (a) in the case when $(B; \wedge, \vee)$ is a compact lattice (this solves, in particular, problem (a) for finite algebras $\mathfrak{B}$). In theorem 3.3 necessary and sufficient conditions are given under which an algebra $\mathfrak{B}$ fulfils the requirements of (a). In 3.4 we prove that to each algebra $\mathfrak{A} = (A; F)$ with a given system of r-ideals B it is possible to construct an algebra $\mathfrak{A}' = (A'; F)$ with a system of r-ideals B' such that the algebras $\mathfrak{A}$ and $\mathfrak{A}'$ are not isomorphic and the systems of r-ideals $\mathscr{I}(\mathfrak{A}, B)$ and $\mathscr{I}(\mathfrak{A}', B')$ are isomorphic.

In section 2 we shall deal with questions depending on the lattice operations only; the problems concerning the operations $f \in F$ are treated in sections 3 and 4.

2. THE LATTICE OF r-IDEALS

2.1. Let A be a non-empty set. If to each subset $X \subset A$ there corresponds a set $X_r \subset A$ such that the conditions 1°, 2° and 3° are satisfied, then the system $B = \{X_r\}_{X \subset A}$ uniquely determines the set X_r for each $X \subset A$, since $X_r = \bigcap Y$, where Y runs over all subsets of A fulfilling $X \subset Y \in B$.

Suppose now that B is a system of subsets of A such that $A \in B$ and $\bigcap\limits_{i \in I} X^i \in B$ for each subsystem $\{X^i\}_{i \in I} \subset B$. For any $X \subset A$ we denote by X_r the meet of all $X^i \in B$ with $X \subset X^i$. The system B is said to satisfy the conditions 1°, 2° and 3°, if the correspondence $X_r \to X$ fulfils these conditions. In such a case the complete lattice $(B; \wedge, \vee)$ (cf. (1) and (2)) is a lattice of r-ideals on the set A; we denote it by $\mathfrak{I}_1(A, B)$. Let $K(A, B)$ be the system of all X_r, where X is a finite set.

We need the following simple lemma (cf. also the remark from [3] cited in 1.2):

2.2. *Let* $(B, \wedge, \vee) = \mathfrak{I}_1(\mathfrak{A}, B)$. *Then the following statements hold*:

(α_1) *Each set* $X_r \in K(A, B)$ *is a compact element of the lattice* B.

(α_1) *Each element of the lattice* B *is a join of some elements of* $K(A, B)$.

(α_3) *If* $X^1, X^2 \in K(A, B)$, *then* $X^1 \vee X^2$ *also belongs to* $K(A, B)$.

Proof. If $X = \emptyset$, then, according to 2°, $X_r \subset Y_r$ for each $Y \subset A$, hence X_r is compact. Let $X \subset A$ be a finite non-empty set. Suppose that $X_r \subset \bigvee\limits_{i \in I} X^i$ and $\{X^i\}_{i \in I} \subset B$. Put $\bigcup\limits_{i \in I} X^i = Y$. Then

$$\bigvee_{i \in I} X^i = Y_r = \bigcup_{j \in J} (Y^j)_r,$$

where $\{Y^j\}_{j \in J}$ is the system of all finite subsets of Y. Therefore for each $x \in X$ there exists a $j(x) \in J$ such that $x \in (Y^{j(x)})_r$. We write $Y^0 = \bigcup\limits_{x \in X} Y^{j(x)}$. Since the sets X and Y^j are finite, the set Y^0 is also finite. For each $y \in Y^0$ there exists a $i(y) \in I$ with $y \in X_{i(y)}$; put $I^0 = \{i(y)\}_{y \in Y^0}$. The set $I^0 \subset I$ is finite and for each $x \in X$ we have

$$x \in (Y^{j(x)})_r \subset (Y^0)_r \subset \Big(\bigcup_{y \in Y^0} X^{i(y)}\Big)_r = \bigvee_{i \in I^0} X^i.$$

Hence $X \subset \bigvee\limits_{i \in I^0} X^i$ and therefore $X_r \subset \bigvee\limits_{i \in I^0} X^i$. This proves ($\alpha_1$). Let now X_r be any element of the set B; for $X_r \in K(A, B)$ condition

(α_2) is trivially satisfied and for $X \notin K(A, B)$ this condition is implied by

$$X_r = \bigvee_{j \in J} (Z_j)_r,$$

where $\{Z_j\}_{j \in J}$ is the system of all finite subsets of the set X.

Let $Y^i \subset A$ $(i = 1, 2)$ be finite sets, $X^i = Y^i_r$. Since $X^1 \vee X^2 = (Y^1 \cup Y^2)_r$, we get $X^1 \vee X^2 \in K(A, B)$.

Remark. The statement dual to (α_3) does not hold in general. To show this let A be an infinite set, $a_1, a_2 \in A$, $a_1 \neq a_2$. Put $Z = A \setminus \{a_1, a_2\}$. If $X \subset Z$, let $X_r = X$. If $X \cap \{a_1, a_2\} \neq \emptyset$, we put $X_r = Z \cup X$. Then conditions 1°, 2° and 3° are satisfied, $\{a_i\}_r \in K(A, B)$, $i = 1, 2$, but $\{a_1\}_r \wedge \{a_2\}_r = Z \notin K(A, B)$.

Throughout the paper let b^0 be the least element of the lattice B.

2.3. THEOREM. *Let B be a complete lattice, $B_0 \subset B$. The following conditions (α) and (β) are equivalent:*

(α) *There exist a set A, a system $\mathfrak{I}_1(A, B_1)$ of r-ideals satisfying 1°, 2°, 3°, and an isomorphism φ of the lattice $\mathfrak{I}_1(A, B_1)$ onto B such that $\varphi(K(A, B_1)) = B_0$.*

(β) *For B_0 the following assertions hold true:*

(i) *each $b \in B_0$ is a compact element of the lattice B;*

(ii) *for any $z \in B$ there exists a subset $Z \subset B_0$ with $z = \sup Z$;*

(iii) *the join of any two elements of B_0 also belongs to B_0.*

The implication $(\alpha) \Rightarrow (\beta)$ has been proved in 2.2. The converse implication is a consequence of the following lemmas 2.4, 2.5 and 2.6.

2.3.1. Remark. If $Z = \emptyset$, then $\sup Z = b^0$. This implies that if a set $B_0 \subset B$ satisfies conditions (i) and (ii) (or (i), (ii) and (iii), respectively) and if $B_0 \neq \{b^0\}$, then the set $B_0 \setminus \{b^0\}$ satisfies these conditions, too. If B fulfils (ii) and $B_0 = \{b^0\}$, then $B = \{b^0\}$.

2.4. *Let B be a lattice, $\emptyset \neq A \subset B$. For each $X \subset A$ put*

$$X_r = \{y \in A\colon \textit{there exists a finite set } X_1 \subset X \textit{ such that } y \leqslant \sup X_1\}.$$

Then the system $B_1 = \{X_r\}_{X \subset A}$ satisfies conditions 1°, 2° and 3°.

Proof. 1° and 2° obviously hold. Let us consider condition 3°. Let $X \subset A$, $X \neq \emptyset$ (for $X = \emptyset$ condition 3° clearly holds) and let $\{Y^i\}_{i \in I}$ be the system of all finite subsets of X. According to 1° and 2°, $Y^i_r \subset X_r$, hence $\bigcup_{i \in I} Y^i_r \subset X_r$. For each $y \in X_r$ there exist elements $x_1, \ldots, x_n \in X$ such that $y \leqslant x_1 \vee \ldots \vee x_n$; put $Y^{i_0} = \{x_1, \ldots, x_n\}$. Then $y \in Y^{i_0}_r$, hence $X_r \subset \bigcup_{i \in I} Y^i_r$.

2.5. *Let B_0 be a subset of a complete lattice B satisfying conditions* (i) *and* (ii). *Let B_1 have the same meaning as in* 2.4, *where $A = B_0$. For each $X_r \in B_1$ and each $b \in B$ let*

$$\varphi(X_r) = \sup X_r, \qquad \tau(b) = \{a \in A : a \leqslant b\}.$$

Then φ is an isomorphism of the lattice B_1 onto B and $\tau = \varphi^{-1}$.

Proof. Let $b \in B$, $\tau(b) = X$. According to the construction from 2.4, $X_r = X$, hence $\tau(b) \in B_1$. By (ii), $\sup \tau(b) = b$, hence $\varphi(\tau(b)) = b$ and therefore φ is a mapping of B_1 onto B. Let $Y \in B_1$, $\varphi(Y) = b$. Then $\sup Y = b$, hence $Y \subset \tau(b)$. Let $a \in \tau(b)$. Since $a \leqslant \sup Y$ and since the element a is compact (by (i)), there exists a finite subset $Y_1 \subset Y$ such that $a \leqslant \sup Y_1$. From this it follows that $a \in Y_r = Y$, hence $\tau(b) \subset Y$. Therefore $\tau(b) = Y$ and the mapping φ is one-to-one. From $\varphi(\tau(b)) = b$ we get now that τ is onto and $\tau = \varphi^{-1}$. According to the definition of φ and τ both φ and τ are isotone. The proof is complete.

2.6. *Let B_0 be a subset of a complete lattice B satisfying* (i), (ii) *and* (iii). *Let A, B_1, φ have the same meaning as in* 2.5. *Then $\varphi(K(A, B_1)) = B_0 \cup \{b^0\}$.*

Proof. Let $X \in K(A, B_1)$. Then either $X = \emptyset_r$ or X has the form $X = \{a_1, \ldots, a_n\}_r$, $a_i \in A$. In the first case X is the least element of B_1, thus according to 2.5 $\varphi(X) = b^0$, $b^0 \in \varphi(K(A, B_1))$. In the other case write $b = a_1 \vee \ldots \vee a_n$. According to 2.4, $X_r = X$ is the set of all $y \in A$ with $y \leqslant b$. By (iii), $b \in A$, hence b is the greatest element of the set X. We get $b = \varphi(X) \in B_0$ and $\varphi(K(A, B_1)) \subset \{b^0\} \cup B_0$. If $b_0 \in B_0$, then $\{b_0\}_r \in K(A, B_1)$ and $\varphi(\{b_0\}_r) = b_0$, hence $\varphi(K(A, B_1)) = B_0 \cup \{b^0\}$.

2.6.1. *The system of r-ideals $\mathfrak{J}_1(A, B_1)$ constructed in* 2.5 *is regular if and only if $b^0 \notin B_0$.*

Proof. As $\sup \emptyset = b^0$, we have, according to 2.4, either $\emptyset_r = \{b^0\}$ (if $b^0 \in B_0$) or $\emptyset_r = \emptyset$ (if $b^0 \notin B_0$).

From 2.2 and 2.5 follows

2.7. THEOREM. *Let B be a complete lattice. The following conditions are equivalent:*

(β_1) *There exist a set A and a lattice of r-ideals $\mathfrak{J}_1(A, B_1)$ such that the lattices B and $\mathfrak{J}_1(A, B_1)$ are isomorphic.*

(β_2) *There exists a subset $B_0 \subset B$ satisfying* (i) *and* (ii).

2.7.1. *Let B be a complete lattice,* card $B > 1$. *Then* (β_2) *is equivalent to the condition*

(β_3) *There exists a set A and a regular system $\mathfrak{J}_1(A, B_1)$ of r-ideals such that B and $\mathfrak{J}_1(A, B_1)$ are isomorphic.*

Proof. According to 2.7, (β_3) $\Rightarrow$ (β_2). Let us suppose that (β_2) holds. Since card $B > 1$, we get $B_0 \neq \{b^0\}$ by 2.3.1. If $b^0 \notin B_0$, then, by 2.6.1, (β_3) is fulfilled. If $b^0 \in B_0$, we take $B_0 \setminus \{b^0\}$ instead of B_0 (cf. 2.3.1).

Let us now suppose that B is a complete lattice satisfying (β_2). The following question seems to be natural: how can all latitces of r-ideals $\mathfrak{J}_1(A^*, B^*)$ be constructed which are isomorphic with the lattice B? An answer is given in 2.8 and 2.9.

2.8. *Let $\mathfrak{J}_1(A, B_1)$ be a lattice of r-ideals. For each $a \in A$ let $M_a \neq \emptyset$ be a set such that $M_{a_1} \cap M_{a_2} = \emptyset$ for distinct $a_1, a_2 \in A$. Let $A' = \bigcup_{a \in A} M_a$ and, for any subset $X \subset A'$, let*

$$X(A) = \{a : a \in A, X \cap M_a \neq \emptyset\}$$

and

$$X_r = \bigcup_{a \in (X(A))_r} M_a.$$

Then

(a) *the system $B_1' = \{X_r\}_{X \subset A'}$ fulfils conditions* 1°, 2° *and* 3°;

(b) *the mapping $\psi(X_r) = X_r(A)$ is an isomorphism of the lattice B_1' onto B_1.*

Proof. The first assertion is an immediate consequence of the fact that for B_1 the conditions 1°, 2° and 3° hold. To prove the other, it suffices to take into account that a set $X \subset A'$ belongs to B_1' if and only if $X(A) \in B_1$ and $X = \bigcup_{a \in X(A)} M_a$.

2.8.1. *Let B be a complete lattice satisfying (β_2). Let A, B_1, φ, τ and A', B_1', ψ have the same meaning as in 2.5 and 2.8. Then $\varphi\psi$ is an isomorphism of the lattice B_1' onto B. For each $X \in B_1'$ and each $b \in B$*

$$(\varphi\psi)(X) = \sup\big(X(A)\big), \qquad (\varphi\psi)^{-1}(b) = \bigcup_{a \in \tau(b)} M_a.$$

This follows from 2.5 and 2.8.

Further we shall prove that if a complete lattice B satisfies (β_2), then every isomorphism $f : \mathfrak{J}_1(A^*, B^*) \to B$ can be composed of suitable isomorphisms φ and ψ that are constructed as in 2.5 and 2.8.

2.9. Theorem. *Let B be a complete lattice and let $\mathfrak{J}_1(A^*, B^*)$ be a lattice of r-ideals. Let f be an isomorphism of B^* onto B. Write*

$$B_0 = \big\{f(\{a^*\}_r)\big\}_{a^* \in A^*}, \qquad M_a = \big\{a_1^* : f(\{a_1^*\}_r) = a\big\}.$$

Let the symbols A, B_1, φ and A', B_1', ψ have the same meaning as in 2.4 and 2.5, respectively. Then

(a) $$A^* = A', \quad B^* = B_1',$$

(b) $$f = \varphi\psi.$$

Proof. Let B_0^* be the system of all sets $\{a^*\}_r$, where $a^* \in A^*$. According to 2.2, $\{a^*\}_r$ is a compact element of the lattice B^*. It is easy to see that

$$X^* = \bigvee_{x^* \in X^*} \{x^*\}_r$$

for each $X^* \in B^*$, $X^* \neq \emptyset$.

It follows now from the isomorphism f that the system $B_0 = f(B_0^*)$ fulfils (i) and (ii), hence, according to 2.5, φ is an isomorphism. Moreover, if $a_1, a_2 \in A = B_0$ and $a_1 \neq a_2$, then $M_{a_1} \cap M_{a_2} = \emptyset$, thus the assumptions of lemma 2.8 hold. At the same time we have $A^* = A'$.

Let us now consider the isomorphisms

$$B_1' \xrightarrow{\psi} B_1 \xrightarrow{\varphi} B \xleftarrow{f} B^*.$$

The mapping $g = \psi^{-1}\varphi^{-1}f$ is an isomorphism of B^* onto B_1'. Let $X^* \in B^*$ and $g(X^*) = Y$. Let us recall that both X^* and Y are subsets of A^*.

Let $a^* \in X^*$. Write $f(X^*) = b$ and $f(\{a^*\}_r) = a$. Since $\{a^*\}_r \subset X^*$, it follows from the isomorphism f that $f(\{a\}_r) \leqslant b$. If $a_1 \in A$ and $a_1 \leqslant b$, then there exists an $a_1^* \in A^*$ such that $a_1 = f(\{a_1^*\}_r)$ and $\{a_1^*\}_r \subset X^*$, hence $a_1^* \in X^*$. This proves the equivalence

$$(4) \qquad a \in \tau(b) \Leftrightarrow a = f(\{a^*\}_n) \quad \text{for some } a^* \in X^*,$$

where τ has the same meaning as in 2.5. Further we have (cf. 2.8.1)

$$g(X^*) = \psi^{-1}(\tau(b)) = \bigcup_{a \in \tau(b)} M_a,$$

hence, according to (4) and by the definition of M_a, we get $g(X^*) = X^*$. This implies $B_1' = B^*$. Moreover, since g is the identity mapping, $f = \varphi\psi$ holds.

2.10. Recall that the subset B_0 of a lattice B is not uniquely determined by conditions (i) and (ii). If, for instance, $B = \{a, b, u, v)$, where $a \wedge b = u$, $a \vee b = v$, then both sets $B_{01} = B$ and $B_{02} = \{a, b, v\}$ fulfil (i) and (ii). In spite of this we have:

2.10.1. *If a subset B_0 of a lattice B satisfies conditions* (i), (ii) *and* (iii), *then $B_0 \cup \{b^0\}$ is the set of all compact elements of B.*

Proof. According to (i) it suffices to verify that each compact element $b \in B$, $b \neq b^0$, belongs to B_0. By (ii) for each compact element $b \in B$, $b \neq b^0$, there exists a finite subset $\{b_1, \ldots, b_n\} \subset B_0$ such that $b = b_1 \vee \ldots \vee b_n$. From (iii) we now get $b \in B_0$.

2.11. *Let B be a lattice of r-ideals of a set A. Then $K(A, B)$ is the set of all compact elements of B.*

Proof. According to 2.2 we have only to show that each compact element of B belongs to $K(A, B)$. Let X be a compact element of the system B. If $X = \emptyset_r$, then clearly $X \in K(A, B)$. If $X \neq \emptyset_r$, then $X = \bigvee_{x \in X} \{x\}_r$, hence there exists a subset $\{x_1, \ldots, x_n\} \subset X$ such that $X = \{x_1\}_r \vee \ldots \vee \{x_n\}_r$. Since $\{x_1\}_r \vee \ldots \vee \{x_n\}_r = \{x_1, \ldots, x_n\}_r$, we get $X \in K(A, B)$.

3. SYSTEMS OF r-IDEALS IN UNIVERSAL ALGEBRAS

3.1. *Let $(B; F, \wedge, \vee)$ be a system of r-ideals of $\mathfrak{A} = (A, F)$. Let $f \in F$ be an n-ary operation, $X^1, \ldots, X^n \in B$ and $B_0 = K(A, B)$. Then*

$$f(X^1, \ldots, X^n) = \bigvee f(Y^1, \ldots, Y^n), \tag{5}$$

where Y^i runs over the system of all subsets of X^i $(i = 1, \ldots, n)$ which belong to B_0.

Remark. We suppose that F does not contain nullary operations (if F would contain a nulary operation $f = a \in A$, then we take instead of f a "constant" unary operation whose value is a; this change is not essential for our considerations).

Proof. Put $X = f(X^1, \ldots, X^n)$, $Y = \bigvee f(Y^1, \ldots, Y^n)$ and write

$$Z = \bigvee f(\{x^1\}_r, \ldots, \{x^n\}_r),$$

where x^i runs over the set X^i $(i = 1, \ldots, n)$. Since $\{x^i\}_r \subset X^i$ and $\{x^i\}_r \in B_0$, we have $Z \subset Y$. For any system of elements $x^i \in X^i$ $(i = 1, \ldots, n)$

$$f(x^1, \ldots, x^n) \in \{f(v^1, \ldots, v^n) \colon v^i \in \{x^i\}_r\}_r = f(\{x^1\}_r, \ldots, \{x^n\}_r)$$

holds true, hence $\{f(x^1, \ldots, x^n) \colon x^i \in X^i\} \subset Z$ and therefore $X = \{f(x^1, \ldots \ldots, x^n) \colon x^i \in X^i\}_r \subset Z$. From $Y^i \subset X^i$ we infer that $f(Y^1, \ldots, Y^n) \subset \subset f(X^1, \ldots, X^n)$, hence $\bigvee f(Y^1, \ldots, Y^n) \subset f(X^1, \ldots, X^n)$. We have shown that $Y \subset X \subset Z \subset Y$; the proof is complete.

3.2. Let $(B; G, \wedge, \vee)$ be an algebra such that $(B; \wedge, \vee)$ is a complete lattice and each $g \in G$ is isotone. Let B_0 be the set of all compact elements of B and suppose that each element $b \in B$ is a join of some elements of B_0. Let us further suppose that the following condition is satisfied:

(iv) If $g \in G$ is an n-ary operation and if $b^1, \ldots, b^n \in B$, then

$$g(b^1, \ldots, b^n) = \bigvee g(a^1, \ldots, a^n), \tag{5'}$$

where a^i runs over the set of all elements which belong to B_0 and are less than or equal to b^i $(i = 1, \ldots, n)$.

The basic idea of the construction we are going to perform follows. We put $B_0 = A$ and construct a system B_1 of subsets of A as in 2.4 and then a system $\mathfrak{I}_1(A', B_1')$ as in 2.8. According to 2.8.1 the lattices B and B_1' are isomorphic. For the sets M_a (that were used in 2.8) we take the sets $M_a = \{(a, m)\}_{m \in M}$, where M is a fixed set with card $M \geqslant$ card B_0. We define in a suitable manner the operations $g \in G$ on the set A', and in this way we get an algebra $\mathfrak{A} = (A', G)$ with the system B_1' of r-ideals. We construct now the algebra $\mathfrak{J}(\mathfrak{A}, B_1') = (B_1'; G, \wedge, \vee)$ as in 1.1 and examine the behaviour of the mapping $\varphi' = \varphi\psi : B_1' \to B$ with respect to the operations $g \in G$.

For $b \in B$ let $\tau(b)$ have the same meaning as in 2.5. Since, by our assumption, card $\tau(b) \leqslant$ card $B_0 \leqslant$ card M, there exists a mapping h_b of the set M onto $\tau(b)$. Let m^0 be a fixed element of M.

Let $g \in G$ be an n-ary operation; we define g on the set A' as follows. Let $x^i \in A'$, $x^i = (a^i, m^i)$, $a^i \in A$, $m^i \in M$ $(i = 1, \ldots, n)$ and $g(a^1, \ldots, a^n) = b$. We put

$$g(x^1, \ldots, x^n) = \big(h_b(m^1), m^0\big); \tag{6}$$

by (6) and (3) the operations $g \in G$ are defined on the set B_1', too. Write $\varphi'^{-1} = \chi$. According to 2.8.1,

$$\chi(b) = \tau(b) \times M.$$

Under our assumptions the following statements 3.2.1 and 3.2.2 hold true:

3.2.1. *If* $b^1, \ldots, b^n \in B$ *and* $g(b^1, \ldots, b^n) = b$, *then* $g\big(\chi(b^1), \ldots, \chi(b^n)\big) = \chi(b)$.

Proof. Let $(a, m) \in \chi(b)$. Then $a \in B_0$ and $a \leqslant b$, hence $a \leqslant \bigvee g(a^1, \ldots, a^n)$, where the meaning of a^i is as in (5'). Since a is compact, there exists a finite subset $\{g(a_j^1, \ldots, a_j^n)\}_{j=1,\ldots,m}$ of the set $\{g(a^1, \ldots, a^n)\}$ such that

$$a \leqslant \bigvee_{j=1,\ldots,m} g(a_j^1, \ldots, a_j^n).$$

Put $\bigvee_{j=1}^{m} a_j^i = d^i$. Since $a_j^i \leqslant b^i$, we get $d^i \leqslant b^i$ and $d^i \in B_0$ (clearly B_0 is closed with respect to finite joins). The isotony of the operation g implies $g(a_j^1, \ldots, a_j^n) \leqslant g(d^1, \ldots, d^n)$, hence

$$\bigvee_{j=1}^{m} g(a_j^1, \ldots, a_j^n) \leqslant g(d^1, \ldots, d^n).$$

We thus obtain

$$a \leqslant g(d^1, \ldots, d^n) = d. \tag{7}$$

From (7) it follows that there exists an $m^1 \in M$ such that $h_d(m^1) = a$. Choose any elements $m^2, \ldots, m^n \in M$ and put $x^i = (d^i, m^i)$ $(i = 1, \ldots, n)$. By (6)

$$g(x^1, \ldots, x^n) = (a, m^0). \tag{8}$$

Since $d^i \leqslant b^i$, we obtain $x^i \in \chi(b^i)$, hence, by (8), $(a, m^0) \in g(\chi(b^1), \ldots, \ldots, \chi(b^n))$. The relation $g(\chi(b^1), \ldots, \chi(b^n)) \in B_1'$ implies (according to 2.8) that $(a, m) \in g(\chi(b^1), \ldots, \chi(b^n))$. We have thus derived $\chi(b) \subset g(\chi(b^1), \ldots, \ldots, \chi(b^n))$.

Let us further write

$$\{g(x^1, \ldots, x^n) : x^i \in \chi(b^i)\} = Z.$$

The elements $x^i \in \chi(b^i)$ are of the form $x^i = (a^i, m^i)$, $a^i \leqslant b^i$, $a^i \in B_0$. By the isotony of g, $g(a^1, \ldots, a^n) \leqslant g(b^1, \ldots, b^n) = b$. Hence, according to (6), $g(x^1, \ldots, x^n) = (b', m^0)$, where $b' \leqslant b$. Therefore $Z(A) \subset \tau(b)$; since $\tau(b) \in B_1$, we obtain $(Z(A))_r \subset \tau(b)$. By 2.8 this implies $Z_r \subset \tau(b) \times \times M = \chi(b)$. Using (3) we get the equality $g(\chi(b^1), \ldots, \chi(b^n)) = Z_r$, hence $g(\chi(b^1), \ldots, \chi(b^n)) \subset \chi(b)$. The proof is complete.

3.2.2. *The mapping $\varphi' : B_1' \to B$ is an isomorphism with respect to the lattice operations and with respect to the operations $g \in G$.*

Proof. The first assertion follows from 2.8.1 and the other from the fact that $\varphi' = \chi^{-1}$ is an isomorphism with respect to each $g \in G$.

3.3. THEOREM. *Let $\mathfrak{B} = (B; G, \wedge, \vee)$ be an algebra, where $(B, \wedge, \vee)$ is a complete lattice and each $g \in G$ is isotone. Then the following conditions are equivalent:*

(γ) *There exists an algebra $\mathfrak{A} = (A, G)$ and a system $\mathfrak{B}_1$ of r-ideals of $\mathfrak{A}$ such that $\mathfrak{B}$ is isomorphic with $\mathfrak{J}(\mathfrak{A}, B_1)$.*

(δ) *Each element of $(B; \wedge, \vee)$ is a join of compact elements and $\mathfrak{B}$ satisfies* (iv).

Proof. The implication $(\gamma) \Rightarrow (\delta)$ follows from 2.2 and 3.1, the converse implication follows from 3.2.2.

Remark. Let $\mathfrak{B} = (B; G, \wedge, \vee)$ be an algebra, where $(B; \wedge, \vee)$ is a complete lattice and each $g \in G$ is isotone. Suppose that each element of $(B; \wedge, \vee)$ is a join of compact elements. In general, $\mathfrak{B}$ need not satisfy condition (iv). For example, let B be the system of all subsets of an infinite set P (B is partially ordered by the inclusion). Obviously, each element of B is a join of compact elements. Let G consist of a single unary operation g defined as follows: $g(X) = \emptyset$ if X is finite and $g(X) = P$ if X is infinite. Then g is isotone and condition (iv) does not hold.

3.3.1. *Let $\mathfrak{B} = (B; G, \wedge, \vee)$ be an algebra, where $(B; \wedge, \vee)$ is a complete lattice, each operation $g \in G$ is isotone and card $B > 1$. The following conditions are equivalent:*

(γ_1) *There exists an algebra $\mathfrak{A} = (A; G)$ and a regular system B_1 of r-ideals such that $\mathfrak{B}$ and $\mathfrak{J}(\mathfrak{A}, B_1)$ are isomorphic.*

(δ_1) *$\mathfrak{B}$ fulfils (δ), and, for any n-ary operation $g \in G$ and any $b^1, \ldots, b^n \in B$, $g(b^1, \ldots, b^n) = b^0 \Rightarrow$ there exists an $i \in \{1, \ldots, n\}$ such that $b^i = b^0$.*

Proof. Let B_1 be a regular system of r-ideals of algebra $\mathfrak{A} = (A; G)$ and let $g \in G$ be an n-ary operation, $X^i \in B_1$ $(i = 1, \ldots, n)$. Clearly, $g(X^1, \ldots, X^n) = \emptyset$ if and only if at least one X^i is the empty set. From this and from 3.3 we infer that $(\gamma_1) \Rightarrow (\delta_1)$. Conversely, suppose that (δ_1) holds and card $B > 1$. Then we can modify the consideration in 3.2 and 3.3 by taking the set $B_0 \setminus \{b^0\}$ instead of B_0 (cf. also 2.3.1). According to 2.6.1 we get a regular system of ideals.

Now we shall make use of the fact that the cardinality of M (cf. 3.2) has to fulfil the condition $\operatorname{card} M \geqslant \operatorname{card} B_0$ only.

3.4. THEOREM. *Let $\mathfrak{A} = (A; G)$ be an algebra with a system B of r-ideals, $\operatorname{card} A = \alpha$, $\max\{\alpha, \aleph_0\} = \beta$. For each cardinal $\gamma \geqslant \beta$ there exists an algebra $\mathfrak{A}_\gamma = (A_\gamma; G)$ with a system B_γ of r-ideals such that $\mathfrak{J}(\mathfrak{A}, B)$ and $\mathfrak{J}(\mathfrak{A}_\gamma, B_\gamma)$ are isomorphic and $\operatorname{card} A_\gamma = \gamma$.*

Proof. We start with the algebra $\mathfrak{A} = (A; G)$ and we construct $\mathfrak{B} = (B; G, \wedge, \vee)$ as in section 1. Let B_0 have the same meaning as in 2.2. The power of the set B_0 is less than or equal to the power of the system of all finite subsets of A, hence card $B_0 \leqslant \beta$. We now consider the algebra $(A', G) = A_\beta$ that is constructed as in 3.2 (with G instead of F), where we choose the set $M = M_\gamma$ so that $\operatorname{card} M_\gamma = \gamma \geqslant \beta$. Since $A' = B_0 \times M$, we obtain $\operatorname{card} A' \leqslant \beta\gamma = \gamma$. According to 3.2.2, the algebras $\mathfrak{J}(\mathfrak{A}) = \mathfrak{B}$ and $\mathfrak{J}(\mathfrak{A}_\beta)$ are isomorphic.

COROLLARY. *To every algebra $\mathfrak{A} = (A; F)$ with a system B of r-ideals there exists an algebra $\mathfrak{A}' = (A'; F)$ with a system B' of r-ideals such that the algebras $\mathfrak{J}(\mathfrak{A}, B)$ and $\mathfrak{J}(\mathfrak{A}', B')$ are isomorphic but $\mathfrak{A}'$ and $\mathfrak{A}$ are not isomorphic.*

3.5. Let $\mathfrak{A} = (A; F)$ be an algebra. If $f \in F$ is an n-ary operation, we put $k(f) = n$. Let $f_1, \ldots, f_n \in F$, $k(f_i) = m_i$; we denote by h the operation satisfying

$$h(a_1^1, \ldots, a_{m_1}^1; \ldots; a_1^n, \ldots, a_{m_n}^n) = f\big(f_1(a_1^1, \ldots, a_{m_1}^1), \ldots, f_n(a_1^n, \ldots, a_{m_n}^n)\big)$$

for any $a_i^j \in A$ $(j = 1, \ldots, n,\ i = 1, \ldots, m_j)$. As usual, the operation h is denoted by $f(f_1, \ldots, f_n)$. Let F_1 be the set of all operations that can be constructed in this way. Let us further put $\bar{F}_1 = F \cup F_1$. By induction we construct the sets F_l and $\bar{F}_l$ $(l = 2, 3, \ldots$; by forming the operations $h \in F_l$ we suppose that $f, f_1, \ldots, f_n \in \bar{F}_{l-1})$. Let $\bar{F} = \bigcup_{n=1,2,\ldots} \bar{F}_n$. Let $R \subset \bar{F} \times \bar{F}$ and suppose that $k(f_1) = k(f_2)$ for each pair $(f_1, f_2) \in R$. The algebra A is said to satisfy the system R if $f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n)$ for any pair $(f_1, f_2) \in R$ and any system $a_1, \ldots, a_2 \in A$ (with $k(f_1) = n$). Let $C(R)$ be the class of all algebras $(A; F)$ which satisfy the system R.

Assertion 3.4 is a corollary of the following theorem which can be proved in a simple way without using 3.2.2 and 3.3 (i.e., without constructing the set A'):

3.5.1. Theorem. *Let $\mathfrak{A}_1 = (A_1, F) \in C(R)$, $\operatorname{card} A_1 = \alpha$. Let B_1 be a system of r-ideals of A. For each cardinal $\beta > 0$ there exists an algebra $\mathfrak{A}_2 = (A_2, F) \in C(R)$ and a system B_2 of r-ideals of A_2 such that the algebras $\mathfrak{J}(\mathfrak{A}_1, B_1)$ and $\mathfrak{J}(\mathfrak{A}_2, B_2)$ are isomorphic and $\operatorname{card} A_2 = \alpha\beta$.*

Proof. Let $\beta > 0$ be a cardinal and let M be a set with card $M = \beta$. Choose a fixed element $m^0 \in M$ and write $A_2 = A_1 \times M$. If $f \in F$, $k(f) = n$ and $y^i = (a^i, m^i) \in A$ $(i = 1, 2, \ldots, n)$, we define $f(y^1, \ldots, y^n) = (a, m^0)$, where $a = f(a^1, \ldots, a^n)$. The algebra $\mathfrak{A}_2 = (A_2, F)$, obviously, satisfies the system R. If $X \subset A_2$, let $X(A_1)$ be the set of all elements $a \in A_1$, such that there exists an $m \in M$ with $(a, m) \in X$. For any $X \subset A_2$, put $X_r = \big(X(A_1)\big)_r \times M$. Let $B_2 = \{X_r : X \subset A_2\}$. It is easy to show that the algebras $\mathfrak{J}(\mathfrak{A}_1, B_1)$ and $\mathfrak{J}(\mathfrak{A}_2, B_2)$ are isomorphic.

4. THE SYSTEMS OF CLOSED SUBSETS

In the previous paragraphs we did not suppose any connection between the system B of r-ideals and the system of operations F of the algebra $\mathfrak{A} = (A; F)$. A subset X of A is *closed* with respect to the system F if $f(a^1, \ldots, a^n) \in X$ for any $f \in F$ and any $a_1, \ldots, a_n \in A$ (where $n = k(f)$). For $X \subset A$ we denote by $[X]_F$ the intersection of all sets Y which are closed with respect to F and satisfy $X \subset Y \subset A$. Let us consider the following conditions for a system B of r-ideals:

(α_1) $$[X]_F \subset X_r \quad \text{for any } X \subset A,$$

(α_2) $$[X]_F = X_r \quad \text{for any } X \subset A, X \neq \emptyset.$$

4.1. *Let $\mathfrak{A} = (A, F)$ be an algebra with a system B of r-ideals satisfying (α_1). Then $f(X, \ldots, X) \subset X$ for any $X \in B$ and any operation $f \in F$.*

Proof. Let $f \in F$, $k(f) = n$, $X \in B$. Since $f(X, \ldots, X) = \{f(x_1, \ldots, x_n) : x_i \in X\}_r$, and $x_i \in X$ implies $f(x_1, \ldots, x_n) \in X$, according to condition (α_1) we get $f(X, \ldots, X) \subset X_r = X$.

4.2. *Let $\mathfrak{B} = (B; G, \wedge, \vee)$ be an algebra, where $(B; \wedge, \vee)$ is a complete lattice and each operation $g \in G$ is isotone. Suppose that condition (δ) from 3.3 is satisfied. If $g(x, \ldots, x) \leqslant x$ for any $x \in B$ and each $g \in G$, then the system B_1' of r-ideals on A' described in 3.2 fulfils the condition $[X]_G \subset X_r$ for any $X \subset A'$.*

Proof. Suppose that $g(x, \ldots, x) \leqslant x$ for any $g \in G$ and any $x \in B$. Let $X \subset A'$. If $X = \emptyset$, then $[X]_G = \emptyset \subset X_r$. Let $X \neq \emptyset$. Since $X \subset X_r$, it suffices to prove that X_r is closed with respect to the system G. Write

$$x = \sup\big(X(A)\big).$$

Then

$$X_r = \tau(x) \times M.$$

Let $g \in G$, $k(g) = n$, $x^i \in X_r$ and $x^i = (a^i, m^i)$, $i = 1, \ldots, n$. Under the same notations as in 3.2 we have $a^i \leqslant x$ (since $a^i \in X(A)$), therefore $b = g(a^1, \ldots, a^n) \leqslant g(x, \ldots, x) \leqslant x$. From this we get $h_b(m^1) \leqslant b \leqslant x$ and hence $g(x^1, \ldots, x^n) = \big(h_b(m^1), m^0\big) \in X_r$.

4.3. Theorem. *Let $\mathfrak{B} = (B; G, \wedge, \vee)$ be an algebra, where $(B; \wedge, \vee)$ is a complete lattice and each operation $g \in G$ is isotone. Then the following conditions are equivalent:*

(β_1) *There exists an algebra $\mathfrak{A} = (A; G)$ and a system B_1 of r-ideals of A such that B_1 satisfies (α_1) (with G instead of F) and B is isomorphic with $\mathfrak{I}(\mathfrak{A}, B_1)$.*

(β_2) *The algebra $\mathfrak{B}$ fulfils condition (δ) from 2.3 and $g(x, \ldots, x) \leqslant x$ for any $x \in B$ and any $g \in G$.*

This follows from 2.3, 4.1 and 4.2.

Remark. An algebra $\mathfrak{A} = (A; F)$ with a system B of r-ideals satisfying (α_1) need not fulfil $[X]_F = X_r$. Let for example card $A \geqslant 2$, $a_0 \in A$, $F = \{f\}$, $f(a) = a_0$ for each $a \in A$, $X_r = A$ for any $X \subset A$. If $X = \{a_0\}$, then $[X]_F = X \neq X_r$ and (α_1) is valid.

4.4. Theorem. *Let $\mathfrak{B} = (B; G, \wedge, \vee)$ be an algebra, where $(B; \wedge, \vee)$ is a complete lattice and each operation $g \in G$ is isotone. Then the following conditions are equivalent:*

(γ_1) *There exists an algebra $\mathfrak{A} = (A; F)$ with $G \subset F$ and a system B_1 of r-ideals of $\mathfrak{A}$ such that B_1 satisfies condition (α_2) and $\mathfrak{B}$ is isomorphic with $\mathfrak{I}\big((\mathfrak{A}; G), B_1\big)$.*

$(\gamma_2) = (\beta_2)$.

Proof. Suppose that (γ_1) holds and let $Y \subset A$. Clearly $[Y]_G \subset [Y]_F$, hence $[Y]_G \subset Y_r$. If $g \in G$, then, by 4.1, $g(X, \ldots, X) \subset X$ for any $X \in B_1$; since B and $\mathfrak{I}\big((\mathfrak{A}; G), B_1\big)$ are isomorphic, we get $g(x, \ldots, x) \leqslant x$. By 2.3, B satisfies (δ).

Conversely, suppose that (β_2) holds true. We construct the algebra $(A; G)$ with a system B_1 of r-ideals as in 4.3. If n is any integer and $x_1, \ldots, x_n \in A$, $y \in \{x_1, \ldots, x_n\}_r$, we define the operation $f_{x_1, \ldots, x_n, y}(t_1, \ldots, t_n) =: h(t_1, \ldots, t_n)$ as follows:

$$h(t_1, \ldots, t_n) = \begin{cases} y, & \text{if } t_i = x_i \ (i = 1, \ldots, n), \\ t_1 & \text{in other cases.} \end{cases}$$

Let G_1 be the set of all operations that we get in this way, and let $F = G \cup G_1$. Consider the algebra $\mathfrak{A} = (A; F)$. Let $Y \in B_1$ and $Y \neq \emptyset$. By 4.3, $[Y]_G \subset Y_r = Y$, hence Y is closed with regard to any operation $g \in G$. If $f \in G_1$, $k(f) = n$, $y_1, \ldots, y_n \in Y$, then either $f(y_1, \ldots, y_n) = y_1$ or $f(y_1, \ldots, y_n) = y \in \{y_1, \ldots, y_n\}_r \subset Y$. Hence $[Y]_F \subset Y_r$.

If $X \subset A$ and $X \neq \emptyset$, then $[X]_F \subset [X_r]_F = X_r$. For any $y \in X_r$ there exists a finite subset $\{x_1, \ldots, x_n\} \subset X$ such that $y \in \{x_1, \ldots, x_n\}_r$. From $f_{x_1,\ldots,x_n,y}(x_1, \ldots, x_n) = y$ it follows that $y \in [X]_F$, and hence $X_r \subset [X]_F$.

REFERENCES

[1] L. Fuchs, *Partially ordered algebraic systems*, Oxford — London — New York — Paris 1963.

[2] — *On partially ordered algebras I*, Colloquium Mathematicum 14 (1966), p. 115-130.

[3] — *On partially ordered algebras II*, Acta Scientiarum Mathematicarum 24 (1965), p. 35-41.

[4] J. Čornayová, *Systems of ideals in universal algebras* (to be published).

[5] L. Nachbin, *On a characterization of the lattice of all ideals of a Boolean ring*, Fundamenta Mathematicae 36 (1949), p. 137-142.

Reçu par la Rédaction le 22. 8. 1967