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Coefficient estimates for spirallike functions

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Abstract. If $\alpha \in [0,1)$, $\varrho > 0$, $\alpha+\varrho \geqslant 1$, $\beta \in (-\pi/2, \pi/2)$ and n is a positive integer, let $S_\beta(\alpha, \varrho, n)$ denote the class of all functions $f(z)$ analytic in the unit disk $E = \{z: |z| < 1\}$, of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ for $z \in E$ and satisfying $e^{i\beta} z \dfrac{f'(z)}{f(z)} = P(z)\cos\beta + i\sin\beta$ for $z \in E$, where $P(z)$ is analytic in E, of the form $P(z) = 1 + \sum_{k=n}^{\infty} b_k z^k$ in E and satisfies $|P(z) - (\alpha+\varrho)| < \varrho$ for $z \in E$. In this paper, bounds are obtained for the coefficients of functions of the class $S_\beta(\alpha, \varrho, n)$ which generalize earlier results of the authors and Plaskota.

Let S denote the class of functions $f(z)$ analytic in the unit disk $E = \{z: |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Throughout this paper, let α, β and ϱ be real numbers satisfying $\alpha \in [0,1)$, $\varrho > 0$, $\alpha + 2\varrho > 1$ and $\beta \in (-\pi/2, \pi/2)$ and n be a positive integer.

Let $\mathfrak{P}(\alpha, \varrho, n)$ denote the class of functions $P(z)$ analytic in E, of the form $P(z) = 1 + b_n z^n + \ldots$ in E and satisfying $|P(z) - (\alpha+\varrho)| < \varrho$ for $z \in E$. Let $S_\beta(\alpha, \varrho, n)$ be the subclass of S consisting of functions $f(z)$ of the form $f(z) = z + a_{n+1} z^{n+1} + \ldots$ in E and satisfying

$$e^{i\beta} z \frac{f'(z)}{f(z)} = (\cos\beta) P(z) + i\sin\beta \quad \text{for } z \in E$$

for some $P(z) \in \mathfrak{P}(\alpha, \varrho, n)$.

For $n = 1$ and ϱ tending to ∞, $S_\beta(\alpha, \varrho, n)$ reduces to the class of β-spiral functions of order α introduced by Špaček [8] for $\alpha = 0$ and extended by Libera [5] for $\alpha \in [0,1)$.

Let $U_\beta(\alpha, \varrho, n)$ denote the class of functions $f(z)$ analytic in the punctured disk $E_1 = \{z: 0 < |z| < 1\}$, of the form $f(z) = \dfrac{1}{z} + \sum_{k=n}^{\infty} a_k z^k$ in E_1 and satisfying $-e^{i\beta} z \dfrac{f'(z)}{f(z)} = (\cos\beta) P(z) + i\sin\beta$ for $z \in E_1$ for

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some $P(z) \in \mathfrak{P}(\alpha, \varrho, n+1)$. $U_0(\alpha, \infty, 1)$ is the class of meromorphic starlike functions of order α introduced by Pommerenke [7].

In this paper we obtain bounds for the coefficients of functions of the classes $S_\beta(\alpha, \varrho, n)$ and $U_\beta(\alpha, \varrho, n)$ if $\alpha+\varrho \geqslant 1$. These results yield, as particular cases, earlier results of the authors [2], Plaskota [6] and Kaczmarski [4].

We use the following lemmas in proving our theorems.

LEMMA 1. *If $\alpha \in [0, 1)$, $\varrho > 0$, $\beta \in (-\pi/2, \pi/2)$ and n and q are positive integers, then*

$$\frac{D}{\varrho^2}\left[D\cos^2\beta + \sum_{m=1}^{q}\{(D-2mnB)\cos^2\beta - m^2n^2\}\left\{\frac{1}{m!}\prod_{j=0}^{m-1} u_j\right\}^2\right] = \left\{\frac{n}{q!}\prod_{j=0}^{q} u_j\right\}^2,$$

where

$$u_j = \left|\frac{De^{-i\beta}\cos\beta}{n\varrho} - \frac{jB}{\varrho}\right| \tag{1}$$

for $j = 0, 1, 2, \ldots$, with $B = 1-\alpha-\varrho$ and $D = (1-\alpha)(2\varrho+\alpha-1)$.

The lemma can be proved by induction on q in the same way as the lemma in [2].

The next lemma gives a representation formula for functions of the class $\mathfrak{P}(\alpha, \varrho, n)$ and can be proved easily by standard methods.

LEMMA 2. *$P(z) \in \mathfrak{P}(\alpha, \varrho, n)$ if and only if $P(z)$ is of the form $P(z) = \dfrac{\varrho+Aw(z)}{\varrho+Bw(z)}$ for $z \in E$, where $A = \varrho+\alpha(1-\alpha-2\varrho)$, $B = 1-\alpha-\varrho$, $w(z)$ is analytic in E, of the form $w(z) = c_n z^n + \ldots$ in E and $|w(z)| < 1$ for $z \in E$.*

THEOREM 1. *If $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in S_\beta(\alpha, \varrho, n)$ and $\alpha+\varrho \geqslant 1$, then*

$$\sum_{k=mn+1}^{(m+1)n} (k-1)^2 |a_k|^2 \leqslant \frac{D^2}{\varrho^2}\cos^2\beta \tag{2}$$

for $m = 1, 2, 3, \ldots$, if $n^2+(2nB-D)\cos^2\beta \geqslant 0$.

If $n^2+(2nB-D)\cos^2\beta < 0$, then

$$\sum_{k=mn+1}^{(m+1)n} (k-1)^2 |a_k|^2 \leqslant \left\{\frac{n}{(m-1)!}\prod_{j=0}^{m-1} u_j\right\}^2 \tag{3}$$

for $m = 1, \ldots, q_0+1$ *and*

(4) $$\sum_{k=mn+1}^{(m+1)n} (k-1)^2 |a_k|^2 \leqslant \left\{\frac{n}{q_0!} \prod_{j=0}^{q_0} u_j\right\}^2$$

for $m = q_0+2, q_0+3, \ldots,$ *where* $D = A-B = (1-a)(2\varrho+a-1)$, A *and* B *being as defined in Lemma* 2, u_j *is given by* (1) *and* q_0 *is the natural number determined by* $q_0 \in [b-1, b)$, *where*

$$b = \cos\beta\left\{\sqrt{D+B^2\cos^2\beta}-B\cos\beta\right\}/n.$$

Estimates (2) *and* (3) *are sharp with equality holding in* (2) *for a given* m *for the function*

$$f_\varepsilon(z) = z\left\{1+\varepsilon\frac{B}{\varrho}z^{mn}\right\}^{\frac{D}{mnB}e^{-i\beta}\cos\beta} \quad \text{if } B \neq 0,$$

$$= z\exp\left\{\frac{\varepsilon(1-a)}{mn}e^{-i\beta}z^{mn}\cos\beta\right\} \quad \text{if } B = 0,$$

if $n^2+(2nB-D)\cos^2\beta \geqslant 0$ *and equality holds in* (3) *for the function*

$$f_\varepsilon(z) = z\left\{1+\frac{\varepsilon B}{\varrho}z^{n}\right\}^{\frac{D}{nB}e^{-i\beta}\cos\beta} \quad \text{if } B \neq 0,$$

$$= z\exp\left\{\frac{\varepsilon(1-a)}{n}e^{-i\beta}z^{n}\cos\beta\right\} \quad \text{if } B = 0,$$

if $n^2+(2nB-D)\cos^2\beta < 0$, *where* $|\varepsilon| = 1$. *Also,*

(5) $$\sum_{k=n+1}^{\infty} [(k-1)^2-\{D-2(k-1)B\}\cos^2\beta]|a_k|^2 \leqslant D\cos^2\beta.$$

Proof. We have

$$e^{i\beta}z\frac{f'(z)}{f(z)} = P(z)\cos\beta+i\sin\beta \quad \text{for } z \in E,$$

where $P(z) \in \mathfrak{P}(a, \varrho, n)$. Hence, by Lemma 2,

(6) $$e^{i\beta}z\frac{f'(z)}{f(z)}\sec\beta-i\tan\beta = P(z) = \frac{\varrho+Aw(z)}{\varrho+Bw(z)}$$

for $z \in E$, where $w(z)$ is analytic in E, of form $w(z) = b_n z^n+\ldots$ in E and $|w(z)| < 1$ for $z \in E$. So,

$$\varrho(1+i\tan\beta)\big(zf'(z)-f(z)\big) = \{(A+iB\tan\beta)f(z)-B(1+i\tan\beta)zf'(z)\}w(z),$$

for $z \in E$.

Substituting the series expansions of $f(z)$ and $w(z)$, we obtain, for $z \in E$,

$$(7) \qquad \varrho(1+i\tan\beta)\sum_{k=n+1}^{\infty}(k-1)a_k z^k$$
$$= \Big[Dz + \sum_{k=n+1}^{\infty}\{A-kB-i(k-1)B\tan\beta\}a_k z^k\Big]\sum_{k=n}^{\infty} b_k z^k.$$

We now proceed by a method introduced by Clunie [1].

Equating the coefficients of z^k on the two sides of (7) for $k = n+1, \ldots$ $\ldots, 2n$, we obtain $\varrho(1+i\tan\beta)(k-1)a_k = Db_{k-1}$ for $k = n+1, \ldots, 2n$. Therefore,

$$(8) \qquad \sum_{k=n+1}^{2n}(k-1)^2|a_k|^2 \leqslant \frac{D^2\cos^2\beta}{\varrho^2}\sum_{k=n+1}^{2n}|b_{k-1}|^2 \leqslant \frac{D^2\cos^2\beta}{\varrho^2},$$

since, we have, for $0 < r < 1$,

$$\sum_{k=n}^{\infty}|b_k|^2 r^{2k} = \frac{1}{2\pi}\int_0^{2\pi}|w(re^{i\varphi})|^2 d\varphi \leqslant 1$$

and letting r tend to 1 we obtain $\sum_{k=n}^{\infty}|b_k|^2 \leqslant 1$.

Again, for $p \geqslant n+1$, (7) can be put in the form

$$G(z) = H(z)w(z) \qquad \text{for } z \in E$$

with

$$G(z) = \varrho(1+i\tan\beta)\sum_{k=n+1}^{n+p}(k-1)a_k z^k + \sum_{k=n+p+1}^{\infty} d_k z^k$$

and $H(z) = Dz + \sum_{k=n+1}^{p}\{A-kB-i(k-1)B\tan\beta\}a_k z^k$, where $\sum_{k=n+p+1}^{\infty} d_k z^k$ converges in E.

Since $|w(z)| < 1$ for $z \in E$, we obtain, for $0 < r < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\varphi})|^2 d\varphi \leqslant \frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\varphi})|^2 d\varphi,$$

so that

$$\varrho^2\sec^2\beta\sum_{k=n+1}^{n+p}(k-1)^2|a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty}|d_k|^2 r^{2k}$$
$$\leqslant D^2 r^2 + \sum_{k=n+1}^{p}|A-kB-i(k-1)B\tan\beta|^2|a_k|^2 r^{2k}.$$

Letting r tend to 1 and rearranging, we obtain for $p \geqslant n+1$,

$$(9) \qquad \sum_{k=p+1}^{p+n} (k-1)^2 |a_k|^2$$

$$\leqslant \frac{D}{\varrho^2}\Big[D\cos^2\beta + \sum_{k=n+1}^{p} \{(D-2(k-1)B)\cos^2\beta - (k-1)^2\}|a_k|^2\Big].$$

If $n^2+(2nB-D)\cos^2\beta \geqslant 0$, then for $k \geqslant n+1$,

$$\{D-2(k-1)B\}\cos^2\beta-(k-1)^2$$

$$= (k-1)^2\left[\left\{\frac{D}{(k-1)^2}-\frac{2B}{k-1}\right\}\cos^2\beta - 1\right]$$

$$\leqslant (k-1)^2\left[\left\{\frac{D}{n^2}-\frac{2B}{n}\right\}\cos^2\beta - 1\right] \quad \text{since } B \leqslant 0,$$

$$= \frac{(k-1)^2}{n^2}[(D-2nB)\cos^2\beta - n^2] \leqslant 0.$$

Hence (9) yields

$$\sum_{k=p+1}^{p+n} (k-1)^2 |a_k|^2 \leqslant \frac{D^2}{\varrho^2}\cos^2\beta \qquad \text{for } p \geqslant n+1.$$

Putting $p = mn$, $m \geqslant 2$ and combining with (8) we obtain (2).

Suppose, now, that $n^2+(2nB-D)\cos^2\beta < 0$. Let q_0 be as defined in the statement of the theorem. Then q_0 is the largest of the natural numbers k for which $k^2n^2+(2knB-D)\cos^2\beta < 0$.

We now establish by an inductive argument inequalities (3) for $m = 1, \ldots, q_0+1$ and the inequalities

$$(10) \qquad \sum_{k=mn+1}^{(m+1)n} [\{D-2(k-1)B\}\cos^2\beta - (k-1)^2]|a_k|^2$$

$$\leqslant [(D-2mnB)\cos^2\beta - m^2n^2]\left\{\frac{1}{m!}\prod_{j=0}^{m-1} u_j\right\}^2$$

for $m = 1, \ldots, q_0$.

For $m = 1$, (3) reduces to (8) whereas the left member of (10)

$$\leqslant \left\{\frac{(D-2nB)\cos^2\beta - n^2}{n^2}\right\} \sum_{k=n+1}^{2n} (k-1)^2|a_k|^2$$

$$\leqslant \{(D-2nB)\cos^2\beta - n^2\}\left\{\frac{D^2\cos^2\beta}{n^2\varrho^2}\right\} \quad \text{by (8)},$$

so that (10) holds for $m = 1$.

Suppose that (3) and (10) hold for $m = 1, \ldots, q-1$, where $2 \leqslant q \leqslant q_0$. For $p = qn$, (9) yields

$$\sum_{k=qn+1}^{(q+1)n} (k-1)^2 |a_k|^2$$

$$\leqslant \frac{D}{\varrho^2}\Big[D\cos^2\beta + \sum_{m=1}^{q-1} \sum_{k=mn+1}^{(m+1)n} \big\{\big(D-2(k-1)B\big)\cos^2\beta - (k-1)^2\big\}\,|a_k|^2\Big]$$

$$\leqslant \frac{D}{\varrho^2}\Big[D\cos^2\beta + \sum_{m=1}^{q-1} \{(D-2mnB)\cos^2\beta - m^2n^2\}\Big\{\frac{1}{m!}\prod_{j=0}^{m-1} u_j\Big\}^2\Big],$$

by (10) for $m = 1, \ldots, q-1$.

Hence, by Lemma 1,

$$\sum_{k=qn+1}^{(q+1)n} (k-1)^2 |a_k|^2 \leqslant \Big\{\frac{n}{(q-1)!}\prod_{j=0}^{q-1} u_j\Big\}^2$$

so that (3) holds for $m = q$. Now,

$$\sum_{k=qn+1}^{(q+1)n} \big[\big(D-2(k-1)B\big)\cos^2\beta - (k-1)^2\big]\,|a_k|^2$$

$$\leqslant \frac{[(D-2qnB)\cos^2\beta - q^2n^2]}{q^2n^2} \sum_{k=qn+1}^{(q+1)n} (k-1)^2 |a_k|^2$$

$$\leqslant \frac{[(D-2qnB)\cos^2\beta - q^2n^2]}{q^2n^2} \Big\{\frac{n}{(q-1)!}\prod_{j=0}^{q-1} u_j\Big\}^2,$$

using (3) with $m = q$ (since $(D-2qnB)\cos^2\beta - q^2n^2 > 0$ because $q \leqslant q_0$). Thus (10) holds for $m = q$.

Hence (3) and (10) hold for $m = 1, \ldots, q_0$. It follows, now, by the argument used above to show that (3) holds for $m = q$, that (3) holds for $m = q_0+1$.

By the definition of q_0, we have

$$\{D-2(q_0+1)nB\}\cos^2\beta - (q_0+1)^2n^2 \leqslant 0.$$

Hence $\{D-2(k-1)B\}\cos^2\beta - (k-1)^2 \leqslant 0$ for $k > (q_0+1)n$. Thus, for $p \geqslant (q_0+1)n$, (9) yields

$$\sum_{k=p+1}^{p+n} (k-1)^2 |a_k|^2$$

$$\leqslant \frac{D}{\varrho^2}\Big[D\cos^2\beta + \sum_{m=1}^{q_0} \sum_{k=mn+1}^{(m+1)n} \{(D-2(k-1)B)\cos^2\beta - (k-1)^2\} |a_k|^2\Big]$$

$$\leqslant \frac{D}{\varrho^2}\Big[D\cos^2\beta + \sum_{m=1}^{q_0} \{(D-2mnB)\cos^2\beta - m^2n^2\}\Big\{\frac{1}{m!}\prod_{j=0}^{m-1} u_j\Big\}^2\Big],$$

by (10) for $m = 1, 2, \ldots, q_0$.

Hence, by Lemma 1,

$$\sum_{k=p+1}^{p+n} (k-1)^2 |a_k|^2 \leqslant \Big\{\frac{n}{q_0!}\prod_{j=0}^{q_0} u_j\Big\}^2 \quad \text{for } p \geqslant (q_0+1)n.$$

Putting $p = mn$, $m = q_0+2, q_0+3, \ldots,$ we obtain (4).

The assertions in Theorem 1 about the sharpness of (2) and (3) are easily verified directly.

Finally, we obtain from (9),

$$\sum_{k=n+1}^{p} [(k-1)^2 - \{D-2(k-1)B\}\cos^2\beta]\,|a_k|^2 \leqslant D\cos^2\beta$$

for $p \geqslant n+1$ and letting p tend to ∞ we obtain (5). This completes the proof of Theorem 1.

Remark 1. Under the hypothesis of Theorem 1, since D tends to ∞ and B tends to $-\infty$ as ϱ tends to ∞, it follows that $n^2+(2nB-D)\cos^2\beta < 0$ for all sufficiently large ϱ and hence (3) and (4) hold for all sufficiently large ϱ. Also, since q_0 tends to ∞, D/ϱ tends to $2(1-\alpha)$ and B/ϱ tends to -1 as ϱ tends to ∞, we obtain the following corollary which is an earlier result of the authors [2], Theorem C.

COROLLARY 1. *If* $f(z) = z+\sum_{k=n+1}^{\infty} a_k z^k \in S_\beta(\alpha, \infty, n)$, *then*

$$\sum_{k=mn+1}^{(m+1)n} (k-1)^2 |a_k|^2 \leqslant \Big\{\frac{n}{(m-1)!}\prod_{j=0}^{m-1} \Big|\frac{2(1-\alpha)e^{-i\beta}\cos\beta}{n} + j\Big|\Big\}^2$$

for $m = 1, 2, \ldots$

The result is sharp.

Remark 2. Choosing $n = 1$ and $\varrho = (1-\alpha)M$, $M \geqslant 1$ in Theorem 1, we obtain a result of Plaskota [6].

THEOREM 2. *Let* $f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k \in U_\beta(\alpha, \varrho, n)$ *and* $\alpha + \varrho \geqslant 1$. *Then*

$$\sum_{k=mn}^{(m+1)n-1} (k+1)^2 |a_k|^2 \leqslant \frac{D^2 \cos^2\beta}{\varrho^2} \tag{11}$$

for $m = 1, 2, \ldots$, *and*

$$\sum_{k=n}^{\infty} [(k+1)^2 - \{D + 2(k+1)B\}\cos^2\beta]|a_k|^2 \leqslant D\cos^2\beta.$$

Estimate (11) *is sharp with equality holding for given m for the function*

$$f_\varepsilon(z) = z^{-1}\left\{1 - \frac{\varepsilon B z^{mn+1}}{\varrho}\right\}^{\frac{D}{(mn+1)B} e^{-i\beta}\cos\beta} \qquad \textit{if } B = 0,$$

$$= z^{-1}\exp\left\{\frac{\varepsilon(\alpha-1)}{mn+1} e^{-i\beta} z^{mn+1}\cos\beta\right\} \qquad \textit{if } B = 0,$$

where $|\varepsilon| = 1$.

The proof is analogous to that of Theorem 1 and is omitted.

For $n = 1$ and $\varrho = (1-\alpha)M$, $M \geqslant 1$, Theorem 2 yields a result of Kaczmarski [4]. Also, for $n = 1$, $\beta = 0$, letting ϱ tend to ∞, Theorem 2 yields a result of Pommerenke [7].

Choosing $n = 1$, the above theorems yield results of Jakubowski [3] obtained by taking $\varrho = (1-\alpha)M$, $M = m \geqslant 1$.

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