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ON CERTAIN COMBINATORIAL IDENTITIES

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The purpose of the present note is to give a probabilistic proof of the combinatorial identities

$$\frac{1}{\binom{m}{s}}\sum_{t=1}^{m-s}\frac{\binom{m-t}{s}}{t}=\sum_{k=s+1}^{m}\frac{1}{k},\qquad 0\leqslant s<m, \tag{1}$$

or, more generally,

$$\frac{1}{\binom{m}{s}}\sum_{t=z}^{m-s}\frac{\binom{m-t}{s}}{t}=\frac{1}{\binom{m}{z-1}}\sum_{k=s+1}^{m-z+1}\frac{\binom{m-k}{z-1}}{k}, \tag{2}$$

valid for $0\leqslant s<m$ and $z=1,\ldots,m-s$.

Although (1) is a special case of (2), it will be more convenient to give first the proof of (1), and then modify it so as to get identity (2).

Let us rewrite (1) in an apparently less natural form

$$\frac{1}{\binom{n}{r-1}}\sum_{t=2}^{n-r+2}\frac{\binom{n-t}{r-2}}{t-1}=\frac{r-1}{n}\sum_{k=r}^{n}\frac{1}{k-1},\qquad 2\leqslant r\leqslant n. \tag{3}$$

We shall show that both sides of (3) represent probability of the same event.

The problem which will serve as a probabilistic set-up is known in the literature under various names and will be presented here in a picturesque terminology, as the so-called *secretary problem* (see [1] and [3]).

In response to an announcement for the vacant secretary position, there appear n candidates. They are interviewed consecutively (in random order) and each candidate may be ranked with respect to those already interviewed, but not with respect to the others. After each interview, the candidate may be accepted or rejected; no change of decision is allowed. The problem is to devise a policy which would maximize the probability of getting the best candidate.

To put it formally, let $1, \ldots, n$ represent the absolute ranks of the candidates, 1 standing for the best candidate, and n for the worst, and let $x_1, \ldots, x_n$ be a permutation of ranks $1, \ldots, n$. Put

$$y_j = \textit{number of terms among } x_1, \ldots, x_j \textit{ which are } \leqslant x_j \quad (j = 1, \ldots, n).$$

Thus, y_j is the relative rank of the j-th candidate with respect to candidates which appeared at places $1, \ldots, j$ (in a given permutation). The problem is then to devise a policy (i. e., a rule which for each $y_1, \ldots, y_j$ tells us whether to employ the j-th candidate or not) which maximizes the probability of getting the best candidate.

One can show (see, for instance, [1], [2], [3]) that the optimal policy is contained in the class of policies defined as follows: interview $r-1$ candidates without employing any of them, and then employ the first candidate superior to all the preceding ones (if such a candidate appears). Formally, the rule is: stop at the first $k \geqslant r$ with $y_k = 1$. Call such a policy π_r, and let p_r be the probability of employing the best candidate (i. e., the one with rank 1) under policy π_r. The problem is thus reduced to finding an r which maximizes the value p_r.

Clearly, for $r = 1$ the policy π_1 leads to employing the first candidate, and $p_1 = 1/n$. Assume, therefore, that $2 \leqslant r \leqslant n$.

Now, in order that the policy π_r be successful, the value 1 must appear in the permutation $x_1, \ldots, x_n$ at a place k with $r \leqslant k \leqslant n$, and if 1 appears at a place k (probability $1/n$), then the minimum among $x_1, \ldots, x_{k-1}$ must occur at one of the places $1, \ldots, r-1$ (probability $(r-1)/(k-1)$). Summing over k we obtain p_r equal to the right-hand side of (3).

Now, we can also argue as follows. Let

$$T = \min(x_1, \ldots, x_{r-1}).$$

Then

$$\text{(4)} \qquad \mathrm{P}(T = t) = \frac{\binom{n-t}{r-2}}{\binom{n}{r-1}}, \qquad t = 1, \ldots, n-r+2.$$

Given that $T = t$, the policy π_r is successful if $t \geqslant 2$ and in the remaining part of the permutation (at places numbered $r, r+1, \ldots, n$) the term 1 precedes the terms $2, \ldots, t-1$, the probability of this event being $1/(t-1)$. Summing over t we obtain p_r equal to the left-hand side of (3), which completes the proof of (1).

Incidentally, using the right-hand side of (3), one can easily show that the value $r(n)$ which maximizes p_r for a given n satisfies the relation $r(n)/n \to e^{-1}$, and also that $\lim_{n\to\infty} p_{r(n)} = e^{-1}$ (see [1]-[3]). Consequently, we have

$$\lim_{n\to\infty} \max_{2\leqslant r\leqslant n} \frac{1}{\binom{n}{r-1}} \sum_{t=2}^{n-r+2} \frac{\binom{n-t}{r-2}}{t-1} = e^{-1}.$$

We can obtain the proof of (2) if we apply the same two techniques of conditioning in computing the probability that the policy π_r $(r \geqslant 2)$ leads to selection of the candidate with rank z. Call this probability $p_r(z)$, where $z = 1, 2, \ldots$; then $p_r(1) = p_r$ (see (3)).

Clearly, the highest possible value of z for which $p_r(z)$ is positive equals $n-r+1$ (if the first $r-1$ candidates, who are not stopped, have the highest ranks $n, n-1, \ldots, n-(r-2)$, and the next candidate — who is then automatically stopped — has the rank $n-r+1$). The sum $p_r(1)+$ $+p_r(2) + \ldots + p_r(n-r+1)$ equals the probability that *some* candidate will be selected, hence equals $(n-r+1)/n$, for no candidate is selected if and only if 1 appears among the first $r-1$ candidates.

Now, let z be one of the integers $1, \ldots, n-r+1$. In order that the process stop at selecting the candidate with rank z, the following conditions must be satisfied:

(a) The term z must appear in the permutation $x_1, \ldots, x_n$ at some place k with $k = r, r+1, \ldots$, and must precede the terms equal to $1, 2, \ldots$ $\ldots, z-1$. This gives the bound $k \leqslant n-z+1$, and the probability of such an event with z appearing at place k equals

$$\frac{1}{n}\left[\binom{n-k}{z-1} \Big/ \binom{n-1}{z-1}\right].$$

(b) Given that the term z appears at place k and precedes the terms $1, \ldots, z-1$, the term z is chosen if and only if at places $1, \ldots, k-1$ the minimum occurs among terms numbered $1, \ldots, r-1$. The probability of this event is equal, as before, to $(r-1)/(k-1)$. Summing over k, we obtain

$$p_r(z) = \frac{r-1}{n\binom{n-1}{z-1}} \sum_{k=r}^{n-z+1} \frac{\binom{n-k}{z-1}}{k-1}. \tag{5}$$

Now, let $q_r(z) = p_r(1) + \ldots + p_r(z)$ be the probability that the selected candidate will have the rank $\leqslant z$.

Let $P(z\,|\,t)$ be the probability of choosing the candidate with rank $\leqslant z$ given $T = t$, i.e., given that the minimum among the terms numbered $1, \ldots, r-1$ is t.

Clearly, $P(z|1) = 0$, for if $T = 1$ no candidate is chosen at all. For $t > 1$ we have $P(z|t) = 1$ if $z \geqslant t$, since the chosen candidate has always rank strictly less than t. For $z < t$ we have $P(z|t) = z/(t-1)$, because in order to choose a candidate with rank $\leqslant z$ it is necessary and sufficient that among the terms equal $1, \ldots, t-1$ (all of them appearing at places numbered $r, r+1, \ldots, n$) one of the terms $\leqslant z$ must appear as the first.

Thus, using (4), we obtain

$$q_r(z) = \sum_{t=1}^{n-r+2} P(z|t)\mathrm{P}(T=t) = \sum_{t=2}^{z}\mathrm{P}(T=t) + \sum_{t=z+1}^{n-r+2} \frac{z}{t-1}\mathrm{P}(T=t)$$

$$= \frac{1}{\binom{n}{r-1}}\left[\sum_{t=2}^{z}\binom{n-t}{r-2} + z\sum_{t=z+1}^{n-r+2}\frac{\binom{n-t}{r-2}}{t-1}\right]$$

for $z = 1, 2, \ldots, n-r+1$, with the convention that if $z = 1$, the first sum is taken as 0.

Since $p_r(z) = q_r(z) - q_r(z-1)$, we obtain from this formula, after some transformations,

$$p_r(z) = \frac{1}{\binom{n}{r-1}}\sum_{t=z+1}^{n-r+2}\frac{\binom{n-t}{r-2}}{t-1}. \tag{6}$$

Comparing (5) and (6), we obtain after some reductions identity (2).

Now, the difference $p_r(z) - p_r(z+1)$ for $z = 1, \ldots, n-r$ is equal, by formula (6), to

$$\frac{\binom{n-z-1}{r-2}}{z\binom{n}{r-1}} \tag{7}$$

and it follows that the probabilities $p_r(z)$ satisfy the inequalities $p_r(1) > p_r(2) > \ldots$, that is, the most probable choice is that of the candidate with the highest rank (provided any choice is made at all).

Expression (7) must be equal to the difference of terms given by (5), which after some transformations yields as a corollary

$$\frac{1}{\binom{n-1}{z-1}}\sum_{k=r}^{n-z+1}\frac{\binom{n-k}{z-1}}{k-1} - \frac{1}{\binom{n-1}{z}}\sum_{k=r}^{n-z}\frac{\binom{n-k}{z}}{k-1} = \frac{\binom{n-z-1}{r-2}}{z\binom{n-1}{r-2}}$$

$$(z = 1, \ldots, n-r).$$

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