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# Uniqueness of solutions of a mixed problem for parabolic differential-functional equations

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**Abstract.** We consider a system of second order differential-functional equations of the type

(1) $u_t^i(t, x) = f^i(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x), u(t, \cdot)) \quad (i = 1, \ldots, m)$,

where $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m)$, $u_x^i = (u_{x_1}^i, \ldots, u_{x_n}^i)$ and $u_{xx}^i$ is the matrix of second order derivatives with respect to $x$. For a fixed $t \in \langle 0, T)$ we denote by $u(t, \cdot) = (u^1(t, \cdot), \ldots, u^m(t, \cdot))$ an element of the space of continuous functions from the closure of an open and bounded set $G \subset R^n$ in $R^m$. For the regular and parabolic solution of (1) in the cylinder $\langle 0, T) \times G$, satisfying adequate initial and boundary conditions following questions are dealt with: estimate of solution, uniqueness of solution and its continuous dependence on the initial and boundary values and on the right-hand sides of (1). Theorems to be proved are known ([1]) if the right-hand sides of (1) do not depend functionally on $u(t, \cdot)$.

**1. Definitions.** In the time-space $(t, x_1, \ldots, x_n)$ put

$$D = (0, T) \times G \quad \text{and} \quad \Sigma = (0, T) \times \partial G,$$

where $G \subset R^n$ is open and bounded and $T \leqslant +\infty$. For a function $a(t, x)$ defined on $\Sigma$ we denote by $\Sigma_a$ the subset of $\Sigma$ in which $a(t, x) \neq 0$.

Let the functions $a^i(t, x)$ $(i = 1, \ldots, m)$ be defined on $\Sigma$ and suppose that

$$a^i(t, x) \geqslant 0 \quad (i = 1, \ldots, m). \tag{1.1}$$

Assumptions H. *For every $(t, x) \in \Sigma_{a^i}$ be given a direction $l^i(t, x)$, so that $l^i$ is orthogonal to the $t$-axis and some segment, with one extremity at $(t, x)$, of the straight half-line from $(t, x)$ in the direction $l^i$ is contained in $\bar{D}$.*

Let $C_m(\bar{G})$ stand for the space of continuous functions $z(x) = (z^1(x), \ldots, z^m(x))$ from $\bar{G}$ in $R^m$ with the norm

$$\|z\| = \max_i \max\{|z^i(x)|: x \in \bar{G}\}.$$

Let $f^i(t, x, u, q, r, z)$ $(i = 1, \ldots, m)$, where $q = (q_1, \ldots, q_n)$, and $r = (r_{jk})$ is a $m \times m$ real symmetric matrix, be defined for $(t, x) \in D$, $u$, $q$, $r$ arbitrary and $z \in C_m(\bar{G})$.

([1]) J. Szarski, *Differential inequalities*, Warszawa 1967.

A solution $u(t, x)$ of (1) is called *regular solution* in $D$ if: $u$ is continuous in $\bar{D}$, $u_t^i$, $u_x^i$, $u_{xx}^i$ are continuous in $D$ and $u$ satisfies (1) for every $(t, x) \in D$. If in addition, for every $i$ the derivative $du^i/dl^i$ exists at each point $(t, x) \in \Sigma_{a^i}$, then a regular solution is said to be *$\Sigma_a$-regular* in $D$.

A regular solution $u(t, x)$ of (1) is called *parabolic* in $D$ if for any two $m \times m$ real symmetric matrices $r = (r_{jk})$, $\tilde{r} = (\tilde{r}_{jk})$ such that $r \leqslant \tilde{r}$ (i.e. such that the quadratic form $\sum_{j,k}(r_{jk} - \tilde{r}_{jk})\lambda_j\lambda_k$ is non-positive) the inequality

$$f^i\big(t, x, u(t, x), u_x^i(t, x), r, u(t, \cdot)\big) \leqslant f^i\big(t, x, u(t, x), u_x^i(t, x), \tilde{r}, u(t, \cdot)\big)$$

is satisfied for $(t, x) \in D$.

Assumptions $\mathrm{H}_1$. *The function $\sigma(t, y)$ will be said to satisfy Assumptions $\mathrm{H}_1$ if it is non-negative and continuous in the domain $t \geqslant 0$, $y \geqslant 0$.*

For $\eta \geqslant 0$ we denote by $\omega(t, \eta)$ the right-hand maximum solution (see ([1]), § 5) through $(0, \eta)$ of the ordinary equation

$$dy/dt = \sigma(t, y). \tag{1.2}$$

Assumptions $\mathrm{H}_2$. *The function $\sigma(t, y)$ is said to satisfy Assumptions $\mathrm{H}_2$ if in the domain $t > 0$, $y \geqslant 0$ it is non-negative and continuous, $\sigma(t, 0) = 0$ and $y(t) = 0$ is the unique solution of* (1.2) *satisfying the condition*

$$\lim_{t \to 0} y(t) = 0.$$

Note that here $\sigma(t, y)$ is not supposed to be continuous for $t = 0$.

**2. Estimate of solution.**

Theorem 2.1. *Assume the right-hand members $f^i(t, x, u, q, r, z)$ of the system*

$$u_t^i = f^i\big(t, x, u, u_x^i, u_{xx}^i, u(t, \cdot)\big) \qquad (i = 1, \ldots, m) \tag{2.1}$$

*to be defined for $(t, x) \in D$, for arbitrary $u, q, r$ and for $z \in C_m(\bar{G})$. Suppose that*

$$f^i(t, x, u, 0, 0, z)\operatorname{sgn} u^i \leqslant \sigma(t, \max_j\{|u^j|, \|z\|\}), \tag{2.2}$$

*where the function $\sigma$ satisfies Assumptions $\mathrm{H}_1$. For $\eta \geqslant 0$ let the right-hand maximum solution $\omega(t, \eta)$ of equation* (1.2) *trough $(0, \eta)$ be defined in an interval $\langle 0, \gamma)$.*

*Let the functions $a^i(t, x)$ satisfying inequalities* (1.1) *and the directions $l^i(t, x)$ satisfying Assumptions* H *be given on $\Sigma$. Let $\beta^i(t, x)$ be defined on $\Sigma_{a^i}$ and assume that inequalities*

$$\beta^i(t, x) > B^i \geqslant 0 \qquad \text{on } \Sigma_{a^i} \tag{2.3}$$

*hold true.*

*Suppose finally that* $u(t,x) = \big(u^1(t,x), \ldots, u^m(t,x)\big)$ *is a* $\Sigma_a$*-regular, parabolic solution of* (2.1) *in* $D$, *satisfying initial inequalities*

$$|u^i(0,x)| \leqslant \eta \quad \textit{for } x \in \bar{G} \tag{2.4}$$

*and boundary inequalities*

$$\begin{aligned} |\beta^i(t,x)u^i(t,x) - a^i(t,x)\,du^i/dl^i| &\leqslant B^i\omega(t,\eta) \quad &\textit{for } (t,x) \in \Sigma_{a^i}, \\ |u_i(t,x)| &\leqslant \omega(t,\eta) \quad &\textit{for } (t,x) \in \Sigma \setminus \Sigma_{a^i}. \end{aligned} \tag{2.5}$$

*Under these assumptions inequalities*

$$|u^i(t,x)| \leqslant \omega(t,\eta) \quad (i = 1, \ldots, m) \tag{2.6}$$

*hold true in* $D$ *for*

$$0 \leqslant t < \delta = \min(\gamma, T). \tag{2.7}$$

Proof. Put

$$M^i(t) = \max\{u^i(t,x)\colon x \in \bar{G}\}, \qquad N^i(t) = \max\{-u^i(t,x)\colon x \in \bar{G}\},$$

$$W(t) = \max_i \max\{|u^i(t,x)|\colon x \in \bar{G}\} = \|u(t,\cdot)\|.$$

The function $W(t)$ is continuous in (2.7) (see ([1]), § 34) and by (2.4) we have

$$W(0) \leqslant \eta.$$

Inequalities (2.6), which are to be proved, are obviously equivalent with

$$W(t) \leqslant \omega(t,\eta) \quad \text{for } 0 \leqslant t < \delta.$$

Now, the last inequality will be proved if we show (see ([1]), § 14) that the differential inequality

$$D_- W(t) \leqslant \sigma\big(t, W(t)\big) \tag{2.8}$$

holds true in the set

$$E = \{t \in (0,\gamma)\colon W(t) > \omega(t,\eta)\}. \tag{2.9}$$

To prove (2.8) in (2.9) fix a $\tilde{t} \in E$; then, we have

$$W(\tilde{t}) > \omega(\tilde{t},\eta). \tag{2.10}$$

There is an index $j$ and a point $\tilde{x} \in \bar{G}$ (see ([1]), § 34), so that either

$$W(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t},\tilde{x}), \qquad D_- W(\tilde{t}) \leqslant D^- M^j(\tilde{t}), \tag{2.11}$$

or

$$W(\tilde{t}) = N^j(\tilde{t}) = -u^j(\tilde{t},\tilde{x}), \qquad D_- W(\tilde{t}) \leqslant D^- N^j(\tilde{t}). \tag{2.12}$$

Suppose we have, for instance, (2.12). Then, in view of (1.1), (2.3), (2.5) and (2.10) we conclude (see ([1]), § 47) that $(\tilde{t}, \tilde{x})$ is an interior point of $D$. Since the function $-u(\tilde{t}, \tilde{x})$ attains its maximum at the interior point $\tilde{x}$ and is of class $C^2$ in its neighbourhood, we have

$$u_x^j(\tilde{t}, \tilde{x}) = 0, \tag{2.13}$$

$$-u_{xx}^j(\tilde{t}, \tilde{x}) \leqslant 0. \tag{2.14}$$

In view of (see ([1]), § 33)

$$D^- N^j(\tilde{t}) \leqslant -u_t^j(\tilde{t}, \tilde{x}),$$

we get by (2.12)

$$D_- W(\tilde{t}) \leqslant -u_t^j(\tilde{t}, \tilde{x}) = -f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), u_{xx}^j(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big). \tag{2.15}$$

Since by (2.12) we have

$$\operatorname{sgn} u^j(\tilde{t}, \tilde{x}) = -1,$$

it follows from (2.15) by (2.13)

$$\begin{aligned} D_- W(\tilde{t}) \leqslant {} & \big[f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), 0, 0, u(\tilde{t}, \cdot)\big) - f^j\big(\tilde{t}, \tilde{x}, \tilde{u}(\tilde{t}, \tilde{x}), 0, \\ & u_{xx}^j(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big)\big] + f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), 0, 0, u(\tilde{t}, \cdot)\big) \operatorname{sgn} u^j(\tilde{t}, \tilde{x}). \end{aligned} \tag{2.16}$$

The difference in brackets is, by the parabolicity of solution $u(t, x)$ and by (2.14), non-positive. Hence, from (2.16) and (2.2) we obtain

$$D_- W(\tilde{t}) \leqslant \sigma\big(\tilde{t}, \max_k \{|u^k(\tilde{t}, \tilde{x})|, \|u(\tilde{t}, \cdot)\|\}\big). \tag{2.17}$$

But, by the definition of $W(t)$ and by (2.12) we have

$$\max_k \{|u^k(\tilde{t}, \tilde{x})|, \|u(\tilde{t}, \cdot)\|\} = W(\tilde{t}),$$

whence inequality (2.17) implies

$$D_- W(\tilde{t}) \leqslant \big(\tilde{t}, W(\tilde{t})\big),$$

what was to be proved.

### 3. Comparison theorem.

THEOREM 3.1. *Suppose the right-hand members* $f^i(t, x, u, q, r, z)$ *of system* (2.1) *and* $g^i(t, x, u, q, r, z)$ *of system*

$$u_t^i = g^i\big(t, x, u, u_x^i, u_{xx}^i, u(t, \cdot)\big) \quad (i = 1, \ldots, m) \tag{3.1}$$

*are defined for* $(t, x) \in D$, *for arbitrary* $u, q, r$ *and* $z \in C_m(\bar{G})$. *Assume that*

$$\begin{aligned} & [f^i(t, x, u, q, r, z) - g^i(t, x, \tilde{u}, q, r, \tilde{z})] \operatorname{sgn}(u^i - \tilde{u}^i) \\ & \qquad \leqslant \sigma\big(\tilde{t}, \max_k \{|u^k - \tilde{u}^k|, \|z - \tilde{z}\|\}\big), \end{aligned} \tag{3.2}$$

*where the function* $\sigma$ *satisfies Assumptions* $H_1$. *For* $\eta \geqslant 0$ *let the right-hand maximum solution* $\omega(t, \eta)$ *of equation* (1.2) *through* $(0, \eta)$ *be defined in an interval* $\langle 0, \gamma)$.

*Let functions* $a^i(t, x)$ *satisfying* (1.1), *the directions* $l^i(t, x)$ *satisfying Assumptions* H *and* $\beta^i(t, x)$ *satisfying* (2.3) *be given on* $\Sigma$.

*Suppose that* $u(t, x) = (u^1(t, x), \ldots, u^m(t, x))$ *is a* $\Sigma_a$*-regular, parabolic solution of* (2.1) *in* $D$ *and* $v(t, x) = (v^1(t, x), \ldots, v^m(t, x))$ *is a* $\Sigma_a$*-regular solution of* (3.1) *in* $D$.

*Assume finally initial inequalities*

$$|u^i(0, x) - v^i(0, x)| \leqslant \eta \quad \text{for } x \in \bar{G} \tag{3.3}$$

*and boundary inequalities*

$$\begin{aligned} &|\beta^i(t, x)[u^i(t, x) - v^i(t, x)] - a^i(t, x) d(u^i - v^i)/dl^i| \leqslant B^i \omega(t, \eta) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t, x) \in \Sigma_{ai}, \\ &|u^i(t, x) - v^i(t, x)| \leqslant \omega(t, \eta) \quad \text{for } (t, x) \in \Sigma \setminus \Sigma_{ai} \end{aligned} \tag{3.4}$$

*to hold true.*

*Under these assumptions we have*

$$|u^i(t, x) - v^i(t, x)| \leqslant \omega(t, \eta) \quad (i = 1, \ldots, m) \tag{3.5}$$

*in* $D$ *for* $t$ *in the interval* (2.7).

Proof. We put

$$\begin{aligned} M^i(t) &= \max\{u^i(t, x) - v^i(t, x) \colon x \in \bar{G}\}, \\ N^i(t) &= \max\{v^i(t, x) - u^i(t, x) \colon x \in \bar{G}\}, \\ W(t) &= \|u(t, \cdot) - v(t, \cdot)\|. \end{aligned} \tag{3.6}$$

The function $W(t)$ is continuous in (2.7) (see ([1]), § 34) and by (3.3) we have $W(0) \leqslant \eta$.

Inequalities (3.5) to be proved are equivalent with

$$W(t) \leqslant \omega(t, \eta) \quad \text{for } 0 \leqslant t < \delta.$$

Now, to prove the last inequality it is sufficient (see ([1]), § 14) to show that the differential inequality (2.8) holds true in the set (2.9). To prove (2.8) in (2.9) for $W(t)$ defined by (3.6), fix a $\tilde{t} \in E$; then, we have (2.10) and there is an index $j$ and a point $\tilde{x} \in \bar{G}$ (see ([1]), § 34) such that either

$$W(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{x}) - v^j(\tilde{t}, \tilde{x}), \quad D_- W(\tilde{t}) \leqslant D^- M^j(\tilde{t}), \tag{3.7}$$

or

$$W(\tilde{t}) = N^j(\tilde{t}) = v^j(\tilde{t}, \tilde{x}) - u^j(\tilde{t}, \tilde{x}), \quad D_- W(\tilde{t}) \leqslant D^- N^j(\tilde{t}). \tag{3.8}$$

Suppose we have, for instance, (3.7). Then, in view of (1.1), (2.3), (3.4) and (2.10) we conclude (see ([1]), § 47) that $(\tilde{t}, \tilde{x})$ is an interior point of $D$.

Since the function $u^j(t, x)-v^j(t, x)$ attains its maximum at the interior point $\tilde{x}$ and is of class $C^2$ in its neighbourhood, we have

(3.9) $$u_x^j(\tilde{t}, \tilde{x}) = v_x^j(\tilde{t}, \tilde{x}),$$

(3.10) $$u_{xx}^j(\tilde{t}, \tilde{x}) \leqslant v_{xx}^j(\tilde{t}, \tilde{x}).$$

In view of (see ([1]), § 33)

$$D^- M^j(\tilde{t}) \leqslant u_t^j(\tilde{t}, \tilde{x}) - v_t^j(\tilde{t}, \tilde{x}),$$

we get by (3.7)

$$\begin{aligned} D_- W(\tilde{t}) &\leqslant u_t^j(\tilde{t}, \tilde{x}) - v_t^j(\tilde{t}, \tilde{x}) \\ &= f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), u_{xx}^j(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big) - \\ &\quad - g^j\big(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), v_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), v(\tilde{t}, \cdot)\big). \end{aligned}$$

From the last inequality it follows by (3.9) that

$$\begin{aligned} D_- W(\tilde{t}) \leqslant {} & \big[f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), u_{xx}^j(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big) - \\ & - f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big)\big] + \\ & + \big[f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big) - \\ & - g^j\big(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), v(\tilde{t}, \cdot)\big)\big]. \end{aligned}$$

The first difference in brackets is, by the parabolicity of solution $u(t, x)$ and by (3.10), non-positive. By (3.7) we have

$$\operatorname{sgn}[u^j(\tilde{t}, \tilde{x}) - v^j(\tilde{t}, \tilde{x})] = 1.$$

Hence, we get in virtue of (3.2)

$$D_- W(\tilde{t}) \leqslant \sigma\big(\tilde{t}, \max_k \{|u^k(\tilde{t}, \tilde{x}) - v^k(\tilde{t}, \tilde{x})|, \|u(\tilde{t}, \cdot) - v(\tilde{t}, \cdot)\|\}\big).$$

But, by the definition of $W(t)$ and by (3.7), we obtain

$$\max_k \{|u^k(\tilde{t}, \tilde{x}) - v^k(\tilde{t}, \tilde{x})|, \|u(\tilde{t}, \cdot) - v(\tilde{t}, \cdot)\|\} = W(\tilde{t}),$$

whence the last inequality implies (2.8) for $t = \tilde{t}$, what was to be proved.

### 4. Uniqueness criteria.

THEOREM 4.1. *Let the right-hand sides $f^i(t, x, u, q, r, z)$ of system* (2.1) *be defined for $(t, x) \in D$, for arbitrary $u, q, r$ and for $z \in C_m(\bar{G})$. Assume that*

(4.1) $$[f^i(t, x, u, q, r, z) - f^i(t, x, \tilde{u}, q, r, \tilde{z})] \operatorname{sgn}(u^i - \tilde{u}^i) \leqslant \sigma(t, \max_k \{|u^k - \tilde{u}^k|, \|z - \tilde{z}\|\}),$$

*where the function $\sigma(t, y)$ satisfies Assumptions* $H_1$.

*Let the functions* $\alpha^i(t, x)$ *satisfying* (1.1), *the directions* $l^i(t, x)$ *satisfying Assumptions* H, $\beta^i(t, x)$ *satisfying* (2.3) *and the functions* $\psi^i(t, x)$ *be given on* $\Sigma$. *Assume the functions* $\varphi^i(x)$ *to be defined on* $\bar{G}$, *and introduce the following initial and boundary conditions:*

$$(4.2)\qquad u^i(0, x) = \varphi^i(x) \quad \text{for } x \in \bar{G},$$

$$(4.3)\qquad \begin{aligned} \beta^i(t, x)u^i(t, x) - \alpha^i(t, x)du^i/dl^i &= \psi^i(t, x) \quad \text{for } (t, x) \in \Sigma_{\alpha^i}, \\ u^i(t, x) &= \psi^i(t, x) \quad \text{for } (t, x) \in \Sigma \setminus \Sigma_{\alpha^i}. \end{aligned}$$

*Let, moreover,* $\sigma(t, 0) = 0$ *and* $y(t) = 0$ *be the unique solution of* (1.2) *through the point* (0, 0), *i.e.*

$$(4.4)\qquad \omega(t, 0) = 0.$$

*Then, the mixed problem* (4.1), (4.2), (4.3) *admits at most one parabolic* $\Sigma_\alpha$*-regular solution in* $D$. *If such a solution exists, then it is unique in the class of* $\Sigma_\alpha$*-regular solutions.*

Proof. Suppose $u(t, x)$ is a parabolic and $\Sigma_\alpha$-regular solution of (4.1), (4.2), (4.3) and $v(t, x)$ is a $\Sigma_\alpha$-regular solution in $D$. Then, they satisfy all the assumptions of Theorem 3.1 with $g^i = f^i$, $\eta = 0$ and $\gamma = +\infty$. Therefore, we obtain in $D$

$$|u^i(t, x) - v^i(t, x)| \leqslant \omega(t, 0),$$

whence, by (4.4), it follows that

$$u^i(t, x) = v^i(t, x) \quad \text{for } (t, x) \in D.$$

Now, we will prove a slightly more general uniqueness criterion.

THEOREM 4.2. *Let the right-hand sides of system* (2.1) *satisfy inequalities* (4.1) *with the function* $\sigma(t, y)$ *satisfying Assumptions* $\mathrm{H}_2$. *This being assumed the mixed problem* (4.1), (4.2), (4.3) *admits at most one parabolic,* $\Sigma_\alpha$*-regular solution; this solution, if it exists, is unique in the class of* $\Sigma_\alpha$*-regular solutions.*

Proof. For $u(t, x)$ and $v(t, x)$ being two solutions of (4.1), (4.2), (4.3) parabolic and $\Sigma_\alpha$-regular respectively $\Sigma_\alpha$-regular in $D$, introduce the notations at the beginning of the proof of Theorem 3.1. The assertion of Theorem 4.2 is then equivalent with

$$(4.5)\qquad W(t) = 0 \quad \text{for } 0 \leqslant t < T.$$

Now, quite similarly as in the proof of Theorem 3.1, we show that $W(t)$ is continuous in $\langle 0, T)$,

$$(4.6)\qquad W(0) = 0$$

and the differential inequality (2.8) is satisfied in the set

$$\{t \in (0, T):\ W(t) > 0\}.$$

From the above facts it follows that (see ([1]), § 14) (4.5) holds true.

Remark. Theorem 4.2 does not follow from Theorem 4.1 since here the function $\sigma(t, y)$ is not supposed to be continuous for $t = 0$.

**5. Continuous dependence of the solution.**

THEOREM 5.1. *Suppose all the assumptions of Theorem* 4.1 *to hold true and*

$$\beta^i(t, x) > B^i > 0. \tag{5.1}$$

*Let $u(t, x)$ be the parabolic, $\Sigma_a$-regular solution of* (4.1), (4.2), (4.3) *in $D$. Denote by $v(t, x, \varepsilon)$ a $\Sigma_a$-regular solution in $D$ of the system*

$$v_t^i = f_\varepsilon^i\big(t, x, v, v_x^i, v_{xx}^i v(t, \cdot)\big) \quad (i = 1, \ldots, m), \tag{5.2}$$

*satisfying initial conditions*

$$v^i(0, x) = \varphi^i(x, \varepsilon) \quad \textit{for } x \in \bar{G} \tag{5.3}$$

*and boundary conditions*

$$\begin{aligned} \beta^i(t, x) v^i(t, x) - a^i(t, x)\, dv^i/dl^i &= \psi^i(t, x, \varepsilon) \quad \textit{for } (t, x) \in \Sigma_a i, \\ v^i(t, x) &= \psi^i(t, x, \varepsilon) \quad \textit{for } (t, x) \in \Sigma \setminus \Sigma_a i. \end{aligned} \tag{5.4}$$

*Suppose that*

$$|f^i(t, x, u, q, r, z) - f_\varepsilon^i(t, x, u, q, r, z)| \leqslant \varepsilon, \tag{5.5}$$

$$|\varphi^i(x, \varepsilon) - \varphi^i(x)| \leqslant \varepsilon, \tag{5.6}$$

$$|\psi^i(t, x, \varepsilon) - \psi^j(t, x)| \leqslant \varepsilon. \tag{5.7}$$

*Under these assumptions*

$$\lim_{\varepsilon \to 0} v(t, x, \varepsilon) = u(t, x) \quad \textit{uniformly in } D. \tag{5.8}$$

Proof. By (4.1) and (5.5) we have

$$\begin{aligned} &[f^i(t, x, u, q, r, z) - f_\varepsilon^i(t, x, \tilde{u}, q, r, \tilde{z})]\operatorname{sgn}(u^i - \tilde{u}^i), \\ &\qquad \sigma(t, \max_k \{|u^k - \tilde{u}^k|, \|z - \tilde{z}\|) + \varepsilon. \end{aligned} \tag{5.9}$$

Denote by $\omega_\varepsilon(t, \eta)$ the right-hand maximum solution through $(0, \eta)$ of the ordinary differential equation

$$dy/dt = \sigma(t, y) + \varepsilon.$$

It is known (see ([1]), § 10) that for $\varepsilon > 0$ and $\eta \geqslant 0$ sufficiently small $\omega_\varepsilon(t, \eta)$ exists in the interval $\langle 0, T)$ and

$$\lim_{\varepsilon \to 0, \eta \to 0} \omega_\varepsilon(t, \eta) = \omega(t, 0),$$

whence, by assumption (4.4) we obtain

$$\lim_{\varepsilon \to 0, \eta \to 0} \omega_\varepsilon(t, \eta) = 0 \quad \text{uniformly for } 0 \leqslant t < T. \tag{5.10}$$

Since $\omega_\varepsilon(t, \eta)$ is increasing with regard to $t$ and $\omega_\varepsilon(0, \eta) = \eta$, we easily check that putting

$$B = \min_k B^k > 0, \quad \eta = \max(\varepsilon, \varepsilon/B),$$

we get

$$\varepsilon \leqslant B^i \omega_\varepsilon(t, \eta),$$

whence in view of (5.6) and (5.7)

$$|\varphi^i(x, \varepsilon) - \varphi^i(x)| \leqslant \eta, \tag{5.11}$$

$$|\psi^i(t, x, \varepsilon) - \psi^i(t, x)| \leqslant \min\big(\omega_\varepsilon(t, \eta), B^i \omega_\varepsilon(t, \eta)\big). \tag{5.12}$$

In virtue of (5.9), (4.2), (4.3), (5.3), (5.4), (5.11) and (5.12) we check that all assumptions of Theorem 3.1 hold true with $g^i = f^i$, $v^i = v^i(t, x, \varepsilon)$, $\eta = \max(\varepsilon, \varepsilon/B)$, $\gamma = T$, with $\sigma$ replaced by $\sigma + \varepsilon$ and $\omega(t, \eta)$ replaced by $\omega_\varepsilon(t, \eta)$. Therefore, we have in $D$

$$|u^i(t, x) - v^i(t, x, \varepsilon)| \leqslant \omega_\varepsilon(t, \eta) \quad (i = 1, \ldots, m).$$

But, $\eta = \max(\varepsilon, \varepsilon/B) \to 0$ as $\varepsilon \to 0$. Hence, the last inequality together with (5.10) implies (5.8).

**6. Final remarks.** All theorems of this paper are valid for more general domains than the cylinder $D$. Indeed, the cylinder may be replaced by a region $\Omega$ having following properties (see ([1]), § 33):

(a) $\Omega$ is open, contained in the zone $0 < t < T$, and the intersection of $\bar{\Omega}$ with any closed zone $0 \leqslant t \leqslant t_1 < T$ is bounded,

(b) the projection $S_t$ on the space $(x_1, \ldots, x_n)$ of the intersection of $\bar{\Omega}$ with the plane $t = t_1$ is, for any $t_1 \in \langle 0, T)$, non-empty,

(c) the point $(t, x)$ being arbitrarily fixed in $\bar{\Omega}$, to every sequence $t_\nu$ such that $t_\nu \in \langle 0, T)$ and $t_\nu \to t$, there is a sequence $x^\nu$ so that $x^\nu \in S_{t_\nu}$ and $x^\nu \to x$.

If we denote now by $\Sigma$ that part of $\partial\Omega$ which is contained in the open zone $0 < t < T$ and if we assume that the right-hand sides $f^i(t, x, u, q, r, z)$ of (2.1) are defined for $(t, x) \in \Omega$ for $u, q, r$ arbitrary and $z \in C_m(S_t)$, then all the proofs of theorems remain unchanged.

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