ANNALES
POLONICI MATHEMATICI
XLI (1983)

Intersection of essential cluster sets

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Abstract. Let $f\colon H \to W$, where W is a topological space and H is the open upper half plane above the real line R. The essential cluster set of f at $x \in R$, denoted by $C_e(f, x)$, is the set of all $w \in W$ such that for every open subset U of W containing w, x is a point of positive upper outer density of the set $f^{-1}(U)$. The directional essential cluster set $C_e(f, x, \theta)$ of f at x in the direction θ is the set of all $w \in W$ such that for every open subset U of W containing w, x is a point of positive upper outer density (linear) of the set $f^{-1}(U) \cap L_\theta(x)$, where $L_\theta(x)$ is the half ray in H emanating from x and having direction θ.

It is proved that if W is compact, normal and second countable and if f is arbitrary, then

(i) for fixed $\theta \in (0, \pi)$

$$C_e(f, x, \theta) \cap C_e(f, x) \neq \emptyset$$

at almost every $x \in R$, and

(ii) for fixed $x \in R$,

$$C_e(f, x, \theta) \cap C_e(f, x) \neq \emptyset$$

for almost every $\theta \in (0, \pi)$.

1. Let H denote the open upper half plane above the real line R and let z and x denote points on H and R, respectively. For a set A, $\mu^* A$ denotes the (Lebesgue) outer measure, linear or planar, according as A is linear or planar. Let, for $x \in R$, $\theta \in (0, \pi)$, and $r > 0$,

$$L_\theta(x) = \{z\colon z \in H;\ \arg(z-x) = \theta\},$$
$$S(x, r) = \{z\colon z \in H;\ |z-x| < r\},$$

and

$$L_\theta(x, r) = L_\theta(x) \cap S(x, r).$$

Let $E \subset H$. Then the upper outer density $\bar{d}^*(E, x)$ and lower outer density $\underline{d}^*(E, x)$ of the set E at x are defined by

$$\bar{d}^*(E, x) = \limsup_{r \to 0} \frac{\mu^*(E \cap S(x, r))}{\mu S(x, r)} \tag{1}$$

and

$$\underline{d}^* G(E, x) = \liminf_{r \to 0} \frac{\mu^*(E \cap S(x, r))}{\mu S(x, r)}, \tag{2}$$

respectively. The definitions of the directional upper [lower] outer density $\bar{d}^*_\theta(E, x)$ [$\underline{d}^*_\theta(E, x)$] of E at x in the direction θ are obtained from (1) [(2)] by replacing $S(x, r)$ by $L_\theta(x, r)$. Furthermore, if $\bar{d}^*(E, x) = \underline{d}^*(E, x)$, the common value is called the *outer density* of E at x and is denoted by $d^*(E, x)$. Similar is the case for directional outer density $d^*_\theta(E, x)$ of E at x in the direction θ. In particular, if the sets concerned are measurable, then μ^*, $\bar{d}^*$ $\bar{d}^*_\theta$, $\underline{d}^*$ and $\underline{d}^*_\theta$ will be replaced by μ, $\bar{d}$, $\bar{d}_\theta$, $\underline{d}$ and $\underline{d}_\theta$, respectively.

A set E is said to *have the Baire property* if $E = G \triangle Q = (G \cup Q)\setminus(G \cap Q)$, where G is open and Q is of the first category. If, in particular, the set Q is countable, then E will be said to *have the restricted Baire property*.

A function $f\colon H \to W$, where W is a topological space, is said to *have restricted Baire property* if for every open set $U \subset W$, $f^{-1}(U)$ has restricted Baire property.

Let $f\colon H \to W$, where W is a topological space. Then the essential cluster set $C_e(f, x)$ of f at x is the set of all $\omega \in W$ such that for every open set U of W, containing ω, $\bar{d}^*(f^{-1}(U), x) > 0$. The definition of directional essential cluster set $C_e(f, x, \theta)$ of f at x in the direction θ is obtained from the definition of $C_e(f, x)$, simply replacing $\bar{d}^*(f^{-1}(U), x)$ by $\bar{d}^*_\theta(f^{-1}(U), x)$.

Throughout the paper W is taken to be a topological space having a countable basis; whenever other restrictions are necessary for W only those will be mentioned therein. The closure of a set A will be denoted by $\bar{A}$.

2. Goffman and Sledd proved that if $f\colon H \to R$ is measurable and if $\theta \in (0, \pi)$ is a fixed direction, then except a measure zero set of points x on R, $C_e(f, x) \subset C_e(f, x, \theta)$. If further f is continuous, then the exceptional set is also of the first category, [2], Theorem 2. Belna, Evans and Humke also proved that if $f\colon H \to W$ is measurable, where W is the Riemann sphere, then except a first category set of measure zero on R, the set $\{\theta\colon \theta \in (0, \pi);\ C_e(f, x) \subset C_e(f, x, \theta)\}$ has measure equal to π. If further f is continuous, then except a first category set of measure zero on R the above set is also residual in $(0, \pi)$, [1], Theorem 2. Authors in [1] also showed that the last part of their result is not true for arbitrary functions and they raised a question whether the first part of their theorem could be proved for arbitrary functions. The following example shows that the first part of either of the above theorems is not true for arbitrary functions.

EXAMPLE 1. There exists a non-measurable function $f\colon H \to R$ such that at every point $x \in R$,

$$C_e(f, x) \not\subset C_e(f, x, \theta)$$

for every direction $\theta \in (0, \pi)$.

Proof. In [6] Sierpiński constructed a non-measurable set S in H with the following properties:

(i) every line in the plane contains almost two points of S, and
(ii) for every measurable set E, $\mu^*(S \cap E) = \mu E$.

Let f be the characteristic function of S. Then evidently

$$1 \in C_e(f, x) \quad \text{and} \quad C_e(f, x, \theta) = \{0\}$$

for each $x \in R$ and for every $\theta \in (0, \pi)$.

In this note we prove in Theorem 1 (resp. Theorem 2) that if f is arbitrary, then for fixed $\theta \in (0, \pi)$ (resp. $x \in R$) $C_e(f, x, \theta)$ intersects $C_e(f, x)$ except a set of points x (resp. θ) of measure zero on R (resp. $(0, \pi)$). In Theorem 3 we also extend the second part of the theorem of Goffman and Sledd cited above to a larger class of functions having restricted Baire property.

3. To prove our results we need the following lemmas.

LEMMA 1. *Let f: $H \to W$ be arbitrary, where W is a compact space, and let $F \subset W$ be a closed set such that $C_e(f, x) \cap F = \emptyset$. Then*

$$d^*(f^{-1}(F), x) = 0.$$

Proof. Since F is closed and disjoint from $C_e(f, x)$ there exist points $w_1, w_2, \ldots, w_k$ in F and corresponding neighbourhoods $V_1, V_2, \ldots, V_k$ of $w_1, w_2, \ldots, w_k$ such that

$$F \subset \bigcup_{i=1}^{k} V_i$$

and

$$\lim_{r \to 0} \frac{\mu^*[f^{-1}(V_i) \cap S(x, r)]}{\mu S(x, r)} = 0$$

for $i = 1, 2, \ldots, k$. Hence

$$\lim_{r \to 0} \frac{\mu^*[f^{-1}(F) \cap S(x, r)]}{\mu S(x, r)} = 0$$

completing the proof.

LEMMA 2. *Let f: $H \to W$ be arbitrary, where W is a compact space, and let $G \subsetneq W$ be an open set such that $C_e(f, x, \theta) \subset G$. Then*

$$d_\theta^*(f^{-1}(G), x) = 1.$$

Since $W \setminus G$ is closed, as in Lemma 1, $d_\theta^*(f^{-1}(W \setminus G), x) = 0$ and so the proof of Lemma 2 is clear.

LEMMA 3. *Let $E \subset H$ be arbitrary and let $\theta \in (0, \pi)$ be a fixed direction; then the set*

$$\mathscr{D}(E, \theta) = \{x\colon\ x \in R;\ \underline{d}_\theta^*(E, x) > 0;\ \underline{d}^*(E, x) = 0\}$$

is of measure zero.

Proof. Let S be a measurable cover of E such that $\mu S = \mu^* E$. Then $S \cap L_\theta(x)$ is measurable for almost all $x \in R$. Let

$$\mathscr{I}(S) = \{x\colon\ x \in R;\ \underline{d}_\theta(S, x) > 0;\ \underline{d}(S, x) = 0\}.$$

Then

$$\mathscr{D}(E, \theta) \subset \mathscr{I}(S) \cup \mathscr{K}, \tag{1}$$

where $\mathscr{K}$ is the set of all $x \in R$ for which $S \cap L_\theta(x)$ is non-measurable.

By Lemma 3 of [5], the set $\mathscr{I}(S)$ is of measure zero and since $\mathscr{K}$ is also of measure zero, by (1) $\mathscr{D}(E, \theta)$ is of measure zero.

LEMMA 4. *Let $E \subset H$ be arbitrary and let for $x \in R$, $\underline{d}^*(E, x) = 0$. Then the set*

$$\vartheta(E, x) = \{\theta\colon\ \theta \in (0, \pi);\ \underline{d}^*_\theta(E, x) > 0\}$$

is of measure zero.

Proof. By considering a measurable cover S of E such that $\mu S = \mu^* E$ and applying Lemma 6 of [5] the proof is completed as in Lemma 3.

LEMMA 5. *If $E \subset H$ has restricted Baire property and if $\theta \in (0, \pi)$ is a fixed direction, then the set*

$$\mathscr{S}(E, \theta) = \{x\colon\ x \in R;\ \bar{d}_\theta(E, x) = 0;\ \bar{d}(E, x) > 0\}$$

is a first category set of measure zero.

Proof. Let $E = G \triangle Q$, where G is open and Q is countable. Let

$$\mathscr{I}(G) = \{x\colon\ x \in R;\ \bar{d}_\theta(G, x) = 0;\ \bar{d}(G, x) > 0\}.$$

Then clearly

$$\mathscr{S}(E, \theta) \subset \mathscr{I}(G). \tag{1}$$

By Lemma 2 and Lemma 3 of [2] it follows that the set $\mathscr{I}(G)$ is a first category set of measure zero. Hence by (1) $\mathscr{S}(E, \theta)$ is a first category set of measure zero.

THEOREM 1. *If $f\colon H \to W$ is arbitrary, where W is compact and normal space, and if $\theta \in (0, \pi)$ is a fixed direction, then except a measure zero set of points x on R*

$$C_e(f, x, \theta) \cap C_e(f, x) \neq \emptyset.$$

Proof. Let

$$\mathscr{D}(\theta) = \{x\colon\ x \in R;\ C_e(f, x, \theta) \cap C_e(f, x) = \emptyset\}$$

and let $\mathscr{B}$ be a countable basis of open sets for the topology of W and also let $\mathscr{G}$ be the collection of all sets G expressible as a finite union of sets in $\mathscr{B}$. Then $\mathscr{G}$ is a countable collection of sets G. For $G \in \mathscr{G}$, let

$$\mathscr{D}(f^{-1}(G), \theta) = \{x\colon\ x \in R;\ \underline{d}^*_\theta(f^{-1}(G), x) > 0;\ \underline{d}^*(f^{-1}(G), x) = 0\}.$$

Let $x_0 \in \mathscr{S}(\theta)$. Then the closed sets $C_e(f, x_0, \theta)$ and $C_e(f, x_0)$ are disjoint. This fact together with the fact that W is compact and normal ensure that there is $G_0 \in \mathscr{G}$ such that $C_e(f, x_0, \theta) \subset G_0$ and $\bar{G}_0 \cap C_e(f, x_0) = \emptyset$. Hence Lemma 2 and Lemma 1 respectively imply that $d_\theta^*(f^{-1}(G_0), x_0) = 1$ and $d^*(f^{-1}(G_0), x_0) = 0$. Thus $x_0 \in \mathscr{S}(f^{-1}(G_0), \theta)$ and consequently

$$\mathscr{S}(\theta) \subset \bigcup \{\mathscr{D}(f^{-1}(G), \theta) \colon G \in \mathscr{G}\}. \tag{1}$$

By Lemma 3 each of the sets $\mathscr{S}(f^{-1}(G), \theta)$ is of measure zero and since $\mathscr{G}$ is a countable collection, by (1), $\mathscr{S}(\theta)$ is of measure zero. This completes the proof of Theorem 1.

THEOREM 2. *If $f\colon H \to W$ is arbitrary, where W is a compact and normal space, then at each point x on R, except a measure zero set of directions $\theta \in (0, \pi)$*

$$C_e(f, x, \theta) \cap C_e(f, x) \neq \emptyset.$$

Proof. Let, for $x \in R$,

$$\vartheta(x) = \{\theta\colon \theta \in (0, \pi);\ C_e(f, x, \theta) \cap C_e(f, x) = \emptyset\}$$

and let $\mathscr{B}$ and $\mathscr{G}$ be the same as taken in the proof of Theorem 1. For $G \in \mathscr{G}$, let

$$\vartheta(f^{-1}(G), x) = \{\theta\colon \theta \in (0, \pi);\ \underline{d}_\theta^*(f^{-1}(G), x) > 0\}.$$

Let $\theta_0 \in \vartheta(x)$. Then the closed sets $C_e(f, x, \theta_0)$ and $C_e(f, x)$ are disjoint. This fact together with the fact that W is compact and normal ensures that there is $G_0 \in \mathscr{G}$ such that $C_e(f, x, \theta_0) \subset G_0$ and $C_e(f, x) \cap \bar{G}_0 = \emptyset$. Hence Lemma 2 and Lemma 1 respectively imply that $d_{\theta_0}^*(f^{-1}(G_0), x) = 1$ and $d^*(f^{-1}(G_0), x) = 0$. Thus $\theta_0 \in \vartheta(f^{-1}(G_0), x)$ and consequently

$$\vartheta(x) \subset \bigcup \{\vartheta(f^{-1}(G), x)\colon G \in \mathscr{G}\}. \tag{1}$$

By Lemma 4, each of the sets $\vartheta(f^{-1}(G), x)$ in the union (1) is of measure zero and hence by (1) $\vartheta(x)$ is of measure zero. This completes the proof of Theorem 2.

We now include two examples to ensure that the exceptional sets of Theorem 1 and Theorem 2 may be any set of measure zero.

EXAMPLE 2. Let $E \subset R$ be any set of measure zero. Also let

$$P = H \setminus \bigcup \{L_{\pi/2}(x)\colon x \in E\}.$$

Let f be the characteristic function of P. Then we have

(i) $\quad C_e(f, x) = \{1\}$ for all $x \in R$,

(ii) $\quad C_e(f, x, \pi/2) = \{0\}$ for $x \in E$,

$\quad = \{1\}$ for $x \in R \setminus E$.

This example shows that the exceptional set of Theorem 1 need not be of the first category.

EXAMPLE 3. Let $x \in R$ be fixed and let $\Theta(x)$ be any set of direction $\theta \in (0, \pi)$ of measure zero. Let Q be a set defined by

$$Q = H \backslash \bigcup \{L_\theta(x) \colon \theta \in \Theta(x)\}.$$

Then, if f be the characteristic function of the set Q, we have

$$\text{(i)} \qquad C_e(f, x) = \{1\}$$

$$\text{(ii)} \qquad \begin{aligned} C_e(f, x, \theta) &= \{0\} \quad \text{for } \theta \in \Theta(x) \\ &= \{1\} \quad \text{for } \theta \in (0, \pi) \backslash \Theta(x). \end{aligned}$$

This example shows that the exceptional set of Theorem 2 need not be of the first category.

THEOREM 3. *If $f \colon H \to W$ has the restricted Baire property and if $\theta \in (0, \pi)$ is a fixed direction, then except a first category set of measure zero on R,*

$$C_e(f, x) \subset C_e(f, x, \theta).$$

Proof. Let $\mathscr{B} = \{V_n\}$ be a countable basis for the topology of W. Also let

$$E_n = f^{-1}(V_n), \qquad T = \{x \colon x \in R;\ C_e(f, x) \not\subset C_e(f, x, \theta)\}$$

and

$$\mathscr{S}(E_n, \theta) = \{x \colon x \in R;\ \bar{d}_\theta(E_n, x) = 0;\ \bar{d}(E_n, x) > 0\}.$$

Then clearly

$$\text{(1)} \qquad T \subset \bigcup_n \mathscr{S}(E_n, \theta).$$

By Lemma 5, each of the sets $\mathscr{S}(E_n, \theta)$ is a first category set of measure zero and hence, by (1), T is a first category set of measure zero and this completes the proof of the theorem.

Remark. In the hypothesis of Theorem 2 of [1], Theorem 1 and Theorem 3 of [3], and also of the theorem of [4] the continuity property can be replaced by the restricted Baire property of the function, by simple modifications of the concerning lemmas.

I am grateful to Dr. S. N. Mukhopadhyay for his kind help and suggestions in the preparation of the paper.

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Reçu par la Rédaction le 17. 4. 1978