COLLOQUIUM MATHEMATICUM

VOL. XXVIII 1973 FASC. 2

A CHARACTERIZATION OF STRONG HOMOMORPHISMS

BY

HARTMUT HÖFT (YPSILANTI, MICHIGAN)

In a recent paper J. Schmidt proved an extension of the Homomorphism Theorem to partial algebras ([2], Theorem 7). One important tool used in the paper we quoted is the kernel of a homomorphism, a generalization of the congruence relation induced by a homomorphism. Some types of homomorphisms are characterized in [2] by properties of their kernels. In this paper, we shall describe strong homomorphisms in terms of their kernels.

Throughout, $(A, \boldsymbol{f})$ and $(B, \boldsymbol{g})$ will be (partial) algebras of type $\Delta = (K_i)_{i \in I}$. A map $\varphi\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ is called a *homomorphism* if

(1) $$\varphi\big(f_i(\boldsymbol{a})\big) = b \text{ implies } g_i(\varphi \circ \boldsymbol{a}) = b.$$

Deviating from the notation used in [2], $\ker\varphi := \{(a, a') \mid a, a' \in A, \varphi(a) = \varphi(a')\}$ — rather than R_φ — shall denote here the *congruence relation* induced by homomorphism φ in A.

We will now list five notions of homomorphisms for partial algebras. A homomorphism $\varphi\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ is called (1):

closed if

(2) $$g_i(\varphi \circ \boldsymbol{a}) = b \text{ implies } \varphi\big(f_i(\boldsymbol{a})\big) = b;$$

closed in itself if

(3) $$g_i(\varphi \circ \boldsymbol{a}) = \varphi(a) \text{ implies } \varphi\big(f_i(\boldsymbol{a})\big) = \varphi(a);$$

initial if

(4) $$g_i(\varphi \circ \boldsymbol{a}) = \varphi(a) \text{ implies } f_i(\boldsymbol{a}) = a;$$

strong if

(5) $g_i(\varphi \circ \boldsymbol{a}) = \varphi(a)$ implies that there is $a' \in A$ and $\boldsymbol{a}'\colon K_i \to A$ such that $a' = f_i(\boldsymbol{a}')$, $\varphi(a) = \varphi(a')$, and $\varphi \circ \boldsymbol{a} = \varphi \circ \boldsymbol{a}'$;

(1) Unfortunately, no standardized terminology has developped in the literature as yet. Grätzer [1] uses "strong" for our "closed" and "full" for our "strong". Słominski [4] uses "strong" in the sense wé do. The terms "final" and "initial" are taken from N. Bourbaki, *Theory of sets,* Chapter IV, § 2, since they describe here analogous situations for algebras.

final if

(6) $g_i(\boldsymbol{b}) = b$ implies that there is $a' \in A$ and $\boldsymbol{a}': K_i \to A$ such that $a' = f_i(\boldsymbol{a}'), \varphi(a') = b$, and $\varphi \circ \boldsymbol{a}' = \boldsymbol{b}$.

As immediate consequences of these definitions we obtain:

(7) "Strong" and "final" as well as "closed in itself" and "closed" are equivalent notions if restricted to surjective homomorphisms.

(8) "Strong", "closed in itself", and "initial" are equivalent notions if restricted to injective homomorphisms.

(9) The following diagram of implications shows how the types of homomorphisms we introduced above are related to each other, in general:

$$\begin{tikzcd}
 & & \text{strong} & & \\
 & \text{closed in itself} \arrow[ur, Rightarrow] & & \text{strong closed image} \arrow[ul, Rightarrow] & \\
\text{initial} \arrow[ur, Rightarrow] & & \text{closed} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & & \text{final} \arrow[ul, Rightarrow]
\end{tikzcd}$$

With each partial algebra $(A, \boldsymbol{f})$ we associate its *free completion* $(\hat{A}, \hat{\boldsymbol{f}})$ which is — up to isomorphism over A — the solution to the following universal problem: $(A, \hat{\boldsymbol{f}})$ is an extension of $(A, \boldsymbol{f})$ such that operation f_i, $i \in I$, on A is the restriction of $\hat{f}_i$ to set A (one might express this property by saying that $(A, \boldsymbol{f})$ is a *relative algebra* of $(\hat{A}, \hat{\boldsymbol{f}})$, and write $\boldsymbol{f} = \hat{\boldsymbol{f}} \| A$), and, for each complete algebra $(B, \boldsymbol{g})$ and homomorphism $\varphi: (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$, there is a unique homomorphic extension $\hat{\varphi}: (\hat{A}, \hat{\boldsymbol{f}}) \to (B, \boldsymbol{g})$ of φ. $(\hat{A}, \hat{\boldsymbol{f}})$ may be characterized internally by axioms (cf. [2], section 5), as a matter of fact, generalizations of Peano's axioms for the natural numbers. One of these axioms states that $(A, \boldsymbol{f})$ is an *initial segment* of $(\hat{A}, \hat{\boldsymbol{f}})$. That is to say:

Every sequence $\boldsymbol{x}: K_i \to \hat{A}$ such that $f_i(\boldsymbol{x}) \in A$ is actually itself in A, $\boldsymbol{x}: K_i \to A$.

An *intermediate* initial segment of algebra $(\hat{A}, \hat{\boldsymbol{f}})$ then is any relative algebra $(D, \hat{\boldsymbol{f}} \| D)$ of $(\hat{A}, \hat{\boldsymbol{f}})$ which is an initial segment and which contains $(A, \boldsymbol{f})$. Every homomorphism $\varphi: (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ into a partial algebra $(B, \boldsymbol{g})$ has a largest homomorphic extension $\tilde{\varphi}: (\operatorname{dom} \tilde{\varphi}, \hat{\boldsymbol{f}} \| \operatorname{dom} \tilde{\varphi}) \to (B, \boldsymbol{g})$ to an intermediate initial segment $\operatorname{dom} \tilde{\varphi}$ of $(\hat{A}, \hat{\boldsymbol{f}})$. This extension $\tilde{\varphi}$ is the only closed homomorphism extending φ to an intermediate initial segment (cf. [2], Theorems 2 to 5). The congruence relation induced by $\tilde{\varphi}$ in $\operatorname{dom} \tilde{\varphi}$, $\ker \tilde{\varphi}$, is called the *kernel* of φ, and we shall refer to its equivalence classes as *kernel classes* of homomorphism φ. It should be noted that the kernel of φ depends on the algebraic structure of algebra B since $\tilde{\varphi}$

does, and that it is uniquely determined by algebra B and by homomorphism φ.

The following observations are obvious (cf. [2], section 10):

(10) φ is injective if and only if $\ker\tilde{\varphi} \cap (A \times A) = \mathrm{id}_A$.

(11) φ is surjective if and only if each kernel class of φ intersects A.

(12) φ is closed if and only if the domain of $\ker\tilde{\varphi}$ equals A.

Before we state our first theorem, we introduce the following notation in $\hat{A}$:

$$D(A) := A \cup B(A), \qquad \text{where } B(A) := \{\hat{f}_i(x) \mid i \in I \text{ and } \boldsymbol{x}\colon K_i \to A\}.$$

THEOREM 1. *For a homomorphism* $\varphi\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ *the following conditions are equivalent*:

(a) φ *is closed in itself*;

(b) $\ker\tilde{\varphi} \cap (A \times D(A)) = \ker\varphi$.

Proof. Suppose φ is closed in itself. Trivially, $\ker\varphi = \ker\tilde{\varphi} \cap (A \times A) \subset \ker\tilde{\varphi} \cap (A \times D(A))$. Let now $(a, x) \in \ker\tilde{\varphi} \cap (A \times D(A))$. If $x \in A$, then $(a, x) \in \ker\varphi$, as just stated. So assume $x = \hat{f}_i(\boldsymbol{x})$ for some $i \in I$ and $\boldsymbol{x}\colon K_i \to A$. Then $\varphi(a) = \tilde{\varphi}(\hat{f}_i(\boldsymbol{x})) = g_i(\tilde{\varphi} \circ \boldsymbol{x}) = g_i(\varphi \circ \boldsymbol{x})$. Therefore, $\varphi(f_i(\boldsymbol{x})) = \varphi(a)$ since φ is closed in itself, and we obtain $\hat{f}_i(\boldsymbol{x}) = f_i(\boldsymbol{x}) = x \in A$ and $(a, x) \in \ker\varphi$. Conversely, suppose (b) holds, and $\varphi(a) = g_i(\varphi \circ \boldsymbol{a})$ for some $i \in I$ and $\boldsymbol{a}\colon K_i \to A$. Since $\tilde{\varphi}$ is a closed homomorphism, $\tilde{\varphi}(a) = \varphi(a) = g_i(\varphi \circ \boldsymbol{a}) = g_i(\tilde{\varphi} \circ \boldsymbol{a}) = \tilde{\varphi}(\hat{f}_i(\boldsymbol{a}))$. By hypothesis, we obtain $(a, \hat{f}_i(\boldsymbol{a})) \in \ker\tilde{\varphi} \cap (A \times D(A)) = \ker\varphi$; and this implies $\hat{f}_i(\boldsymbol{a}) = f_i(\boldsymbol{a})$, proving that φ is closed in itself.

Applying (8) to Theorem 1 we get as a special case Theorem 10 in [2].

COROLLARY. *For a homomorphism* $\varphi\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$, *the following conditions are equivalent*:

(a) φ *is strong and injective*;

(b) $\ker\tilde{\varphi} \cap (A \times D(A)) = \mathrm{id}_A$.

Condition (b) of the next theorem is in a sense remarkable. Recall that every congruence relation is induced by a strong homomorphism, e.g. the natural projection. Therefore, strongness of a homomorphism φ cannot be characterized by $\ker\varphi$; however, condition (b) expresses the strongness of homomorphism φ in terms of $\ker\tilde{\varphi}$.

THEOREM 2. *For a homomorphism* $\varphi\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$, *the following conditions are equivalent*:

(a) φ *is strong*;

(b) *for each sequence* $\boldsymbol{x}$ *of type* K_i *in* A *such that* $x = \hat{f}_i(\boldsymbol{x}) \in \operatorname{dom}\tilde{\varphi}$, *the kernel class of* x *contains an element* $f_i(\boldsymbol{a})$ *for some* $\boldsymbol{a}\colon K_i \to A$ *such that* $\varphi \circ \boldsymbol{a} = \varphi \circ \boldsymbol{x}$, *provided that the kernel class of* x *intersects* A.

Proof. Suppose φ is strong. Consider a sequence $\boldsymbol{x}: K_i \to \boldsymbol{A}$ such that $x = \hat{f}_i(\boldsymbol{x}) \in \operatorname{dom} \tilde{\varphi}$, and suppose the kernel class of x intersects A. Then there is a $y \in A$ such that $\varphi(y) = \tilde{\varphi}(x) = g_i(\varphi \circ \boldsymbol{x})$. But since φ is strong, $a \in A$ and $\boldsymbol{a}: K_i \to A$ exist such that $a = f_i(\boldsymbol{a})$, $\varphi(a) = \varphi(y)$, and $\varphi \circ \boldsymbol{a} = \varphi \circ \boldsymbol{x}$, proving (b). Conversely, suppose (b) holds and $\varphi(a) = g_i(\varphi \circ \boldsymbol{a})$ with $a \in A$ and $\boldsymbol{a}: K_i \to A$. Since $\tilde{\varphi}$ is closed, $\varphi(a) = g_i(\tilde{\varphi} \circ \boldsymbol{a}) = \tilde{\varphi}(\hat{f}_i(\boldsymbol{a}))$, so that the kernel class of $\hat{f}_i(\boldsymbol{a})$ intersects A. Therefore, it contains $a' = f_i(\boldsymbol{a}')$, $\boldsymbol{a}': K_i \to A$, satisfying $\varphi \circ \boldsymbol{a} = \varphi \circ \boldsymbol{a}'$. Finally, $\varphi(a') = \varphi(f_i(\boldsymbol{a}')) = g_i(\varphi \circ \boldsymbol{a}') = g_i(\varphi \circ \boldsymbol{a}) = \varphi(a)$, and we have verified that φ is strong.

We may also express condition (b) of Theorem 2 in this way:

For every sequence $\boldsymbol{x}: K_i \to A$, if $\hat{f}_i(\boldsymbol{x})$ belongs to $\operatorname{dom} \tilde{\varphi}$ *and is congruent* (modulo $\ker \tilde{\varphi}$) *to some element in A, then the sequence $\boldsymbol{x}$ itself is congruent* (modulo $\ker \tilde{\varphi}$) *to some sequence $\boldsymbol{a}$ in the domain of operation f_i on A.*

An immediate consequence of Theorem 2 and (11) is the following

COROLLARY. *For a homomorphism $\varphi: (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$, the following conditions are equivalent:*

(a) *φ is strong and surjective;*

(b) *for each sequence $\boldsymbol{x}$ of type K_i in A such that $x = \hat{f}_i(\boldsymbol{x}) \in \operatorname{dom} \tilde{\varphi}$, the kernel class of x contains an element $f_i(\boldsymbol{a}) \in A$ with $\varphi \circ \boldsymbol{a} = \varphi \circ \boldsymbol{x}$.*

Finally, we want to make some remarks on initial and final homomorphisms. We put $\operatorname{im} \boldsymbol{f} := \{f_i(\boldsymbol{a}) \mid i \in I \text{ and } \boldsymbol{a}: K_i \to A\} = \bigcup \{\operatorname{im} f_i \mid i \in I\}$ and we will say that $\boldsymbol{f}$ *covers* A whenever $A \subset \operatorname{im} \boldsymbol{f}$. From (3), (4), and (9) we infer that φ is initial if and only if φ is closed in itself and $\ker \varphi \cap (A \times \operatorname{im} \boldsymbol{f}) \subset \mathrm{id}_A$, and so, in fact, we characterize an initial homomorphism by its congruence relation rather than its kernel. As a corollary (cf. [3], Corollary to Theorem 6.5) we get:

If $\boldsymbol{f}$ covers A, then φ is initial if and only if φ is strong and injective.

It is impossible to characterize final homomorphisms by their kernels alone. An immediate consequence of the definition of a final homomorphism $\varphi: (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ is that the relative algebra $B - \operatorname{im} \varphi$ of B has empty structure. Therefore, a characterization for final homomorphisms will have to include a condition on $\operatorname{dom} \boldsymbol{g}$ which, of course, is very unsatisfactory. One such characterization is, for instance, the following: φ is final if and only if φ is strong, $\operatorname{im} \tilde{\varphi} \subset \operatorname{im} \varphi$, and every sequence in $\operatorname{dom} \boldsymbol{g}$ is a sequence in $\operatorname{im} \varphi$.

REFERENCES

[1] G. Grätzer, *Universal algebra*, Princeton 1968.

[2] J. Schmidt, *A homomorphism theorem for partial algebras*, Colloquium Mathematicum 21 (1970), p. 5-21.

[3] — *Allgemeine Algebra*, Lecture Notes, Universität Bonn, 1966.
[4] J. Słomiński, *A theory of extensions of quasi-algebras to algebras*, Rozprawy Matematyczne 40, Warszawa 1964.

DEPARTMENT OF MATHEMATICS
EASTERN MICHIGAN UNIVERSITY

Reçu par la Rédaction le 20. 6. 1972