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On the stability of a functional equation related to associativity*

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Abstract. We solve, under some regularity conditions, the functional inequality $|F(Kx, KF(y, z))-F(KF(x, y), Kz)| \leqslant \varepsilon$, where F is a binary operation on $\boldsymbol{R}^+$ to be found. In the case $K = 1$ we exhibit some non-associative solutions of this inequality corresponding to the stability of the associativity.

After the celebrated paper by Hyers [5] and the ideas introduced by Ulam [6] the study of stability of functional equations has become an important field of research.

The aim of this paper is to solve the functional inequality

$$|F(Kx, KF(y, z))-F(KF(x, y), Kz)| \leqslant \varepsilon \tag{1}$$

where ε is a given positive constant, the variables x, y, z and K run over $\boldsymbol{R}^+ := [0, \infty)$ and the unknown function F is a continuous strictly increasing binary operation on $\boldsymbol{R}^+$ with 0 as a unit element.

A surprising result is that (1) is equivalent to the corresponding equation $F(Kx, KF(y, z)) = F(KF(x, y), Kz)$ which yields the associativity and homogeneity of F. But if we fix $K = 1$ in (1) then we have the stability of the associativity equation and it is possible to find non-associative solutions.

We begin with the following

THEOREM 1. *Given $\varepsilon > 0$ let F be a binary operation on $\boldsymbol{R}^+$ with* 0 *as a unit element. If F satisfies the inequality* (1):

$$|F(Kx, KF(y, z))-F(KF(x, y), Kz)| \leqslant \varepsilon$$

for all x, y, z and K in $\boldsymbol{R}^+$, then F must satisfy the functional equation

$$F(Kx, KF(y, z)) = F(KF(x, y), Kz) \tag{2}$$

for all x, y, z and K in $\boldsymbol{R}^+$.

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Proof. If $u > 1$ the substitution $K = u^n$, $n \in N$ and $x = 0$ in (1) yields

$$|u^n F(y, z) - F(u^n \cdot y, u^n \cdot z)| \leqslant \varepsilon,$$

or equivalently

$$|F(y, z) - F(u^n \cdot y, u^n \cdot z)/u^n| \leqslant \varepsilon/u^n,$$

whence

$$F(y, z) = \lim_{n\to\infty} F(u^n \cdot y, u^n \cdot z)/u^n \tag{3}$$

for all y, z in R^+ and for all $u > 1$. From (3) we obtain

$$F(y, z) = \lim_{n\to\infty} u^n \cdot F(y/u^n, z/u^n), \tag{4}$$

whenever y, z are in R^+ and $u < 1$. Now we proceed to prove the associativity of F. Using (1) and (3) we have for all x, y, and z in R^+

$$\begin{aligned} &\big|F(x, F(y, z)) - F(F(x, y), z)\big| \\ &\qquad = \left| \lim_{n\to\infty} \frac{F(2^n x, 2^n F(y, z))}{2^n} - \lim_{n\to\infty} \frac{F(2^n F(x, y), 2^n z)}{2^n} \right| \\ &\qquad = \lim_{n\to\infty} \frac{|F(2^n x, 2^n F(y, z)) - F(2^n F(x, y), 2^n z)|}{2^n} \\ &\qquad \leqslant \lim_{n\to\infty} \varepsilon/2^n = 0, \end{aligned}$$

whence the associativity of F follows, i.e., (2) is true for $K = 1$. Using this and (3) we have for all x, y, z in R^+ and for all $K > 1$

$$\begin{aligned} &\big|F(Kx, KF(y, z)) - F(KF(x, y), Kz)\big| \\ &\quad = \lim_{n\to\infty} \frac{|F(K^{n+1} x, K^{n+1} F(y, z)) - F(K^{n+1} F(x, y), K^{n+1} z)|}{K^n} \\ &\quad = K \cdot \left| \lim_{n\to\infty} \frac{F(K^{n+1} x, K^{n+1} F(y, z))}{K^{n+1}} - \lim_{n\to\infty} \frac{F(K^{n+1} F(x, y), K^{n+1} z)}{K^{n+1}} \right| \\ &\quad = K\big|F(x, F(y, z)) - F(F(x, y), z)\big| = 0, \end{aligned}$$

whence (2) holds whenever $K > 1$. In a similar way using (4) and the associativity of F we show that (2) holds for $K < 1$.

THEOREM 2. *Given $\varepsilon > 0$ let F be a continuous, strictly increasing two-argument function from $R^+ \times R^+$ onto R^+ with 0 as a unit element. Then F satisfies the inequality* (1):

$$\big|F(Kx, KF(y, z)) - F(KF(x, y), Kz)\big| \leqslant \varepsilon \quad \textit{for all } x,\ y,\ z \textit{ and } K \textit{ in } R^+$$

if and only if there exists a positive constant $c > 0$ such that

$$F(x, y) = \sqrt[c]{x^c + y^c}.$$

Proof. By the previous theorem, (1) holds if and only if the corresponding equation (2) is satisfied by F:

$$F(Kx,\ KF(y,\ z)) = F(KF(x,\ y),\ Kz).$$

The substitution $z = 0$ in (2) yields the homogeneity of F:

$$F(Kx,\ Ky) = KF(x,\ y). \tag{5}$$

Also the substitution $K = 1$ yields the associativity of F, which together with the conditions assumed on F implies [1] that F can be represented in the form $F(x,\ y) = f^{-1}(f(x)+f(y))$, where f is a monotonic continuous function from $\boldsymbol{R}^+$ onto $\boldsymbol{R}^+$ such that $f(0) = 0$. It is known [1] that the only associative operations representable in this form which are homogeneous are $F(x,\ y) = \sqrt[c]{x^c+y^c}$ for some $c > 0$.

Now we turn our attention to (1) in the case where $K = 1$.

DEFINITION 1. Given $\varepsilon > 0$, a binary operation F on $\boldsymbol{R}^+$ is said to be *ε-associative* if (1) holds for $K = 1$ and for all x, y, and z in $\boldsymbol{R}^+$, i.e.,

$$|F(x,\ F(y,\ z)) - F(F(x,\ y),\ z)| \leqslant \varepsilon. \tag{6}$$

Obviously any associative operation satisfies (6), but there are non-associative solutions of (6). An easy way to construct such examples is to apply the following

THEOREM 3. *Let F and L be two binary operations on $\boldsymbol{R}^+$ and let $\varepsilon > 0$ be given. Assume that the following conditions hold for all a, b, c in $\boldsymbol{R}^+$:*

$$L(a,\ L(b,\ c)) = L(L(a,\ b),\ c); \tag{i}$$

$$L(a,\ b) = L(b,\ a); \tag{ii}$$

$$L(a,\ b) \leqslant L(c,\ b) \quad \text{whenever } a \leqslant c; \tag{iii}$$

$$L(a+\varepsilon,\ c) \leqslant L(a,\ c)+L(\varepsilon,\ 0); \tag{iv}$$

$$|L(a,\ b)-F(a,\ b)| \leqslant \varepsilon; \tag{v}$$

$$F(a,\ b) = F(b,\ a). \tag{vi}$$

Then F is $2(\varepsilon+L(0,\ \varepsilon))$-associative.

Proof. Using the hypotheses on L and F we have

$$\begin{aligned} L(F(a,\ b),\ c) &\leqslant L(L(a,\ b)+\varepsilon,\ c) \leqslant L(L(a,\ b),\ c)+L(\varepsilon,\ 0) \\ &= L(a,\ L(b,\ c))+L(\varepsilon,\ 0) \leqslant L(a,\ F(b,\ c)+\varepsilon)+L(\varepsilon,\ 0) \\ &\leqslant L(a,\ F(b,\ c))+L(0,\ \varepsilon)+L(\varepsilon,\ 0) \\ &= L(a,\ F(b,\ c))+2L(0,\ \varepsilon). \end{aligned}$$

Then we also have

$$L(F(c, b), a) \leqslant L(c, F(b, a)) + 2L(0, \varepsilon),$$

so we can conclude using (vi) that

$$(7) \qquad |L(F(a, b), c) - L(a, F(b, c))| \leqslant 2L(0, \varepsilon).$$

Finally, using (v) and (7) we obtain

$$\begin{aligned} |F(a, F(b, c)) - F(F(a, b), c)| &\leqslant |F(a, F(b, c)) - L(a, F(b, c))| \\ &\quad + |L(a, F(b, c)) - L(F(a, b), c)| \\ &\quad + |L(F(a, b), c) - F(F(a, b), c)| \\ &\leqslant \varepsilon + 2L(0, \varepsilon) + \varepsilon = 2(\varepsilon + L(0, \varepsilon)), \end{aligned}$$

i.e., F is $2(\varepsilon + L(0, \varepsilon))$-associative.

COROLLARY 1. *Given $\varepsilon \geqslant 0$ let F be a binary operation on $\boldsymbol{R}^+$ satisfying one of the following inequalities:*

$$(1) \qquad |F(a, b) - \mathrm{Max}(a, b)| \leqslant \varepsilon/4;$$

$$(2) \qquad |F(a, b) - (a + b)| \leqslant \varepsilon/4.$$

Then F is ε-associative.

Thus we have shown that "near" maximum or addition it is trivial to find an ε-associative operation. A natural question arises: what happens "near" product? The answer is that the only ε-associative operation close to product is just product. This fact will be an immediate consequence of the following theorem whose proof follows the method introduced by Baker in [3]:

THEOREM 4. *Given $\varepsilon > 0$ let F be an ε-associative operation on $\boldsymbol{R}^+$. Suppose that there exist a positive real number δ and a bijective function f from $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ such that $|f(x) - f(y)| \leqslant g(|x-y|)$ for some non-decreasing function g from $\boldsymbol{R}^+$ into itself, and*

$$|f(F(x, y)) - f(x)f(y)| < \delta \quad \text{for all } x, y \text{ in } \boldsymbol{R}^+.$$

Then necessarily $F(x, y) = f^{-1}(f(x) \cdot f(y))$.

Proof. For any x, y and z in $\boldsymbol{R}^+$ we have

$$\begin{aligned} |f(F(x, y))f(z) - f(x)f(y)f(z)| &\leqslant |f(F(x, y))f(z) - f(x)f(F(y, z))| \\ &\quad + |f(x)f(F(y, z)) - f(x)f(y)f(z)| \\ &\leqslant |f(F(F(x, y), z)) - f(F(x, y)) \cdot f(z)| \\ &\quad + |f(F(F(x, y), z)) - f(F(x, F(y, z)))| \end{aligned}$$

$$
\begin{aligned}
&+\left|f\big(F(x, F(y, z))\big)-f(x)f\big(F(y, z)\big)\right|\\
&+\left|f(x)f\big(F(y, z)\big)-f(x)f(y)f(z)\right|\\
&\leqslant \delta+g\Big(\big|F\big(F(x, y), z\big)-F\big(x, F(y, z)\big)\big|\Big)+\delta+|f(x)|\delta\\
&\leqslant 2\delta+g(\varepsilon)+|f(x)|\delta.
\end{aligned}
$$

If we substitute $z=f^{-1}(n)$ we obtain

$$
\left|f\big(F(x, y)\big)-f(x)f(y)\right| \leqslant \frac{2\delta+g(\varepsilon)+|f(x)|\delta}{n}. \tag{8}
$$

Thus when n goes to infinity we obtain

$$
0 \leqslant \left|f\big(F(x, y)\big)-f(x)f(y)\right| \leqslant 0,
$$

i.e., $f\big(F(x, y)\big)=f(x)f(y)$ and consequently $F(x, y)=f^{-1}\big(f(x)f(y)\big)$.

COROLLARY 2. *Given $\varepsilon>0$ and $\delta>0$ let F be an ε-associative operation on $\boldsymbol{R}^+$ such that $|F(x, y)-xy|<\delta$ for all x, y in $\boldsymbol{R}^+$. Then $F(x, y)=x\cdot y$.*

Proof. Substitute $f(x)=g(x)=x$ in the previous theorem.

Finally, we point out an open question: Given an ε-associative function F on $\boldsymbol{R}^+$ is there an associative operation A such that $|F(x, y)-A(x, y)|\leqslant K\varepsilon$ for some $K>0$? The answer to this problem would complete the study of the stability of the associativity equation.

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