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## AN ANALYTICAL METHOD FOR CALCULATION OF INTEGRALS APPEARING IN APPROXIMATE SOLUTIONS OF DEFORMABLE BODY MECHANICS

The paper presents an analytical method for calculation of integrals of the form

$$\int_0^1 (a+bx)^n \prod_{j=1}^p \sin(m_j \pi x) \prod_{k=1}^r \cos(n_k \pi x) dx.$$

The calculation of values of these integrals is necessary when using some approximate methods to solve problems in deformable body mechanics. These integrals, particularly for nonlinear problems, are commonly calculated using classical numerical methods. Multiple calculation of the integral values by known methods increases significantly the computer time. Thus, taking into account the homogeneous form of the integrals appearing in the solution of definite type problems, we may use an analytical method which shortens considerably the calculation time.

**1. Introduction.** The application of approximate methods (e.g., the well-known orthogonalization Bubnov-Galerkin technique or Ritz technique) in many problems studied now in the deformable body mechanics ([1], [2]) as well as in other domains leads to calculation of integrals of the form (see [3])

$$(1) \quad T^{(n)}(m_1, m_2, \dots, m_p; n_1, n_2, \dots, n_r) \\ = \int_0^1 (a+bx)^n \prod_{j=1}^p \sin(m_j \pi x) \prod_{k=1}^r \cos(n_k \pi x) dx,$$

where  $a, b \in R$ ,  $n, p, r \in \{0\} \cup N$ ,  $m_j, n_k \in I$ . These integrals are determined using classical numerical methods. Because of the properties of the subintegral function (a large number of zero positions), as well as the significant number of integrals occurring in the approximate solution, the calculations are rather tedious.

Taking into account the homogeneous form of integrals in the approximate solution, we should note the possibility and advantage of using analytical methods of determining these integrals.

**2. Basic trigonometric identities.** As the subintegral function (1) is a product of the power and trigonometric functions, the determination of the integral makes no troubles when replacing the product of trigonometric functions with a sum of these functions with proper arguments.

Deriving the basic identities we use the following relations:

$$(2) \quad \sin x = \frac{1}{2i}(e^{xi} - e^{-xi}), \quad \cos x = \frac{1}{2}(e^{xi} + e^{-xi}).$$

Thus we have

$$(3) \quad \prod_{j=1}^p \sin x_j = \frac{1}{(2i)^p} \prod_{j=1}^p (\exp\{x_j i\} - \exp\{-x_j i\}) \\ = \frac{1}{(2i)^p} \sum_{(z_1, z_2, \dots, z_p)} \prod_{j=1}^p (z_j \exp\{z_j x_j i\}),$$

where  $z_j \in Z = \{-1, 1\}$  and  $(z_1, z_2, \dots, z_p)$  is a  $p$ -element variation with repetitions of the set  $Z$ . The number of variations is  $2^p$ , so the sum on the right-hand side of (3) has  $2^p$  terms.

The elements of the set  $Z$  are reverse numbers. Thus, for each variation  $(z_1, z_2, \dots, z_p)$  we shall find a reverse variation  $(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_p)$  satisfying the condition  $\bar{z}_j = -z_j$ . Hence

$$\prod_{j=1}^p \sin x_j = \frac{1}{(2i)^p} \sum_{(1, z_2, \dots, z_p)} \left( \prod_{j=1}^p z_j \prod_{j=1}^p \exp\{z_j x_j i\} + \prod_{j=1}^p \bar{z}_j \prod_{j=1}^p \exp\{\bar{z}_j x_j i\} \right),$$

where  $z_1 = 1$ .

Let  $l$  denote the number of negative elements in the variation  $(z_1, z_2, \dots, z_p)$ . Then

$$\prod_{j=1}^p \sin x_j = \frac{1}{(2i)^p} \sum_{(1, z_2, \dots, z_p)} (-1)^l [\exp\{(\sum_{j=1}^p z_j x_j) i\} + (-1)^p \exp\{-(\sum_{j=1}^p z_j x_j) i\}].$$

Taking into account relation (2) we receive

$$(4) \quad \prod_{j=1}^p \sin x_j = \begin{cases} \frac{(-1)^{(p-1)/2}}{2^{p-1}} \sum_{(1, z_2, \dots, z_p)} (-1)^l \sin(x_1 + z_2 x_2 + \dots + z_p x_p) & \text{for odd value of } p, \\ \frac{(-1)^{p/2}}{2^{p-1}} \sum_{(1, z_2, \dots, z_p)} (-1)^l \cos(x_1 + z_2 x_2 + \dots + z_p x_p) & \text{for even value of } p. \end{cases}$$

The sums on the right-hand side of (4) have  $2^{p-1}$  terms.

Proceeding in the same manner we obtain

$$\prod_{j=1}^p \sin x_j \prod_{k=1}^r \cos y_k$$

$$\begin{aligned}
 &= \frac{1}{(2i)^p} \frac{1}{2^r} \prod_{j=1}^p (\exp\{x_j i\} - \exp\{-x_j i\}) \prod_{k=1}^r (\exp\{y_k i\} + \exp\{-y_k i\}) \\
 &= \frac{1}{(2i)^p \cdot 2^r} \sum_{\substack{(z_1, z_2, \dots, z_p, \\ z_{p+1}, \dots, z_{p+r})}} \prod_{j=1}^p (z_j \exp\{z_j x_j i\}) \prod_{k=1}^r \exp\{z_{p+k} y_k i\}.
 \end{aligned}$$

Putting  $x_j = y_{j-p}$  for  $j = p+1, p+2, \dots, p+r$  we have

$$\prod_{j=1}^p \sin x_j \prod_{k=1}^r \cos y_k = \frac{1}{2^{p+r} i^p} \sum_{(z_1, z_2, \dots, z_{p+r})} \prod_{j=1}^p z_j \prod_{j=1}^{p+r} \exp\{z_j x_j i\}.$$

Adding pairwise the terms determined by reverse variations, we receive

$$\begin{aligned}
 &\prod_{j=1}^p \sin x_j \prod_{k=1}^r \cos y_k \\
 &= \frac{1}{2^{p+r} i^p} \sum_{(1, z_2, \dots, z_{p+r})} \left( \prod_{j=1}^p z_j \prod_{j=1}^{p+r} \exp\{z_j x_j i\} + \prod_{j=1}^p \bar{z}_j \prod_{j=1}^{p+r} \exp\{\bar{z}_j x_j i\} \right),
 \end{aligned}$$

where  $\bar{z}_j = -z_j$  and  $z_1 = 1$ .

Let  $l$  denote the number of negative elements in places from 1 to  $p$  in the variation  $(1, z_2, \dots, z_p, z_{p+1}, \dots, z_{p+r})$ . Then

$$\begin{aligned}
 &\prod_{j=1}^p \sin x_j \prod_{k=1}^r \cos y_k \\
 &= \frac{1}{2^{p+r} i^p} \sum_{(1, z_2, \dots, z_{p+r})} (-1)^l [\exp\{(\sum_{j=1}^{p+r} z_j x_j) i\} + (-1)^p \exp\{-(\sum_{j=1}^{p+r} z_j x_j) i\}].
 \end{aligned}$$

Finally, we have

$$(5) \quad \prod_{j=1}^p \sin x_j \prod_{k=1}^r \cos y_k = \begin{cases} \frac{(-1)^{(p-1)/2}}{2^{p+r-1}} \sum_{(1, z_2, \dots, z_{p+r})} (-1)^l \sin(x_1 + z_2 x_2 + \dots + z_p x_p + z_{p+1} y_1 + \dots + z_{p+r} y_r) & \text{for odd value of } p, \\ \frac{(-1)^{p/2}}{2^{p+r-1}} \sum_{(1, z_2, \dots, z_{p+r})} (-1)^l \cos(x_1 + z_2 x_2 + \dots + z_p x_p + z_{p+1} y_1 + \dots + z_{p+r} y_r) & \text{for even value of } p. \end{cases}$$

The sums on the right-hand side of (5) have  $2^{p+r-1}$  terms.

**3. Definite integral of the product of the power function and an arbitrary number of trigonometric functions. Putting**

$$(6) \quad S^{(n)}(m) = \int_0^1 (a+bx)^n \sin(m\pi x) dx, \quad C^{(n)}(m) = \int_0^1 (a+bx)^n \cos(m\pi x) dx$$

and using identities (5) we may write the definite integral (1) in the form

$$(7) \quad T^{(n)}(m_1, m_2, \dots, m_p; n_1, n_2, \dots, n_r) \begin{cases} \frac{(-1)^{(p-1)/2}}{2^{p+r-1}} \sum_{(1, z_2, \dots, z_{p+r})} (-1)^l S^{(n)}(m_1 + z_2 m_2 + \dots + z_p m_p \\ + z_{p+1} n_1 + \dots + z_{p+r} n_r) & \text{for odd value of } p, \\ \frac{(-1)^{p/2}}{2^{p+r-1}} \sum_{(1, z_2, \dots, z_{p+r})} (-1)^l C^{(n)}(m_1 + z_2 m_2 + \dots + z_p m_p + z_{p+1} n_1 + \dots + z_{p+r} n_r) \\ & \text{for even value of } p, \end{cases}$$

where  $z_j \in \{-1, 1\}$  for  $j = 2, 3, \dots, p+r$ , and  $l$  denotes the number of negative elements in places 1 to  $p$  in the variation  $(1, z_2, \dots, z_{p+r})$ .

Integrals (6) for  $n \in N$  are to be determined using recurrent relations. For  $m \in I$  and  $m \neq 0$  we have

$$(8) \quad \begin{aligned} S^{(n)}(m) &= \frac{1}{m\pi} [bnC^{(n-1)}(m) + a^n - (-1)^m(a+b)^n], \\ C^{(n)}(m) &= -\frac{bn}{m\pi} S^{(n-1)}(m), \end{aligned}$$

and for  $m = 0$

$$(9) \quad \begin{aligned} S^{(n)}(0) &= 0, \\ C^{(n)}(0) &= \begin{cases} \frac{1}{b(n+1)} [(a+b)^{n+1} - a^{n+1}] & \text{for } b \neq 0, \\ a^n & \text{for } b = 0. \end{cases} \end{aligned}$$

For  $n = 0$ ,  $m \in I$  and  $m \neq 0$  we have

$$(10) \quad S^{(0)}(m) = \frac{1}{m\pi} [1 - (-1)^m], \quad C^{(0)}(m) = 0,$$

whereas for  $n = 0$  and  $m = 0$  we obtain

$$(11) \quad S^{(0)}(0) = 0, \quad C^{(0)}(0) = 1.$$

Basing on (7)–(11), the exact values of integrals (1) are easy to find. It is worth noticing that the solution obtained here is true for an arbitrary number of trigonometric subintegral functions. The advantages of formula (7) stem from identity (5) which permits to replace the product of any number of trigonometric functions by a sum. Using (5) and proceeding in a similar manner we can find, if necessary, in the simplest way the values of the integrals occurring in the product of elementary functions other than the power function and an arbitrary number of trigonometric functions.

Some general conclusions are drawn now from the formulae derived above, concerning the fact of taking any fixed value by the integrals (especially, the zero values) for a set of parameters characterized by special properties, very easy to formulate for  $n = 0$ . These conclusions may be helpful when selecting the functions which approximate the solution, as well as in the construction of simplified solutions and in the explanation of the considered phenomena on the base of approximate solutions.

**4. Final remarks.** The integrals (1) exist in the approximate solutions of nonlinear problems and, in consequence, are computed many times for different values of parameters. Recurrent relations are also present in formulae (8), derived to enable the determination of the integral (7). So, it seems to be useful to prepare a program which would carry out the repeated calculations.

The program built by the author in BASIC for the microcomputer SINCLAIR ZX 81 consists of two principal subroutines: AUXILIARY TABLES and INTEGRAL. According to the subroutine AUXILIARY TABLES the integrals (6) are computed for  $m, n \in \{0\} \cup N$ ,  $m \leq m_{\max}$ ,  $n \leq n_{\max}$ , taking into account formulae (8)–(11). For  $m_{\max}$  and  $n_{\max}$  we take the greatest value of the sum  $\sum_j |m_j| + \sum_k |n_k|$  and the greatest value for  $n$  in the set of computed integrals. In the INTEGRAL subroutine, the integral (1) is calculated by using (7). The running time of the subroutine AUXILIARY TABLES is much longer than that for the INTEGRAL subroutine. For example, the preparation of auxiliary tables for  $m_{\max} = 54$ ,  $n_{\max} = 5$ , i.e., tables with dimensions  $2 \cdot 6 \cdot 55$ , takes about 2 minutes. Thus, calculations for a larger number of integrals are to be performed in two stages. At the first stage, the auxiliary tables are prepared. At the second stage we find the values of successive integrals. Due to this, the calculation time of integrals of functions being the product of the power function and 4 trigonometric functions in the nonlinear problem of degree 3 (see [3]) lasts only about 2 seconds although the subintegral function has 54 zero points in the interval  $\langle 0, 1 \rangle$ . Of course, the calculation time at the second stage, when the number of subintegral product factors (1) is given, does not depend on the number of zero positions. The preparation time of the auxiliary table elongates approximately in proportion to the number of zero positions (the first stage). However, this is not of great importance because the auxiliary tables are prepared only once, i.e., before starting the calculations of an arbitrary number of integrals. Thus, the analytical method proposed here enables us to find the exact values of integrals of the power function and an arbitrary number of trigonometric functions, shortening greatly the calculation time. It may also be an important source of information and facilitates the proper choice of functions which approximate the solutions.

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