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ON ENDOMORPHISM STRUCTURE FOR ALGEBRAS OVER A FIXED SET

BY

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1. Introduction. The problem of determining for a fixed but arbitrary set A which transformation monoids on A can equal $\mathrm{End}\,\mathfrak{A}$ for a suitably chosen algebraic structure $\mathfrak{A}$ over A has received attention from several investigators (e.g. [1], [3] and [6]). A successful solution for the corresponding problem for automorphisms was given by Jónsson in [2]. Since the problem is trivial for $|A| \leqslant 2$, we assume throughout that $|A| \geqslant 3$. The purpose of this note is to provide a characterization for a certain class of semigroups which generalizes a result announced by Grätzer and Lampe in [1] and [3], and to present some work on the general problem.

2. Background and definitions. By a *monoid* we mean a semigroup with identity. A transformation monoid $M \subseteq A^A$ is said to be *algebraic* if $M = \mathrm{End}\,\mathfrak{A}$, where $\mathfrak{A} = \langle A, F\rangle$ is a universal algebra (1). For M to be algebraic it is necessary, in general, that M satisfy a "local" closure property [2] and that certain "constants of M" be included in M [3]. If M has a special structure, these conditions may also be sufficient; if, for example, M consists of a group G and a set of constant maps K, the above-given conditions are necessary and sufficient [3]. Our Theorem 1 is a more general result of this kind.

K is used throughout to denote a set of constant maps.

We say that a monoid $E \subseteq A^A$ is *locally invertible* provided

$$\forall_{n\in\omega}\ \forall_{a,b\in A^n}\ \forall_{\sigma,\tau\in E}\Big(\big(\sigma(a) = \tau(b)\big) \Rightarrow \big(\exists_{\varphi\in E}\ \varphi(a) = b\big)\Big).$$

Note that in a locally invertible monoid E each map is one-to-one. Indeed, given $x, y \in A$ and $\sigma\in E$ with $\sigma(x) = \sigma(y)$, we have $\sigma(\langle x, x\rangle) = \sigma(\langle x, y\rangle)$; hence $\varphi(\langle x, x\rangle) = \langle x, y\rangle$ for some $\varphi\in E$, so that $x = y$. In

(1) Here a universal algebra means each operation has finite rank.

case A is finite, a locally invertible monoid is a group. Every group is a locally invertible monoid.

Every algebraic monoid satisfies the following local closure condition (see [2] and [3]):

$$\forall_{\varphi\in A^A}\Big(\big(\forall_{X \text{ finite} \subset A}\ \exists_{\sigma\in M}\ \sigma\restriction X = \varphi\restriction X\big) \Rightarrow \varphi\in M\Big).$$

The set of constants of any algebraic monoid M must include the set (see [3])

$$C(M) = \Big\{\varphi\in A^A \mid \varphi(A) = \{a\} \ \&\ \Big(\forall_{A\ni b\neq a}\ \exists_{\sigma,\tau\in M}\ \big(\sigma(a) = \tau(a)\ \&\ \sigma(b)\neq\tau(b)\big)\Big)\Big\}.$$

3. Characterization of certain algebraic monoids.

THEOREM 1. *Let E be locally invertible and K a set of constant maps. Then a monoid $M = E\cup K$ is algebraic iff $C(M)\subseteq M$ and M is locally closed.*

Proof. The necessity of the conditions is obvious.

To prove the converse, assume that $C(M)\subseteq M$ and M is locally closed. Let $\mathfrak{A} = \langle A, F\rangle$ be the algebra of all (finitary) operations substitutive over M. Clearly, $M\subseteq \mathrm{End}\,\mathfrak{A}$. The proof of the converse inclusion will be carried out in three steps.

Step I. If $\varphi\in\mathrm{End}\,\mathfrak{A}$, then φ is one-to-one or constant.

To see this, suppose φ is neither one-to-one nor constant. Then $\varphi\notin M$, so there exists a finite set $X\subseteq A$ with $\sigma\restriction X\neq\varphi\restriction X$ for any $\sigma\in M$, and $|X| = n\geqslant 2$. We shall prove that φ must be constant on X. To this end, fix a one-to-one sequence y with range X and define an n-ary operation f as follows:

$$f(z) = \begin{cases} z_0 & \text{if } z = \sigma y \text{ for some } \sigma\in E, \\ z_1 & \text{otherwise.}\end{cases} \tag{i}$$

Note that $f\in F$, since it has the substitution property for M; that is, $af(z) = f\big(a(z)\big)$ for any $a\in E$ and $z\in A^n$. Indeed, if $z = \sigma(y)$ for some $\sigma\in E$, then $a(z) = a\sigma(y)$, so that $af(z) = a(z_0) = f\big(a(z)\big)$. On the other hand, if $z\neq\sigma(y)$ for any $\sigma\in E$, then, inasmuch as E is locally invertible and $a(z) = \sigma(y)$ implies $z = \xi(y)$ for some $\xi\in E$, we have $a(z)\neq\eta(y)$ provided $\eta\in E$. Therefore, $af(z) = a(z_1) = f(az)$. Likewise, if $a\in K$, then $af(z) = a = f(az)$, where $\{a\} = a(A)$. Now, if $a, b\in X$ and $y = (a, b, \ldots)$, then $\varphi(a) = \varphi f(y) = f(\varphi y) = \varphi(b)$. Thus φ is constant on X. Let $\varphi(x) = a$ for $x\in X$. Since no member of M agrees with φ on X, the constant map $\psi : A\to\{a\}$ is not a member of M. Let g be some n-ary operation for which $a\neq g(a, \ldots, a)$. If φ were constant on $S_g^{\mathfrak{A}}(X)$, where $S_g^{\mathfrak{A}}(X)$ denotes the subalgebra of $\mathfrak{A}$ generated by X, then

$$\varphi g(x_0, \ldots, x_0) = \psi g(x_0, \ldots, x_0) = a = g(\varphi x_0, \ldots, \varphi x_0) = g(a, \ldots, a),$$

a contradiction. So there exists a $b \in S_g^{\mathfrak{A}}(X)$ with $\varphi(b) \neq a$.

Again, let $y \in A^n$ be a one-to-one sequence with range X, and let $y = (a, b, \ldots)$. Define an n-ary operation f_b as follows:

$$f_b(z) = \begin{cases} \sigma(b) & \text{if } z = \sigma(y) \text{ for some } \sigma \in E, \\ z_0 & \text{otherwise.} \end{cases}$$

Note that f_b is well defined, since $\sigma, \tau \in E$ agree on $S_g^{\mathfrak{A}}(X)$ if they agree on X. It is routine to verify that $f \in F$; that is, $\alpha f_b(z) = f_b(\alpha z)$ for any $\alpha \in E$ and $z \in A$. Indeed, if $z = \sigma(y)$ for some $\sigma \in E$, then $\alpha(z) = \alpha\sigma(y)$, so that $\alpha f_b(z) = \alpha\sigma(b) = f_b(\alpha z)$. On the other hand, if $z \neq \sigma(y)$ for any $\sigma \in E$, then, by the local invertibility of E, we have $\alpha(z) \neq \eta(y)$ provided $\eta \in E$. Therefore, $\alpha f_b(z) = \alpha(z_0) = f_b(\alpha(z))$. Further, if $\alpha \in K$ and $\alpha(x) = d$ for $x \in A$, then we have $\alpha f_b(z) = d$ and $f_b(\alpha z) = f_b(d, \ldots, d)$. Now $(d, \ldots, d) \neq \sigma y$ for some $\sigma \in E$ since each σ is one-to-one, y is one-to-one and $y \in A^n$, where $n > 1$. Thus $f_b(\alpha z) = d = \alpha f_b(z)$. But now $a = \varphi(y_0) = f_b(\varphi y) = \varphi(b)$, a contradiction. This completes the proof of Step I.

To establish Steps II and III we shall use the methods and operations introduced by Jónsson [2].

Step II. If $\varphi \in \operatorname{End}\mathfrak{A}$ is one-to-one, then $\varphi \in M$.

If $\varphi \notin M$, then there exists a finite set $X \subseteq A$ with $\sigma \restriction X \neq \varphi \restriction X$ for any $\sigma \in M$. Let $y \in A^n$ be a one-to-one sequence with range X and let f be defined by (i). The local invertibility of E shows that $f \in F$. But $\varphi(y_0) = \varphi f(y) = f(\varphi y) = \varphi y_1$ and, on the other hand, $y_0 \neq y_1$. Since φ is one-to-one, $\varphi(y_0) \neq \varphi(y_1)$, a contradiction. So we must have $\varphi \in M$.

Step III. If $\varphi \in \operatorname{End}\mathfrak{A}$ is constant, then $\varphi \in M$.

Suppose $\varphi(x) = 0$ for $x \in A$. If $\varphi \notin M$, then $\varphi \notin C(M)$. Thus there is some $b \in A$, $b \neq a$, such that $\sigma(b) = \tau(b)$ whenever $\sigma, \tau \in M$ and $\sigma(a) = \tau(a)$. Define a unary operation $f_{\langle a,b \rangle}$ as follows:

$$f_{\langle a,b \rangle}(z) = \begin{cases} \sigma b & \text{if } z = \sigma a \text{ for some } \sigma \in E, \\ z & \text{otherwise.} \end{cases}$$

Using the same argument as in the proof of Step I, we get $f_{\langle a,b \rangle} \in F$. But now $a = \varphi f_{\langle a,b \rangle}(a) = f_{\langle a,b \rangle}(\varphi(a)) = b$, a contradiction. So we must have $\varphi \in M$. This completes the proof of Theorem 1.

It is easy to see that certain monoids are algebraic by means of this theorem. For example, we have

COROLLARY 1. *The monoid of all one-to-one maps and all constant maps on a given set is algebraic.*

Lampe's characterization of algebraic monoids of the form $G \cup K$, where G is a group (see [3]), also is a corollary to Theorem 1. Algebraic monoids of the form $G \cup K \cup \{\varphi\}$, where G is a group and φ is arbitrary, are characterized in [7]. It is shown there that the local closure plus the inclusion of sufficiently many constant maps is both necessary and sufficient. A generalization of Theorem 1 replaces the local closure by the m-local closure after the manner of Płonka [5] to characterize those monoids of the form $E \cup K$, where E is locally invertible, which equal $\mathrm{End}\,\mathfrak{A}$ for some algebra with at most m-ary operations; again the m-local closure together with the inclusion of constants is both necessary and sufficient [7].

4. Remarks on the general problem. It is worth-while to observe that the local closure together with the inclusion of enough constant maps is not, in general, necessary and sufficient for M to be algebraic. The following result shows that it is sometimes necessary to include many non--constant maps to make a given monoid algebraic:

THEOREM 2. *Let the kernels of M exhaust all non-trivial partitions on A and let M include two constant maps. Then M is algebraic iff $M = A^A$.*

Proof. A^A is obviously algebraic. Conversely, let $M = \mathrm{End}\,\mathfrak{A}$, where $\mathfrak{A} = \langle A, F\rangle$. Since the kernels of M exhaust all non-trivial partitions on A, every equivalence relation on A is a congruence. Hence $\mathfrak{A}$ (with $|A| \geqslant 3$) can have as operations only constant operators and projections (cf. [4], Exercise 2, p. 38). In fact, no $f \in F$ is a constant operation, since no element of A is fixed under all endomorphisms (this could also be insured by some explicit assumption on M other than the inclusion of two constant maps). Thus F consists entirely of projections and every map is an endomorphism, so $M = A^A$. This completes the proof of Theorem 2.

Theorem 2 enables one to produce easily a variety of non-algebraic monoids; we obtain, for example (for $|A| \geqslant 3$),

COROLLARY 2. *For any different $a, b \in A$ the monoid consisting of the identity together with all constant maps and all maps into $\{a, b\}$ is not algebraic.*

If A is finite, the monoids in Corollary 2 are locally closed and include all constant maps, but fail to be algebraic.

The interrelation of subsets of A and the structure of M can be explored further to yield a sufficient condition for M to be algebraic. We say that a subset $B \subseteq A$ is *M-independent* if, for each one-to-one sequence $b \in B^n$ and each sequence $d \in A^n$, there exists a $\sigma \in M$ with $\sigma(b) = d$. A subset $B \subseteq A$ *M-spans A* provided, for any $\sigma, \tau \in M$, the equality $\sigma \restriction B = \tau \restriction B$ implies $\sigma = \tau$. An *M-basis* for A is an M-independent set which M-spans A.

THEOREM 3. *If A has a finite M-basis, then M is algebraic.*

Proof. Let $\mathfrak{A} = \langle A, F\rangle$ be the algebra of all operations substitutive over M. Then $M \subseteq \mathrm{End}\,\mathfrak{A}$. Suppose B is a (finite) M-basis for A, say $|B| = n$. For each one-to-one sequence $b \in B^n$ and $a \in A$ define an n-ary operation f_b^a as follows:

$$f_b^a(X) = f_b^a\big(\sigma(b)\big) = \sigma(a), \quad \text{where } \sigma \in M \text{ and } X = \sigma(b).$$

Note that f_b^a is well defined, since B spans A. Moreover, $f_b^a \in F$, since

$$\tau\big(f_b^a(X)\big) = \tau\sigma(a) = f_b^a\big(\tau\sigma(b)\big) = f_b^a\big(\tau(X)\big) \quad \text{for } \tau \in M.$$

Fix $\varphi \in \mathrm{End}\,\mathfrak{A}$. Since B is M-independent, there exists a $\sigma \in M$ with $\sigma \restriction B = \varphi \restriction B$. In fact,

$$\varphi(a) = \varphi f_b^a(b) = f_b^a(\varphi b) = f_b^a\big(\sigma(b)\big) = \sigma(a) \quad \text{for } a \in A.$$

Thus $\varphi = \sigma$, so that $\varphi \in M$. Hence $M = \mathrm{End}\,\mathfrak{A}$.

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