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DENDRITIC DECOMPOSITIONS OF GENERALIZED SIMPLE CLOSED CURVES

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A *dendritic continuum*, or *dendrite*, is a continuum such that each two of its points are separated by a third point. A (*generalized*) *simple closed curve* is a Hausdorff continuum that is separated by each pair of its points. An *arc* is a Hausdorff continuum with only two non-cut points. If p is a point of the dendrite Y, then $c(p)$ denotes the cardinal number of components of $Y-\{p\}$. The point p of Y is a *branch point* of Y if $c(p) \geqslant 3$, and p is an *infinite branch point* of Y if $c(p) \geqslant \aleph_0$. A map f of X onto Y is *light* if, for each point y of Y, $f^{-1}(y)$ is totally disconnected, and f is *non-alternating* if $\{f^{-1}(y) \mid y \in Y\}$ is *non-separating*, i.e., for each two points y and z of Y neither of the sets $f^{-1}(y)$ and $f^{-1}(z)$ separates two points of the other set in X. The cardinal of a set M is denoted by $|M|$.

We showed in [7] that for each dendrite Y there exist a simple closed curve X and a continuous light non-alternating map f of X onto Y such that, for each point y of Y, $c(y) = |f^{-1}(y)|$. It is the main purpose of this paper to prove that for each simple closed curve X there exist a dendrite Y and a map f of X onto Y having the above properties. Now, the proof of this theorem is trivial in the metric case; indeed, Y may be required to be an arc. However, in the non-metric case, it is by no means clear how to obtain Y and f. For example, X may be so complicated that no two arcs in X are homeomorphic (cf. [6]), or X may fail to have a countable base at any point (see [5], Theorem 5, p. 61). Furthermore, we shall show that the dendrite Y may be required to have the property that the set of all its branch points is dense in Y. The dendrite Y may also be required to have the property that each of its branch points is infinite, which is of some interest even in the metric case, for it then follows that Y is a universal metric dendrite. Finally, we shall consider an application of the case in which, for some point x of X, $X-\{x\}$ is a Suslin space.

1. The main theorem. This section is devoted to a proof of the main theorem. The proof involves a tedious transfinite induction and is broken up into several lemmas.

THEOREM 1. *For each simple closed curve X there exist a dendrite Y and a continuous light non-alternating map f of X onto Y such that the set of all branch points of Y is dense in Y and, for each point y of Y, $c(y) = |f^{-1}(y)|$.*

Definition. Suppose that X is a simple closed curve, p and q are two points of X, and $\mathscr{S}$ is a collection of subsets of X such that

(1) the elements of $\mathscr{S}$ are pairwise disjoint,

(2) each element of $\mathscr{S}$ is finite,

(3) $\bigcup \mathscr{S}$ is perfect,

(4) if U is a component of $X - \bigcup \mathscr{S}$, then ∂U is a subset of some element of $\mathscr{S}$,

(5) $\mathscr{S}$ is non-separating,

(6) if H and K are two elements of $\mathscr{S}$, then some element of $\mathscr{S}$ containing at least four points separates H from K in X,

(7) if $x = p$ or $x = q$, then either $\{x\} \in \mathscr{S}$ or there is a point y of $X - \{p, q\}$ such that $\{x, y\} \in \mathscr{S}$.

Then $\mathscr{S}$ is said to have *property* X_{pq}.

LEMMA 1. *If X is a simple closed curve, and p and q are two points of X, then there exists a collection $\mathscr{S}$ of subsets of X such that $\mathscr{S}$ has property X_{pq}.*

Proof. Let U and V denote the two components of $X - \{p, q\}$. Let $\mathscr{P}$ be the collection of all ordered pairs (A, B) such that A is an arc lying in U, and B is an arc lying in V. We order $\mathscr{P}$ as follows. Let $(A, B) < (C, D)$ if C separates A from q in $\bar{U}$, and D separates B from q in $\bar{V}$. Then $\mathscr{P}$ is clearly partially ordered by the relation $\leqslant$. Let $\mathscr{I}$ be a maximal chain in $\mathscr{P}$. Let H be the union of all first terms of the elements of $\mathscr{I}$, and let K be the union of all second terms of the elements of $\mathscr{I}$. For each (A, B) in $\mathscr{I}$ let A' be the component of $(U - H) \cup A$ containing A. It follows from the maximality of $\mathscr{I}$ that there is neither a last element of $\mathscr{I}$ preceding (A, B) nor a first element of $\mathscr{I}$ following (A, B), so that A' is an arc. Let $\mathscr{I}'$ be the collection of all such ordered pairs (A', B), and let H' be the union of all first terms of the elements of $\mathscr{I}'$. Hence, if $(A', B) \in \mathscr{I}'$, then each end point of A' is a limit point of $H' - A'$. Now, for each (A', B) in $\mathscr{I}'$ let B' be the component of $(V - K) \cup B$ containing B. It follows, as before, that B' is an arc. Let $\mathscr{I}''$ be the collection of all such ordered pairs (A', B'), and let K' be the union of all second terms of the elements of $\mathscr{I}''$. If $(A', B') \in \mathscr{I}''$, then each end point of B' is a limit point of $K' - B'$. We now define $\mathscr{S}$ to be the collection consisting of the following sets. If $(A', B') \in \mathscr{I}''$, then the set consisting of the four end points of A' and B' is in $\mathscr{S}$. If $a \in \overline{H'} - H'$ and $a \neq q$, then there is a well-ordered decreasing chain $\{(A_\xi, B_\xi) \mid \xi < \lambda\}$ in $\mathscr{I}''$ such that the net $\{A_\xi, \xi < \lambda\}$ converges to a. The net $\{B_\xi, \xi < \lambda\}$ then converges to a point b, and we let $\{a, b\}$ be in $\mathscr{S}$. Thus, if $p \in \overline{H'}$, then there is a point r of $\overline{K'}$, different from p

or not, such that $\{p, r\} \in \mathscr{S}$. If $p \notin \overline{H'}$, then it follows from the maximality of $\mathscr{S}$ that $p \in \overline{K'}$, and hence there exists a point r of $\overline{H'}$ such that $\{p, r\} \in \mathscr{S}$. Similarly, there exists a point s such that $\{q, s\} \in \mathscr{S}$. It is easily seen that $\mathscr{S}$ satisfies all the conditions for having property X_{pq}.

Now let X denote a definite simple closed curve, and let p and q denote two points of X. Let $\mathscr{S}_0$ be a collection of subsets of X with property X_{pq}. Suppose that $\mathscr{S}_\alpha$ has been defined and has property X_{pq}. For each component U of $X - \bigcup \mathscr{S}_\alpha$ let x and y be the end points of $\overline{U}$, let z be a point of U, and let $\mathscr{S}_U$ be a collection of subsets of $\overline{U}$ such that if $\overline{U}$ is considered as a simple closed curve by identifying x and y, then $\mathscr{S}_U$ has property $\overline{U}_{xz}$. In what follows, S_α denotes $\bigcup \mathscr{S}_\alpha$, and S_U denotes $\bigcup \mathscr{S}_U$. The closure of a point set M will sometimes be denoted by ClM. If $M \in \mathscr{S}_\alpha$, then $C_\alpha(M)$ denotes the union of M and all components of $X - S_\alpha$ with end points in M, and if $M \in \mathscr{S}_U$, then $C_U(M)$ denotes the union of M and all components of $\overline{U} - S_U$ with end points in M. Let $\mathscr{S}_{\alpha+1}$ be the collection of all subsets M of X such that either (i) or (ii) is satisfied:

(i) for some component U of $X - S_\alpha$, $M \in \mathscr{S}_U$ and $M \cap \partial U = \emptyset$;

(ii) M is the union of some element S of $\mathscr{S}_\alpha$ and all subsets T of X such that, for some component U of $X - S_\alpha$, $\partial U \subseteq S$, $T \in \mathscr{S}_U$, and $S \cap T \neq \emptyset$; if no such T exists, we take $M = S$.

Now suppose that γ is a limit ordinal and that $\mathscr{S}_\alpha$ has been defined and has property X_{pq} for $\alpha < \gamma$. Let $\mathscr{S}_\gamma$ be the collection of all subsets M of X such that either (iii) or (iv) is satisfied:

(iii) there exist an $\alpha < \gamma$ and a collection $\{M_\xi \mid \alpha \leqslant \xi < \gamma\}$ such that, for each ξ,

$$M_\xi \in \mathscr{S}_\xi, \qquad \bigcap \{M_\xi \mid \alpha \leqslant \xi < \gamma\} \neq \emptyset,$$

and

$$M = \mathrm{Cl} \bigcup \{M_\xi \mid \alpha \leqslant \xi < \gamma\};$$

(iv) for each $\xi < \gamma$ there is a component U_ξ of $X - S_\xi$ such that if M_ξ is an element of $\mathscr{S}_\xi$ containing ∂U_ξ, then, for each $\alpha < \xi$,

$$\bigcap \{M_\xi \mid \alpha \leqslant \xi < \gamma\} = \emptyset \quad \text{and} \quad M = \partial \bigcap \{U_\xi \mid \xi < \gamma\}.$$

LEMMA 2. *If* $\alpha < \beta$, *then* $S_\alpha \subseteq S_\beta$.

Proof. The proof is by induction on β. Suppose that $S_\alpha \subseteq S_\beta$. It follows from (i) and (ii) that $S_\beta \subseteq S_{\beta+1}$, and hence $S_\alpha \subseteq S_{\beta+1}$. Now suppose that γ is a limit ordinal and $S_\alpha \subseteq S_\beta$ for each β such that $\alpha < \beta < \gamma$. Let x be a point of S_α. For each β such that $\alpha < \beta < \gamma$ there is an element M_β of $\mathscr{S}_\beta$ containing x. Hence

$$\bigcap \{M_\beta \mid \alpha \leqslant \beta < \gamma\} \neq \emptyset,$$

and it then follows from (iii) that

$$\mathrm{Cl}\bigcup\{M_\beta \mid \alpha \leqslant \beta < \gamma\} \in \mathscr{S}_\gamma.$$

Therefore $S_\alpha \subseteq S_\gamma$.

Remark. It follows that the collection $\{U_\xi \mid \xi < \gamma\}$ defining M in (iv) is a *decreasing chain*, i.e., if $\alpha < \beta < \gamma$, then $U_\alpha \supseteq U_\beta$.

LEMMA 3. *If γ is a limit ordinal and, for each $\xi < \gamma$, U_ξ is a component of $X - \mathscr{S}_\xi$, then $\partial\bigcap\{U_\xi \mid \xi < \gamma\} \subseteq S_\gamma$.*

Proof. Let $U = \bigcap\{U_\xi \mid \xi < \gamma\}$, and assume that $U \neq \varnothing$. Suppose that there is an $\alpha < \gamma$ such that for each ξ with $\alpha \leqslant \xi < \gamma$ there is an element M_ξ of $\mathscr{S}_\xi$ containing ∂U_ξ such that

$$\bigcap\{M_\xi \mid \alpha \leqslant \xi < \gamma\} \neq \varnothing.$$

If M denotes $\mathrm{Cl}\bigcup\{M_\xi \mid \alpha \leqslant \xi < \gamma\}$, then $\partial U \subseteq M$ and it follows from (iii) that $M \in \mathscr{S}_\gamma$. If no such α exists, then, for each $\xi < \gamma$, each M_ξ in $\mathscr{S}_\xi$ containing ∂U_ξ, and each $\alpha < \gamma$,

$$\bigcap\{M_\xi \mid \alpha \leqslant \xi < \gamma\} = \varnothing,$$

and it follows from (iv) that $\partial U \in \mathscr{S}_\gamma$. In either case, $\partial U \subseteq S_\gamma$.

LEMMA 4. *If $H \in \mathscr{S}_\alpha$, U is a component of $X - S_\alpha$ with $\partial U \subseteq H$, and $K \in \mathscr{S}_{\alpha+1}$ is such that $H \subseteq K$, then U contains at most one point of K.*

Proof. It follows from (ii) that there is an element L of $\mathscr{S}_U$ such that $L = K \cap \bar{U}$. Hence $\partial U \subseteq L$, and it then follows from condition (7) for $\mathscr{S}_U$ that U contains at most one point of L.

LEMMA 5. *For each α, S_α is closed and each element of $\mathscr{S}_\alpha$ is closed.*

Proof. The proof is by induction on α. Let the lemma be denoted by $P(\alpha)$. Now $P(0)$ follows from the definition of $\mathscr{S}_0$. We shall assume $P(\alpha)$ and prove $P(\alpha+1)$. Let M be an element of $\mathscr{S}_{\alpha+1}$. If $M \in \mathscr{S}_U$ for some component U of $X - S_\alpha$, then M is closed in $\bar{U}$ and hence in X. Suppose that, for each component U of $X - S_\alpha$, $M \notin \mathscr{S}_U$, and assume that $x \in \bar{M} - M$. It then follows from (ii) and $P(\alpha)$ that, for some element H of $\mathscr{S}_\alpha$, $H \subseteq M$, $M \subseteq C_\alpha(H)$, and since H is closed, for some component U of $X - S_\alpha$, we have $x \in U$. Now, since S_α is closed, U is open, and hence x is a limit point of $M \cap U$. But, by Lemma 4, $M \cap U$ is degenerate. Therefore, each element of $\mathscr{S}_{\alpha+1}$ is closed. Suppose that $x \in \overline{S_{\alpha+1}} - S_{\alpha+1}$. Since $S_\alpha \subseteq S_{\alpha+1}$, there is a component U of $X - S_\alpha$ such that $x \in U$. Hence x is a limit point of $S_{\alpha+1} \cap U$, since U is open. But it follows from (i) and (ii) that $S_{\alpha+1} \cap U = S_U - \partial U$, and S_U is closed in $\bar{U}$, and hence in X, so that $x \in \partial U$, a contradiction. Therefore $S_{\alpha+1}$ is closed.

Suppose that γ is a limit ordinal and $P(\alpha)$ is true for each $\alpha < \gamma$. By definition, each element of $\mathscr{S}_\gamma$ is closed. If $M \in \mathscr{S}_\gamma$ obtained from (iii), then, clearly,

$$M \subseteq \mathrm{Cl} \bigcup \{S_\xi \mid \xi < \gamma\}.$$

If $M = \partial \bigcap \{U_\xi \mid \xi < \gamma\}$ as in (iv), then, for each $\xi < \gamma$, $\partial U_\xi \subseteq S_\xi$ since S_ξ is closed, and hence $M \subseteq \mathrm{Cl} \bigcup \{S_\xi \mid \xi < \gamma\}$. It follows that

$$\overline{S_\gamma} \subseteq \mathrm{Cl} \bigcup \{S_\xi \mid \xi < \gamma\}.$$

Suppose that $x \in \overline{S_\gamma} - S_\gamma$. For each $\xi < \gamma$ there is a component U_ξ of $X - S_\xi$ containing x. Let $U = \bigcap \{U_\xi \mid \xi < \gamma\}$. Then $x \in \partial U$, and it follows from Lemma 3 that $x \in S_\gamma$. Therefore S_γ is closed.

LEMMA 6. *For each α and each component U of $X - S_\alpha$ there is an element M of $\mathscr{S}_\alpha$ such that $\partial U \subseteq M$.*

Proof. Let the lemma be denoted by $P(\alpha)$. Suppose that $P(\alpha)$ is true and U is a component of $X - S_{\alpha+1}$. Then U is a subset of some component V of $X - S_\alpha$, and hence U is a component of $\overline{V} - S_V$. There is an element K of $\mathscr{S}_V$ such that $\partial U \subseteq K$. If $K \cap \partial U = \emptyset$, then $K \in \mathscr{S}_{\alpha+1}$. If $K \cap \partial U \neq \emptyset$, then there is an element H of $\mathscr{S}_\alpha$ such that $\partial V \subseteq H$, and since $H \cap K \neq \emptyset$, it follows from (ii) that $H \cup K$ is a subset of some element of $\mathscr{S}_{\alpha+1}$. In either case, ∂U is a subset of some element of $\mathscr{S}_{\alpha+1}$. Therefore $P(\alpha)$ implies $P(\alpha+1)$. Suppose that γ is a limit ordinal and $P(\alpha)$ is true for each $\alpha < \gamma$. Let U be a component of $X - S_\gamma$. For each $\xi < \gamma$ let U_ξ be the component of $X - S_\xi$ containing U. Hence

$$U = \bigcap \{U_\xi \mid \xi < \gamma\}.$$

It then follows from Lemma 3 that there is an element of $\mathscr{S}_\gamma$ containing ∂U.

LEMMA 7. *For each α, S_α is perfect.*

Proof. S_0 is perfect by definition. Suppose that S_α is perfect. Let x be a point of $S_{\alpha+1}$. If $x \in S_\alpha$, then x is a limit point of $S_{\alpha+1}$ since $S_\alpha \subseteq S_{\alpha+1}$. If $x \in X - S_\alpha$, then it follows from (ii) that, for some component U of $X - S_\alpha$, $x \in S_U$, and, since S_U is perfect and $S_U \subseteq S_{\alpha+1}$, x is a limit point of $S_{\alpha+1}$. Now suppose that γ is a limit ordinal and S_α is perfect for each $\alpha < \gamma$. Let x be a point of S_γ, and M an element of $\mathscr{S}_\gamma$ containing x. First, suppose that for some $\alpha < \gamma$ there is a collection $\{M_\xi \mid \alpha \leqslant \xi < \gamma\}$ as in (iii) such that

$$M = \mathrm{Cl} \bigcup \{M_\xi \mid \alpha \leqslant \xi < \gamma\}.$$

Then, for some ξ, $x \in M_\xi$ or x is a limit point of M. Therefore, in either case, x is a limit point of S_γ. Now suppose that for each $\xi < \gamma$ there is a component U_ξ of $X - S_\xi$ as in (iv) such that

$$M = \partial \bigcap \{U_\xi \mid \xi < \gamma\}.$$

If, for some $\xi < \gamma$, $x \in \partial U_\xi$, then $x \in S_\xi$, and hence is a limit point of S_ξ. If, for each $\xi < \gamma$, $x \notin \partial U_\xi$, then x is a limit point of $\bigcup\{\partial U_\xi \mid \xi < \gamma\}$, which is a subset of $\bigcup\{S_\xi \mid \xi < \gamma\}$ by Lemma 3, and $\bigcup\{S_\xi \mid \xi < \gamma\} \subseteq S_\gamma$ by Lemma 2. It follows that x is a limit point of S_γ.

LEMMA 8. *If $\mathscr{S}_a$ is disjoint, then $\mathscr{S}_{a+1}$ is disjoint.*

Proof. Suppose that H and K are two elements of $\mathscr{S}_{a+1}$. If H and K are subsets of different components of $X - S_a$, then $H \cap K = \emptyset$. If H and K are subsets of the same component U of $X - S_a$, then H and K are in $\mathscr{S}_U$, and hence $H \cap K = \emptyset$. If H is a subset of some component of $X - S_a$ and K is not a subset of any component of $X - S_a$, then it follows from (ii) that $H \cap K = \emptyset$. Suppose that neither H nor K is a subset of any component of $X - S_a$. If L denotes either H or K, then it follows from (ii) that L is the union of an element L' of S_a and the collection $\mathscr{I}_L$ of all sets M such that, for some component U of $X - S_a$, $M \in \mathscr{S}_U$ and $L \cap M \neq \emptyset$. Thus

$$H = H' \cup \bigcup \mathscr{I}_H \quad \text{and} \quad K = K' \cup \bigcup \mathscr{I}_K.$$

Since H' and K' are in $\mathscr{S}_a$, $H' \cap K' = \emptyset$. Suppose that $M \in \mathscr{I}_H$ and $M \cap K' \neq \emptyset$. Then, for some component U of $X - S_a$, $M \in \mathscr{S}_U$ and $M \cap H' \neq \emptyset$. Hence ∂U is a subset of both H' and K', a contradiction. Therefore $\bigcup \mathscr{I}_H \cap K' = \emptyset$ and, similarly, $\bigcup \mathscr{I}_K \cap H' = \emptyset$. Finally, suppose that $L \in \mathscr{I}_H$ and $M \in \mathscr{I}_K$ are such that $L \cap M \neq \emptyset$. There exist components U and V of $X - S_a$ such that $L \in \mathscr{S}_U$ and $M \in \mathscr{S}_V$. If $U = V$, then $L = M$ and we get a contradiction as above. Hence $U \neq V$, and it follows from Lemma 7 that $\bar{U} \cap \bar{V} = \emptyset$. But $L \subseteq \bar{U}$ and $M \subseteq \bar{V}$. Therefore,

$$\bigcup \mathscr{I}_H \cap \bigcup \mathscr{I}_K = \emptyset,$$

and it follows that $H \cap K = \emptyset$.

LEMMA 9. *If $a < \beta$, $M \in \mathscr{S}_a$, and $\mathscr{S}_\xi$ is disjoint for $a \leqslant \xi < \beta$, then there exists a chain $\{M_\xi \mid a \leqslant \xi \leqslant \beta\}$ such that $M_\xi \in \mathscr{S}_\xi$ for each ξ and $M_a = M$.*

Proof. The proof is by induction on β. Suppose that $\{M_\xi \mid a \leqslant \xi \leqslant \beta\}$ is a chain such that $M_\xi \in \mathscr{S}_\xi$ for each ξ and $M_a = M$. Now M_β is a subset of some element $M_{\beta+1}$ of $\mathscr{S}_{\beta+1}$, so that $\{M_\xi \mid a \leqslant \xi \leqslant \beta+1\}$ is the desired chain. Now suppose that γ is a limit ordinal and, for each β with $a < \beta < \gamma$, $\{M_\xi^\beta \mid a \leqslant \xi \leqslant \beta\}$ is a chain such that $M_\xi^\beta \in \mathscr{S}_\xi$ for each ξ and $M_a^\beta = M$. Now, for $a < \zeta < \eta < \gamma$ and for each ξ, $M_\xi^\zeta \cap M_\xi^\eta \neq \emptyset$, and hence $M_\xi^\zeta = M_\xi^\eta$. For each β such that $a < \beta < \gamma$ denote M_ξ^β by M_ξ. Hence $\{M_\xi \mid a \leqslant \xi < \gamma\}$ is a chain. Let

$$M_\gamma = \mathrm{Cl} \bigcup \{M_\xi \mid a \leqslant \xi < \gamma\}.$$

It follows from (iii) that $M_\gamma \in \mathscr{S}_\gamma$, and hence $\{M_\xi \mid a \leqslant \xi \leqslant \gamma\}$ is the desired chain.

LEMMA 10. *If* $\mathscr{S}_\xi$ *is disjoint and* $M_\xi \in \mathscr{S}_\xi$ *for* $\alpha \leqslant \xi < \gamma$ *and* $\mathrm{Cl}\bigcup\{M_\xi \mid \alpha \leqslant \xi < \gamma\} \in \mathscr{S}_\gamma$, *then* $\{M_\xi \mid \alpha \leqslant \xi < \gamma\}$ *is a chain.*

Proof. By Lemma 9, there is a chain $\{M'_\xi \mid \alpha \leqslant \xi < \gamma\}$ such that $M'_\xi \in \mathscr{S}_\xi$ for each ξ and $M'_\alpha = M_\alpha$. Then, since $M_\xi \cap M'_\xi \neq \emptyset$, we have $M_\xi = M'_\xi$.

LEMMA 11. *If* $H \in \mathscr{S}_\alpha$, $K \in \mathscr{S}_{\alpha+1}$, *and* $H \subseteq K$, *then* $C_\alpha(H) \supseteq C_{\alpha+1}(K)$.

Proof. Clearly, there is no component U of $X - S_\alpha$ such that $K \in \mathscr{S}_U$ and $K \cap \partial U = \emptyset$. It then follows from (ii) that $K \subseteq C_\alpha(H)$. Let V be a component of $X - S_{\alpha+1}$ such that $\partial V \subseteq K$. If $\partial V \subseteq H$, then V is a component of $X - S_\alpha$, and hence $V \subseteq C_\alpha(H)$. Suppose that $x \in \partial V - H$. Then $x \in K - H$, and it follows from (ii) that there is a component U of $X - S_\alpha$ such that $\partial U \subseteq H$ and $x \in U$. Now V is a subset of some component of $X - S_\alpha$, and since $U \cap V \neq \emptyset$, we have $V \subseteq U$, so that $V \subseteq C_\alpha(H)$. Therefore $C_{\alpha+1}(K) \subseteq C_\alpha(H)$.

LEMMA 12. *For each* β *and each two elements* H *and* K *of* $\mathscr{S}_\beta$,

$$C_\beta(H) \cap C_\beta(K) = \emptyset,$$

and if $\alpha < \beta$, $H \in \mathscr{S}_\alpha$, $K \in \mathscr{S}_\beta$, *and* $H \subseteq K$, *then*

$$C_\alpha(H) \supseteq C_\beta(K).$$

Proof. The proof is by induction on β. Denote the lemma by $P(\beta)$, and suppose that $P(\xi)$ holds for each $\xi \leqslant \beta$. Hence, for each $\xi \leqslant \beta$, $\mathscr{S}_\xi$ is disjoint. Suppose that $H \in \mathscr{S}_\alpha$ and $L \in \mathscr{S}_{\beta+1}$ are such that $H \subseteq L$. By Lemma 8, $\mathscr{S}_{\beta+1}$ is disjoint. It then follows from Lemma 9 that there is an element K of $\mathscr{S}_\beta$ such that $H \subseteq K \subseteq L$. By Lemma 11, $C_{\beta+1}(L) \subseteq C_\beta(K)$, and $P(\beta)$ implies that $C_\beta(K) \subseteq C_\alpha(H)$. Hence

$$C_{\beta+1}(L) \subseteq C_\alpha(H).$$

Now suppose that H and K are two elements of $\mathscr{S}_{\beta+1}$.

Case 1. For some component U of $X - S_\beta$, we have $H \subseteq U$ and $K \subseteq U$. Hence both H and K are elements of $\mathscr{S}_U$. Now, if V is a component of $X - S_{\beta+1}$ such that $\partial V \subseteq H$, then $V \subseteq U$, since ∂U separates X, and ∂U is a subset of some element of $\mathscr{S}_{\beta+1}$. Hence V is a component of $\overline{U} - S_U$, and $\partial V \subseteq H$. It follows that $C_U(H) = C_{\beta+1}(H)$ and, similarly, $C_U(K) = C_{\beta+1}(K)$. Since $\mathscr{S}_U$ is non-separating,

$$C_U(H) \cap C_U(K) = \emptyset.$$

Case 2. For some component U of $X - S_{\beta+1}$, we have $H \subseteq U$ and $\partial U \subseteq K$. Then $K \cap \overline{U} \in \mathscr{S}_U$, and some element L of $\mathscr{S}_U$ separates H from $K \cap \overline{U}$ in $\overline{U}$. It follows that L separates H from K in X, and since $L \in \mathscr{S}_{\beta+1}$, L separates $C_{\beta+1}(H)$ from $C_{\beta+1}(K)$ in X.

Case 3. For each component U of $X-S_\beta$, neither H nor K is a subset of U. It follows from (ii) that there exist elements L and M of $\mathscr{S}_\beta$ such that $L \subseteq H$ and $M \subseteq K$. Then

$$C_{\beta+1}(H) \subseteq C_\beta(L), \quad C_{\beta+1}(K) \subseteq C_\beta(L), \quad \text{and} \quad C_\beta(L) \cap C_\beta(M) = \emptyset.$$

Thus we have shown that if $P(\xi)$ holds for each $\xi \leqslant \beta$, then $P(\beta+1)$ holds.

Suppose that γ is a limit ordinal and $P(\xi)$ holds for each $\xi < \gamma$. Assume that $\alpha < \gamma$, $H \in \mathscr{S}_\alpha$, and $K \in \mathscr{S}_\gamma$ are such that $H \subseteq K$.

Case 1. For each $\xi < \gamma$ there is a component U_ξ of $X - S_\xi$ such that $K = \partial\{U_\xi \mid \xi < \gamma\}$. Hence $H \subseteq \overline{U_{\alpha+1}}$, and since $U_{\alpha+1} \subset U_\alpha$ and $H \subseteq X - U_\alpha$, there is a point x of ∂U_α such that $H = \{x\}$. But it follows from Lemma 6 and $P(\alpha)$ that $\partial U_\alpha \subseteq H$.

Case 2. There exist a $\beta < \gamma$ and, for each ξ with $\beta \leqslant \xi < \gamma$, an element K_ξ of $\mathscr{S}_\xi$ such that

$$K = \mathrm{Cl}\bigcup\{K_\xi \mid \beta \leqslant \xi < \gamma\}.$$

By Lemma 10, $\{K_\xi \mid \beta \leqslant \xi < \gamma\}$ is a chain, so that we may take $\beta > \alpha$. Thus, for each ξ,

$$K_\xi \subseteq C_\beta(K_\beta) \subseteq C_\alpha(H),$$

and hence $K \subseteq C_\alpha(H)$. Let U be a component of $X - S_\gamma$ such that $\partial U \subseteq K$. For each $\xi < \gamma$, U is contained in a component U_ξ of $X - S_\xi$, and ∂U_ξ is a subset of some element L_ξ of $\mathscr{S}_\xi$. Suppose that, for some $\zeta < \gamma$, $K_\zeta \neq L_\zeta$. Now

$$K = \mathrm{Cl}\bigcup\{K_\xi \mid \zeta \leqslant \xi < \gamma\}$$

and, for each ξ, $K_\xi \subseteq C_\zeta(K_\zeta)$, so that $\partial U \subseteq K \subseteq C_\zeta(K_\zeta)$. But

$$U \subseteq U_\zeta \subseteq C_\zeta(L_\zeta),$$

and $C_\zeta(K_\zeta)$ and $C_\zeta(L_\zeta)$ are disjoint closed sets. Hence, for each $\xi < \gamma$, $K_\xi = L_\xi$, so that $U \subseteq C_\xi(K) \subseteq C_\alpha(H)$. Therefore $C_\alpha(H) \supseteq C_\gamma(K)$.

Now suppose that H and K are two elements of $\mathscr{S}_\gamma$.

Case 1. For each $\xi < \gamma$ there exist components U_ξ and V_ξ of $X - S_\xi$ such that

$$H = \partial\bigcap\{U_\xi \mid \xi < \gamma\} \quad \text{and} \quad K = \partial\bigcap\{V_\xi \mid \xi < \gamma\}.$$

For some $\alpha < \gamma$, $U_\alpha \neq V_\alpha$, and hence $\overline{U_\alpha} \cap \overline{V_\alpha} = \emptyset$. Clearly,

$$C_\gamma(H) \subseteq \overline{U_\alpha} \quad \text{and} \quad C_\gamma(K) \subseteq \overline{V_\alpha}.$$

Case 2. There exist an $\alpha < \gamma$ and, for each ξ with $\alpha \leqslant \xi < \gamma$, elements H_ξ and K_ξ of $\mathscr{S}_\xi$ such that

$$H = \mathrm{Cl}\bigcup\{H_\xi \mid \alpha \leqslant \xi < \gamma\} \quad \text{and} \quad K = \mathrm{Cl}\bigcup\{K_\xi \mid \alpha \leqslant \xi < \gamma\}.$$

For some β such that $a \leqslant \beta < \gamma$, we have $H_\beta \neq K_\beta$. Then

$$C_\gamma(H) \subseteq C_\beta(H_\beta), \quad C_\gamma(K) \subseteq C_\beta(K_\beta), \quad \text{and} \quad C_\beta(H_\beta) \cap C_\beta(K_\beta) = \emptyset.$$

Case 3. For each $\xi < \gamma$ there is a component U_ξ of $X - S_\xi$ with

$$H = \partial \bigcap \{U_\xi \mid \xi < \gamma\},$$

and there exist an $a < \gamma$ and, for each ξ with $a \leqslant \xi < \gamma$, an element K_ξ of $\mathscr{S}_\xi$ such that

$$K = \mathrm{Cl} \bigcup \{K_\xi \mid a \leqslant \xi < \gamma\}.$$

Since

$$\bigcap \{H_\xi \mid a \leqslant \xi < \gamma\} = \emptyset \quad \text{and} \quad \bigcap \{K_\xi \mid a \leqslant \xi < \gamma\} \neq \emptyset,$$

there is a β such that $a \leqslant \beta < \gamma$ and $H_\beta \neq K_\beta$. Then

$$C_\gamma(H) \subseteq C_\beta(H_\beta), \quad C_\gamma(K) \subseteq C_\beta(K_\beta), \quad \text{and} \quad C_\beta(H_\beta) \cap C_\beta(K_\beta) = \emptyset.$$

Therefore, in any case, $C_\gamma(H) \cap C_\gamma(K) = \emptyset$.

LEMMA 13. *For each a, each element of $\mathscr{S}_a$ is totally disconnected.*

Proof. Clearly, each element of $\mathscr{S}_0$ is totally disconnected. It follows from Lemma 4 that if each element of $\mathscr{S}_a$ is totally disconnected, then each element of $\mathscr{S}_{a+1}$ is totally disconnected. Suppose that γ is a limit ordinal, $M \in \mathscr{S}_\gamma$ and, for each $a < \gamma$, each element of $\mathscr{S}_a$ is totally disconnected. If

$$M = \partial \bigcap \{U_\xi \mid \xi < \gamma\},$$

then M consists of at most two points. Suppose that

$$M = \mathrm{Cl} \bigcup \{M_\xi \mid a \leqslant \xi < \gamma\}$$

and L is an arc in M. It follows from Lemma 12 that $M \subseteq C_\xi(M_\xi)$ for each ξ such that $a \leqslant \xi < \gamma$, and since M_ξ is totally disconnected, there is a component U_ξ of $X - S_\xi$ such that $\partial U_\xi \subseteq M_\xi$ and $L \subseteq \overline{U_\xi}$. Hence

$$L \subseteq \bigcap \{\overline{U_\xi} \mid a \leqslant \xi < \gamma\},$$

and since $\partial \bigcap \{\overline{U_\xi} \mid a \leqslant \xi < \gamma\} \in \mathscr{S}_\gamma$, we have

$$L \subseteq \bigcap \{U_\xi \mid a \leqslant \xi < \gamma\}.$$

But if x is an interior point of L, then x is not a limit point of $\bigcup \{M_\xi \mid a \leqslant \xi < \gamma\}$, and hence $x \notin M$. Thus M is totally disconnected.

LEMMA 14. *For each a, no element of $\mathscr{S}_a$ separates two points of any other element of $\mathscr{S}_a$.*

Proof. For each a and each component U of $X - S_a$ let J_U denote the simple closed curve obtained from $\overline{U}$ by identifying the end points of $\overline{U}$. Let $P(a)$ denote the lemma. Clearly, $P(0)$ is true. Suppose that $P(a)$ is true. Assume that H and K are two elements of $\mathscr{S}_{a+1}$, and x and y are two points of K such that H separates x from y in X.

Case 1. For some component U of $X-S_a$, H and K are subsets of U. It is easily seen that H separates x from y in J_U. But H and K are elements of $\mathscr{S}_U$.

Case 2. For some component U of $X-S_a$, we have $H \subseteq U$ and $K \nsubseteq U$. If x and y are in $X-U$, then $X-U$ is a connected subset of $X-H$ containing x and y. Suppose that one of the points x and y, say x, is in U. Since $K \cap U$ is degenerate, $y \in X-U$. But then H separates x from ∂U in J_U, $H \in \mathscr{S}_U$, and $\partial U \cup \{x\} \in \mathscr{S}_U$.

Case 3. For some component U of $X-S_a$, we have $H \nsubseteq U$ and $K \subseteq U$. It is easily seen that $H \cap \overline{U}$ separates x from y in J_U. But $H \cap \overline{U} \in \mathscr{S}_U$ and $K \in \mathscr{S}_U$.

Case 4. For each component U of $X-S_a$ neither H nor K is a subset of U. Hence there exist two elements H_a and K_a of $\mathscr{S}_a$ such that $H_a \subseteq H$ and $K_a \subseteq K$. It follows that $H \subseteq C_a(H_a)$ and $K \subseteq C_a(K_a)$. If $x \in K_a$, let $x' = x$. If $x \notin K_a$, then let U be the component of $X-S_a$ with boundary in K_a containing x and let x' be a point of ∂U. Let y' be defined similarly. Since

$$C_a(H_a) \cap C_a(K_a) = \emptyset,$$

H separates x' from y', and hence H_a separates x' from y'. But x' and y' are two points of K_a, which contradicts $P(a)$. Therefore $P(a)$ implies $P(a+1)$.

Now suppose that γ is a limit ordinal and $P(a)$ is true for each $a < \gamma$. Assume that H and K are two elements of $\mathscr{S}_\gamma$.

Case 1. We have

$$H = \partial \bigcap \{U_\xi \mid \xi < \gamma\} \quad \text{and} \quad K = \partial \bigcap \{V_\xi \mid \xi < \gamma\}.$$

For some a, $U_a \neq V_a$, and hence $\overline{U_a} \cap \overline{V_a} = \emptyset$. Since $H \subseteq \overline{U_a}$ and $K \subseteq \overline{V_a}$, neither H nor K separates two points of the other.

Case 2. We have

$$H = \partial \bigcap \{U_\xi \mid \xi < \gamma\} \quad \text{and} \quad K = \operatorname{Cl} \bigcup \{K_\xi \mid a \leqslant \xi < \gamma\}.$$

For each ξ such that $a \leqslant \xi < \gamma$, $U_\xi \cap K_\xi = \emptyset$, and hence

$$\bigcap \{U_\xi \mid \xi < \gamma\} \cap K = \emptyset.$$

It follows that neither H nor K separates two points of the other.

Case 3. We have

$$H = \operatorname{Cl} \bigcup \{H_\xi \mid a \leqslant \xi < \gamma\} \quad \text{and} \quad K = \operatorname{Cl} \bigcup \{K_\xi \mid a \leqslant \xi < \gamma\}.$$

Suppose that H separates two points x and y of K. Then for some β with $\alpha \leqslant \beta < \gamma$ there exist two points x' and y' of K_β such that H separates x' from y'. For each ξ such that $\beta \leqslant \xi < \gamma$, x' and y' are in K_ξ, and hence H_ξ does not separate x' from y'. Therefore, for each such ξ there exists a component U_ξ of $X - H_\xi$ containing x' and y'. Let

$$U = \bigcap \{U_\xi \mid \beta \leqslant \xi < \gamma\}.$$

Thus

$$U \subseteq X - \bigcup \{H_\xi \mid \beta \leqslant \xi < \gamma\}.$$

Now U contains an open set V such that $\bar{V}$ is an arc from x' to y'. Then $\bar{V} \subseteq X - H$, and hence H does not separate x' from y', a contradiction.

LEMMA 15. *For each α and each two elements H and K of $\mathscr{S}_\alpha$, some element of $\mathscr{S}_\alpha$ contains at least four points and separates H from K.*

Proof. Denote the lemma by $P(\alpha)$. Clearly, $P(0)$ is true. Suppose that $P(\alpha)$ is true, and let H and K be two elements of $\mathscr{S}_{\alpha+1}$.

Case 1. For some component U of $X - S_\alpha$, we have $H \subseteq U$ and $K \subseteq U$. Some element L of $\mathscr{S}_U$ contains at least four points and separates H from K in $\bar{U}$. Hence L separates H from K in X.

Case 2. For some component U of $X - \mathscr{S}_\alpha$, we have $H \subseteq U$ and $K \subseteq X - \bar{U}$. Now ∂U separates H from K in X, and some element L of $\mathscr{S}_U$ contains at least four points and separates H from ∂U in $\bar{U}$. Hence L separates H from K in X.

Case 3. For some component U of $X - S_\alpha$, we have $H \subseteq U$, and $K \cap \partial U \neq \emptyset$. There is an element K' of $\mathscr{S}_U$ such that $\partial U \subseteq K' \subseteq K$. Some element L of $\mathscr{S}_U$ contains at least four points and separates H from K' in $\bar{U}$. Hence $L \in \mathscr{S}_{\alpha+1}$, and L separates H from K in X.

Case 4. For each component U of $X - S_\alpha$ neither H nor K is a subset of U. There exist elements H_α and K_α of $\mathscr{S}_\alpha$ such that $H_\alpha \subseteq H$ and $K_\alpha \subseteq K$. Some element L_α of $\mathscr{S}_\alpha$ contains at least four points and separates H_α from K_α in X. Thus L_α is a subset of some element L of $\mathscr{S}_{\alpha+1}$. Now L_α separates $C_\alpha(H_\alpha)$ from $C_\alpha(K_\alpha)$ in X, and hence L_α separates H from K in X. Then, since $\mathscr{S}_{\alpha+1}$ is disjoint, L separates H from K in X.

Now suppose that γ is a limit ordinal and $P(\alpha)$ is true for each $\alpha < \gamma$. Let H and K be two elements of $\mathscr{S}_\gamma$.

Case 1. We have

$$H = \mathrm{Cl} \bigcup \{H_\xi \mid \alpha \leqslant \xi < \gamma\} \quad \text{and} \quad K = \mathrm{Cl} \bigcup \{K_\xi \mid \alpha \leqslant \xi < \gamma\}.$$

Some element L_α of $\mathscr{S}_\alpha$ separates H_α from K_α in X, and L_α is a subset of some element L of $\mathscr{S}_\gamma$. It follows that L separates H from K in X.

Case 2. We have

$$H = \partial\bigcap\{U_\xi \mid \xi < \gamma\} \quad \text{and} \quad K = \partial\bigcap\{V_\xi \mid \xi < \gamma\}.$$

For each $\xi < \gamma$ there exist elements H_ξ and K_ξ of $\mathscr{S}_\xi$ such that $\partial U_\xi \subseteq H_\xi$ and $\partial V_\xi \subseteq K_\xi$. First, suppose that, for each $\xi < \gamma$, $H_\xi = K_\xi$. For some $\alpha < \gamma$, $U_\alpha \neq V_\alpha$, and hence $\overline{U_\alpha} \cap \overline{V_\alpha} = \emptyset$. Since

$$\bigcap\{H_\xi \mid \alpha \leqslant \xi < \gamma\} = \emptyset,$$

for some β such that $\alpha < \beta < \gamma$ we have $\overline{U_\beta} \subseteq U_\alpha$ and $\overline{V_\beta} \subseteq V_\alpha$. Then H_α contains at least four points and separates H from K in X. Thus H_α is a subset of some element L of $\mathscr{S}_\gamma$, and hence L separates H from K in X. Now suppose that, for some $\alpha < \gamma$, $H_\alpha \neq K_\alpha$. Some element L_α of $\mathscr{S}_\alpha$ contains four points and separates H_α from K_α in X, and L_α is a subset of some element L of $\mathscr{S}_\gamma$. It follows that L separates H from K in X.

Case 3. We have

$$H = \partial\bigcap\{U_\xi \mid \xi < \gamma\} \quad \text{and} \quad K = \mathrm{Cl}\bigcup\{K_\xi \mid \alpha \leqslant \xi < \gamma\}.$$

For each $\xi < \gamma$ let H_ξ be the element of $\mathscr{S}_\xi$ containing ∂U_ξ. There exists a β such that $\alpha \leqslant \beta < \gamma$ and $H_\beta \neq K_\beta$. Some element L_β of $\mathscr{S}_\beta$ contains four points and separates H_β from K_β in X, and L_β is a subset of some element L of $\mathscr{S}_\gamma$. It follows that L separates H from K in X.

We are now in a position to prove Theorem 1. Clearly, for some limit ordinal λ, $X = \bigcup\mathscr{S}_\lambda$. It follows from the lemmas that $\mathscr{S}_\lambda$ is a non--separating collection of pairwise disjoint closed totally disconnected subsets of X and, furthermore, $\mathscr{S}_\lambda$ is *saturated*, i.e., for each M in $\mathscr{S}_\lambda$ and each x in $X-M$ some element of $\mathscr{S}_\lambda$ separates x from M. It then follows as in Whyburn [8] that $\mathscr{S}_\lambda$ is upper semi-continuous, the decomposition space $Y = \mathscr{S}_\lambda$ is a dendrite, and the natural map f of X onto Y is light and non-alternating. It also follows from Lemma 15 that the set of all branch points of Y is dense in Y. It remains to prove that, for each y in Y, $c(y) = |f^{-1}(y)|$.

Let M be an element of $\mathscr{S}_\lambda$. If M is finite, then $c(M) = |M|$. Suppose that M is infinite. Since the boundary of each component of $X-M$ is in M and each two components of $X-M$ have at most one common boundary point, $c(M) \leqslant |M|$. Let $P(\alpha)$ denote the statement that, for each infinite set M in $\mathscr{S}_\alpha$, $c(M) = |M|$. Since each element of $\mathscr{S}_0$ is finite, $P(0)$ is true. Suppose that $P(\alpha)$ is true and M is an infinite element of $\mathscr{S}_{\alpha+1}$. If, for some component U of $X-S_\alpha$, $M \in \mathscr{S}_U$, then M is finite. Hence, for some element L of $\mathscr{S}_\alpha$, $L \subseteq M$. If U is a component of $X-L$ containing a point of M, then U is a component of $X-S_\alpha$ such that $\partial U \subseteq L$, and hence U contains at most one point of M. Therefore $c(M) = c(L)$. It also follows that $|M| = |L|$, so that $c(M) = |M|$. Now suppose that γ is a limit ordinal and $P(\alpha)$ is true for each $\alpha < \gamma$. Let M be an infinite element of $\mathscr{S}_\gamma$. If

$$M = \partial \bigcap \{U_\xi \mid \xi < \gamma\},$$

then M is finite. Hence there exists a chain $\{M_\xi \mid \alpha \leqslant \xi < \gamma\}$ such that if

$$L = \bigcup \{M_\xi \mid \alpha \leqslant \xi < \gamma\},$$

then $M = \bar{L}$. Since $M_\xi \subseteq M$ and $M \subseteq C_\xi(M_\xi)$, we have $c(M_\xi) \leqslant c(M)$. Then

$$|L| = \sup\{|M_\xi| \mid \alpha \leqslant \xi < \gamma\} = \sup\{c(M_\xi) \mid \alpha \leqslant \xi < \gamma\} \leqslant c(M).$$

Let x be a point of $M - L$. For each ξ let U_ξ be the component of $X - \mathscr{S}_\xi$ containing x. Suppose that, for each ξ, $\partial U \subseteq X - S_\xi$. Then $\partial U \in \mathscr{S}_\gamma$, and since $x \in \bar{L}$, we have $x \in \partial U$. It follows that $M = \partial U$, which is a contradiction since M is infinite. Hence for some β such that $\alpha < \beta < \gamma$ there is a point y in $\partial U \cap S_\beta$. It follows that $\partial U = \{x, y\}$. The interior of U is then a component of $X - M$ with x as a boundary point. Thus each point of $M - L$ is a boundary point of one and only one component of $X - M$. Hence $|M - L| \leqslant c(M)$. Then

$$|M| = |L| + |M - L| \leqslant 2c(M) = c(M).$$

Therefore $|M| = c(M)$. This completes the proof of Theorem 1.

2. Universal dendrites. Let $\mathscr{C}$ be a class of dendrites. We say that the element M of $\mathscr{C}$ is a *universal dendrite with respect to* $\mathscr{C}$ if each dendrite in $\mathscr{C}$ is homeomorphic to a subdendrite of M. Menger [3] has shown the existence of universal dendrites for the class $\mathscr{C}$ of:

(1) all metric dendrites;

(2) all metric dendrites with only finite branch points;

(3) all metric dendrites X such that, for each point x of X and for each integer $n > 2$, $c(x) \leqslant n$.

Menger also proved that a dendrite M is a universal metric dendrite if the set of all infinite branch points of M is dense in M and gave an elegant construction for a universal metric dendrite in the plane. We indicate in the sequel how to modify the construction used in the proof of Theorem 1 to obtain a universal metric dendrite. Universal dendrites for classes of (2) and (3) may also be obtained by suitable modifications.

Let X be a metric simple closed curve. In the definition following Theorem 1, replace condition (2) by the statement that each element of $\mathscr{S}$ is countable and closed, replace condition (6) by the statement that if H and K are two elements of $\mathscr{S}$, then some infinite element of $\mathscr{S}$ separates H from K, and replace condition (7) by the statement that $\{p\} \in \mathscr{S}$ and $\{q\} \in \mathscr{S}$. Define $\mathscr{S}_\alpha$ only if $\alpha \leqslant \omega$, and if $0 < \alpha < \omega$, require that each element of $\mathscr{S}_a$ has diameter less than $1/a$. The decomposition space $\mathscr{S}_\infty$ is then a dendrite having a dense set of infinite branch points and is therefore a universal metric dendrite.

It would seem to be of interest to consider the existence of universal dendrites for other classes of dendrites. The construction used in the proof of Theorem 1 should be useful in such considerations.

3. Propositions related to the Suslin conjecture. A space is *separable* if it has a countable dense subset. We shall call a space *paraseparable* if it does not contain uncountably many disjoint open sets. Thus a *Suslin space* is a fully ordered space which is connected, non-separable, and paraseparable in its order topology. In [1] Eberhart proves that a dendrite is metrizable if and only if it is separable and asks whether every paraseparable dendrite in which each arc is separable is metrizable. He conjectured that an affirmative answer to this question is equivalent to the non-existence of a Suslin space. Miller [4] proved the conjecture to be correct. Miller called a non-separable paraseparable dendrite in which each arc is separable an *Eberhart continuum*. In this section we shall give an alternative proof, based on Theorem 1, of the hard part of Miller's theorem, stating that the existence of a Suslin space implies the existence of an Eberhart continuum. We shall also obtain another metrization theorem for dendrites which avoids the Suslin question.

THEOREM 2. *If X is a non-separable paraseparable simple closed curve, Y is a dendrite such that the set of all branch points of Y is dense in Y, and f is a continuous light non-alternating map of X onto Y such that, for each point y of Y, $c(y) = |f^{-1}(y)|$, then Y is an Eberhart continuum.*

Proof. Since X is paraseparable and f is continuous, Y is paraseparable. Suppose that Y is separable. Then Y is metrizable [1], and hence the set K of all branch points of Y is countable. Since Y is paraseparable, for each point y of Y we have $c(y) \leqslant \aleph_0$. Let

$$H = \bigcup \{f^{-1}(y) \mid c(y) > 2\}.$$

Hence H is countable. We show that H is dense in X. Suppose that $w \in X$ and U is a connected open set in X containing w. Since $f^{-1}(f(w))$ is totally disconnected, there is a point x of U such that $f(w) \neq f(x)$. Let wx denote the arc in X from w to x lying in U, and let V denote the set of all interior points of the arc in Y from $f(w)$ to $f(x)$. Since $f(wx)$ is a continuum containing $f(w)$ and $f(x)$, and Y is dendritic, we have $V \subseteq f(wx)$. Now, if V contains no point of K, then V is open. Hence V contains a point v of K. There is a point u of U such that $f(u) = v$. It follows that H is dense in X. Therefore Y is non-separable.

Now suppose that A is an arc in Y. Let K be the set of all branch points of Y belonging to A. For each point y of K, let C_y denote the component of $X-A$ with boundary point y. For each y in K, C_y is open, and if x and y are distinct points of K, then $C_x \cap C_y = \emptyset$. Hence $\{f^{-1}(C_y) \mid y \in K\}$ is a collection of disjoint open sets in X and is therefore countable. It follows that K is countable. Furthermore, if K is not dense in A, then A contains an open set having no branch point of Y. Therefore, each arc in Y is separable.

A continuum will be called *Suslinian* if it does not contain uncountably many disjoint non-degenerate continua. This term was introduced by Lelek in [2] for metric curves. Note that a dendrite is Suslinian if and only if it does not contain uncountably many disjoint arcs.

THEOREM 3. *The dendrite Y is metrizable if and only if Y is Suslinian and each arc in Y is separable.*

Proof. Suppose that Y is metrizable. Clearly, each arc in Y is metrizable, and hence separable. If Y is not Suslinian, then for some $\varepsilon > 0$ there is an infinite sequence of disjoint arcs in Y each of diameter greater than ε, and we easily get a contradiction to the fact that Y is dendritic.

Suppose that Y is Suslinian and each arc in Y is separable. Clearly, Y is paraseparable. Let A_0 be a maximal arc in Y, and let $\mathscr{S}_0 = \{A_0\}$. Suppose that $\mathscr{S}_\alpha$ has been defined for $\alpha < \beta$. Let

$$Y_\beta = \mathrm{Cl}\bigcup\{\bigcup\mathscr{S}_\alpha \mid \alpha < \beta\},$$

and let $\mathscr{S}_\beta$ be the collection of all maximal arcs A such that A has one end point p in Y_β and $A-\{p\} \subseteq X - Y_\beta$. Let $\mathscr{S}$ be the union of all such collections $\mathscr{S}_\alpha$. It follows that if A and B are two arcs in $\mathscr{S}$, then either $A \cap B = \emptyset$ or, for some point p, $A \cap B = \{p\}$ and p is an end point of one of the arcs A and B. Moreover, each point of $Y - \bigcup\mathscr{S}$ is a non-cut point of Y. Since each arc in $\mathscr{S}$ contains an arc in its interior and Y is Suslinian, $\mathscr{S}$ is countable. Hence, if Y has uncountably many branch points, then some arc in $\mathscr{S}$ contains uncountably many branch points of Y, so that Y contains uncountably many disjoint arcs. Therefore, Y has at most countably many branch points. It then follows from Eberhart's theorem that Y is metrizable.

REFERENCES

[1] C. Eberhart, *Metrizability of trees*, Fundamenta Mathematicae 65 (1969), p. 42-50.

[2] A. Lelek, *On the topology of curves II*, ibidem 70 (1971), p. 131-138.

[3] K. Menger, *Kurventheorie*, Leipzig-Berlin 1932.

[4] G. G. Miller, *Dendritic continua*, Doctoral Dissertation, University of Missouri, Kansas City, 1968.

[5] J. Novák, *On partition of an ordered continuum*, Fundamenta Mathematicae 39 (1952), p. 53-64.

[6] B. J. Pearson, *Mapping an arc onto a dendritic continuum*, Colloquium Mathematicum 30 (1974), p. 237-243.

[7] — *On the existence of rigid arcs*, The Journal of the Australian Mathematical Society 18 (1974), p. 306-309.

[8] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Publications 28 (1942).

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