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On the reduction formulas

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Abstract. In this note we shall give explicit formulas for the components of partial derivatives of quantities in the normal coordinates. These formulas are the main tool of the proofs of reduction theorems in [3] and [4]. On the other hand, they have applications in differential geometry and the theory of relativity. For references see [2], [3], [4], and [5].

1. An algebraic lemma ([3], p. 127).

Lemma 1. *If the numbers* $X^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu}$ $(\lambda, \nu_1, \ldots, \nu_l, \varrho, \mu = 1, \ldots, n)$ *fulfil the equations*

$$
(1)\qquad
\begin{aligned}
X^{\lambda}_{\nu_l,\ldots,\nu_2[\nu_1\varrho]\mu} &= 2U^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu},\\
X^{\lambda}_{(\nu_l,\ldots,\nu_1)\varrho\mu} &= X^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu},\\
X^{\lambda}_{\nu_l,\ldots,\nu_1(\varrho\mu)} &= X^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu},\\
X^{\lambda}_{(\nu_l,\ldots,\nu_1\varrho\mu)} &= 0,
\end{aligned}
$$

where $U^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu}$ *are given real numbers, then* $X^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu}$ *are linear functions of* $U^{\lambda}_{\nu_l,\ldots,\nu_1\varrho\mu}$.

We express this fact by

$$
(2)\qquad X = E_l(U), \quad l = 1, \ldots
$$

2. Normal coordinates of order p for an affine connexion. Let $\Gamma^{\lambda}_{\mu\nu}$ be an object of an affine connexion. A coordinate system is called a *normal coordinate system of order p* ([3], p. 129) *for the affine connexion* $\Gamma^{\lambda}_{\mu\nu}$ *at a point* ξ_0 if the equations

$$
(3)\qquad
\begin{aligned}
\Gamma^{\lambda}_{\mu\nu} &= 0\\
\partial_{(\nu_1}\Gamma^{\lambda}_{\mu\nu)} &= 0\\
&\ldots\ldots\ldots\ldots\\
\partial_{(\nu_p,\ldots,\nu_1}\Gamma^{\lambda}_{\mu\nu)} &= 0
\end{aligned}
$$

are satisfied in these coordinates at the point ξ_0.

We recall the following

LEMMA 2 ([3], p. 129). *The normal coordinate system of order p at a point ξ_0 can be introduced by C_p-coordinate transformation $\xi^{\lambda'} = \varphi^{\lambda'}(\xi^\lambda)$ with partial derivatives at $\xi_0(\delta^\lambda_{\lambda'}, A^\lambda_{\nu'_1\lambda'}, \ldots, A^\lambda_{\nu'_{p-1},\ldots,\nu'_1\lambda'})$, where $\partial\xi^\lambda/\partial\xi^{\lambda'} = \delta^\lambda_{\lambda'}$ is the Kronecker symbol. Partial derivatives at ξ_0 of $\xi^{\lambda'} = \varphi^{\lambda'}(\xi^\lambda)$, which transform $\Gamma^\lambda_{\mu\nu}$ into normal from* (3) *are determined uniquely if $\partial\xi^\lambda/\partial\xi^{\lambda'} = \delta^\lambda_{\lambda'}$.*

Proof. In view of the proof of Lemma 2 in [3], p. 129, where the existence was shown, it suffices to show that the element $(\delta^\lambda_{\lambda'}, A^\lambda_{\nu'\lambda'_1}, \ldots$ $\ldots, A^\lambda_{\nu'_{p-1},\ldots,\nu'_1\lambda'})$ is unique.

We notice that, by a coordinate transformation with partial derivatives at $\xi_0(\delta^\lambda_{\lambda'}, 0, \ldots, 0, A^\lambda_{\nu'_k,\ldots,\nu'_1\mu'\nu'}, \ldots, A^\lambda_{\nu'_p,\ldots,\nu'_1\mu'\nu'})$, we have

$$\partial_{\nu'_k,\ldots,\nu'_1}\Gamma^{\lambda'}_{\mu'\nu'} = \partial_{\nu_k,\ldots,\nu_1}\Gamma^\lambda_{\mu\nu} + A^\lambda_{\nu'_k,\ldots,\nu'_1\mu'\nu'},$$

where $\nu_j = \nu'_j$, $\lambda = \lambda'$, $\mu = \mu'$, $\nu = \nu'$, $k = 0, 1, \ldots, p$ and the partial derivatives of $\Gamma^\lambda_{\mu\nu}$ of order $q < k$ do not change. According to the above equalities, the element $(\delta^\lambda_{\lambda'}, A^\lambda_{\lambda'\nu'}, A^\lambda_{\nu'_1\nu'\lambda'}, \ldots, A^\lambda_{\nu'_l,\ldots,\nu'_1\nu'\lambda'})$ which does not change the components of $(\Gamma^\lambda_{\mu\nu}, \partial_{\nu_1}\Gamma^\lambda_{\mu\nu}, \ldots, \partial_{\nu_p,\ldots,\nu_1}\Gamma^\lambda_{\mu\nu})$ has the form $(\delta^\lambda_{\lambda'}, 0, \ldots, 0)$. This completes the proof.

Remark. The coordinate transformation $\xi^{\lambda'} = \varphi^{\lambda'}(\xi^\lambda)$ is a composition of coordinate transformations $\varphi_0, \ldots, \varphi_p$ with the partial derivatives φ_k at ξ_0 equal to $(\delta^\lambda_{\lambda'}, 0, \ldots, 0, \partial_{(\nu_k,\ldots,\nu_1}\overset{k}{\Gamma}{}^\lambda_{\mu\nu)}, 0, \ldots, 0)$, where

$$\overset{0}{\Gamma}{}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}, \ldots, \partial_{\nu_k,\ldots,\nu_1}\overset{k}{\Gamma}{}^\lambda_{\mu\nu} \tag{4}$$

are the partial derivatives at ξ_0 of $\Gamma^\lambda_{\mu\nu}$ after the coordinate transformation $\varphi_{k-1}\circ\ldots\circ\varphi_0$.

3. Reduction formulas for the affine connexion. Let $\partial^p\Lambda$ be a differential extension of the quantity Λ (cf. [2] and [3]). By the differentiation of the formula defining the curvature tensor $R^\lambda_{\varrho\mu\nu}$ of the affine connexion $\Gamma^\lambda_{\mu\nu}$ we obtain (cf. [3], p. 130)

$$\nabla_{\nu_{k-1},\ldots,\nu_1}R^\lambda_{\varrho\mu\nu} = 2\partial_{\nu_{k-1},\ldots,\nu_1[\varrho}\Gamma^\lambda_{\mu]\nu} + W_{k-1}(\partial^{k-1}\Gamma), \tag{5}$$

where W_{k-1} is a polynomial.

Reduction formulas for the affine connexion $\Gamma^\lambda_{\mu\nu}$ are as follows ([3], p. 130)([1]):

([1]) $\overset{*}{\Gamma}{}^\lambda_{\mu\nu} = 0$, $\partial_{\nu_1}\overset{*}{\Gamma}{}^\lambda_{\mu\nu} = E_1(R)$, $\partial_{\nu_2\nu_1}\overset{*}{\Gamma}{}^\lambda_{\mu\nu} = E_2\big(\nabla_1 R - W_1(0, E_1(R))\big), \ldots$

$$
\begin{aligned}
&\overset{*}{\Gamma} = 0,\\
&\partial_1 \overset{*}{\Gamma} = E_1(R),\\
&\partial_2 \overset{*}{\Gamma} = E_2\big(\nabla_1 R - W_1(0, E_1(R))\big),\\
(6)\quad &\dots\dots\dots\dots\dots\dots\dots\dots\\
&\partial_{p+1} \overset{*}{\Gamma} = E_{p+1}\Big(\nabla_p R - W_p\Big(0, E_1(R), E_2\big(\nabla_1 R - W_1(0, E_1(R))\big), \dots\\
&\qquad\qquad \dots, E_p\big(\nabla_{p-1} R - W_{p-1}(0, \dots, 0)\big)\Big)\Big).
\end{aligned}
$$

where $\partial_l \overset{*}{\Gamma}$ denotes the set of partial derivatives $\partial_{\nu_l,\dots,\nu_1} \Gamma^{\lambda}_{\mu\nu}$ in the normal coordinates, $\nabla_l R$ denotes the set of the covariant derivatives $\nabla_{\nu_l,\dots,\nu_1} R^{\lambda}_{\varrho\mu\nu}$ of the curvature tensor, and functions E_i, W_i are determined by (2) and (5). In formulas (6) we substitute number 0 for $\Gamma^{\lambda}_{\mu\nu}$ according to equality (3).

4. Components of the partial derivatives of quantities in normal coordinates of order p. Let Φ be a tensor density of the valence (q, r) and of the weight a and let $\Gamma^{\lambda}_{\mu\nu}$ be an affine connexion. Denote by $\partial_p \Phi$ the set of partial derivatives $\partial_{\nu_p,\dots,\nu_1} \Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r}$ and denote by $\nabla_p \Phi$ the set of covariant derivatives (determined by $\Gamma^{\lambda}_{\mu\nu}$) of the form $\nabla_{\nu_p,\dots,\nu_1} \Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r}$.

Let $\partial^{p-1}\Phi$ and $\partial^{p-1}\Gamma$ be differential extensions of objects Φ and $\Gamma^{\lambda}_{\mu\nu}$, respectively.

We have the formula (cf. [3], p. 132, and [4]):

$$(7)\qquad \partial_p \Phi = \nabla_p \Phi - P_p(\partial^{p-1}\Phi, \partial^{p-1}\Gamma),$$

where P_p is a polynomial.

From equalities (5), (6) and (4.7), [3], p. 132 (see also [4]), we obtain in the normal coordinates of order p for the affine connexion $\Gamma^{\lambda}_{\mu\nu}$ the equalities

$$
\begin{aligned}
&\partial_1 \Phi = \nabla_1 \Phi,\\
&\partial_2 \Phi = \nabla_2 \Phi - P_2\big(\Phi, \nabla_1 \Phi, 0, E_1(R)\big),\\
(8)\quad &\dots\dots\dots\dots\dots\dots\dots\dots\\
&\partial_p \Phi = \nabla_p \Phi - P_p\Big(\Phi, \nabla_1 \Phi, \nabla_2 \Phi - P_2\big(\Phi, \nabla_1 \Phi, 0, E_1(R)\big), \dots\\
&\qquad \dots, \nabla_{p-1}\Phi - P_{p-1}(\Phi, \nabla_1 \Phi, \dots), 0, E_1(R),\\
&\qquad E_2\big(\nabla_1 R - W_1(0, E_1(R))\big), \dots, E_p\big(\nabla_{p-1} R - W_{p-1}(\dots)\big)\Big),
\end{aligned}
$$

where we substitute number 0 for $\Gamma^{\lambda}_{\mu\nu}$ according to equality (3).

5. Reduction formulas for the linear connexion. Let $L^{\lambda}_{\mu\nu}$ be an object of linear connexion. Denote by $\Gamma^{\lambda}_{\mu\nu}$ and $T^{\lambda}_{\mu\nu}$ the symmetric part of $L^{\lambda}_{\mu\nu}$ and its torsion, respectively, $\Gamma^{\lambda}_{\mu\nu} = L^{\lambda}_{(\mu\nu)}$, $T^{\lambda}_{\mu\nu} = L^{\lambda}_{[\mu\nu]}$.

Denote by $\overset{s}{R}{}^{\varrho}_{\mu\nu\lambda}$, $\overset{s}{\nabla}$ the curvature tensor and the covariant derivative determined by $\Gamma^{\lambda}_{\mu\nu}$. In the normal coordinates of order p of the affine connexion $\Gamma^{\lambda}_{\mu\nu}$ we obtain from (6) and (8) (for the case $p = 1, 2$ see [1]) the formulas

$$
(9)\quad
\begin{aligned}
L &= T,\\
\partial_1 L &= E_1(\overset{s}{R}) + \overset{s}{\nabla}_1 T,\\
\partial_2 L &= E_2\big(\overset{s}{\nabla}_1\overset{s}{R} - W_1(0, E_1(\overset{s}{R}))\big) + \overset{s}{\nabla}_2 T - P_2(T, \overset{s}{\nabla}_1 T, 0, E_1(\overset{s}{R})),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\partial_{p+1} L &= E_{p+1}\big(\overset{s}{\nabla}_p\overset{s}{R} - W_p(0, E_1(\overset{s}{R}), \ldots), E_p(\overset{s}{\nabla}_{p-1}\overset{s}{R} - W_{p-1}(\ldots))\big)\\
&\quad + \overset{s}{\nabla}_{p+1} T - P_{p+1}\big(T, \overset{s}{\nabla}_1 T, \ldots, P_p(T, \overset{s}{\nabla}_1 T, \ldots, 0, E_1(\overset{s}{R}), \ldots)\big),
\end{aligned}
$$

where P_i are polynomials corresponding to $T^{\lambda}_{\mu\nu}$ which appeared in equality (6).

6. Reduction formulas for a non-singular metrical tensor. Let $g_{\lambda\mu}$ be a non-singular metrical tensor and let $\Gamma^{\lambda}_{\mu\nu}$ be the Christoffel symbols belonging to the tensor $g_{\lambda\mu}$. In the normal coordinates of order p of the connexion $\Gamma^{\lambda}_{\mu\nu}$ we get

$$
(10)\quad
\begin{aligned}
\partial_1 g &= 0,\\
\partial_2 g &= -P_2(g, 0, E_1(R)),\\
&\ldots\ldots\ldots\ldots\\
\partial_p g &= -P_p\Big(g, 0, -P_2(g, 0, E_1(R)), \ldots, -P_{p-1}\big(g, \ldots, 0, E_1(R),\\
&\qquad E_2(\nabla_1 R - W_1, 0, E_1(R))\big), \ldots, E_{p-1}(\nabla_{p-2} R - W_{p-2}(\ldots))\Big),
\end{aligned}
$$

where R and ∇ denote the curvature tensor and the covariant derivative with respect to $\Gamma^{\lambda}_{\mu\nu}$, respectively. Polynomials P_i are determined by equality (6) applied to $g_{\lambda\mu}$.

References

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