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Lifting of vector fields and diffeomorphisms between manifolds*

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Abstract. In Section 1 we extend a result obtained by Arnold (see [1]) on liftable holomorphic germs of vector fields to the case of smooth vector fields on manifolds (Theorem 1.1).

In Section 2 we give a necessary and sufficient condition for an analytic function to be reduced and this condition is useful for verifying the hypotheses of Theorem 1.1.

In Section 3 we prove a similar theorem for liftable diffeomorphisms between manifolds (Theorem 3.1).

1. Lifting of vector fields. Let $f\colon X \to Y$ be a smooth(1) map and let v be a smooth vector field on X, w a smooth vector field on Y.

Let $f_*\colon TX \to TY$ be the map induced by f between the tangent bundles.

DEFINITION 1.1. The vector fields w on Y and v on X are said to be *f-agreeing* if and only if $w = f_*(v)$.

We say also that the vector field w is *f-liftable* and the vector field v is an *f-lifting* of w.

Remark 1.1. Note how the above condition can be written in local coordinates.

If we introduce local coordinates $\{x_1, \ldots, x_n\}$ on a neighbourhood of x_0 such that $x_0 \equiv 0$, and $\{y_1, \ldots, y_n\}$ on a neighbourhood of $y_0 = f(x_0)$ such that $y_0 \equiv 0$, $f(U) \subset W$, and call $f_1, \ldots, f_n$ the components of f on $U, v_1, \ldots, v_n$ the components of v on U and $w_1, \ldots, w_n$ the components of w on W, the f-agreeing condition of Definition 1.1 is:

$$\textbf{1.1.}\quad w_i(f(x)) = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}(x) v_j(x) \ \forall\, x \in U,\ 1 \leqslant i \leqslant n.$$

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* Work executed in the field of GNSAGA of CNR.

(1) "Smooth" always means: differentiable of class C^∞.

If we use the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \leqslant i,j \leqslant n} = (Jf)_x$, condition 1.1 obtains the form: $w(f(x)) = (Jf)_x v(x)\ \forall\, x \in U$.

Let $\Sigma = \{\text{critical points of } f\}$, $\Delta = \{\text{critical values of } f\}$, $\Delta_r = \{y \in \Delta \mid \exists$ a nbhd U_y of y in Y: $U_y \cap \Delta$ is a submanifold of $Y\}$.

The problem we are interested in consists in giving sufficient conditions on f in order that a smooth vector field w on Y be f-liftable if and only if w is tangent to Δ_r.

Let $f_0\colon (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ be a germ of a smooth map and let f be a representative defined on a neighbourhood U of the origin.

Definition 1.2. The origin is a *simple fold point for* f_0 if and only if, for suitable local coordinates $\{x_1, \ldots, x_n\}$ at the source and $\{y_1, \ldots, y_n\}$ at the range, we can write f as:

$$y_1 = x_1^2, \quad y_i = x_i, \quad 2 \leqslant i \leqslant n.$$

Let $A(f, 0) = \varepsilon(n)/(f^* m(n) \cdot \varepsilon(n))$ be the local ring of the singularity at 0.

Lemma 1.1 (well known). *Let* $f_0\colon (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ *be a germ of a smooth map. The following conditions are equivalent*:

(a) $\dim_{\boldsymbol{R}} A(f, 0) = 2$,

(b) 0 *is a simple fold point for* f_0.

Proof. (b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b): We claim that $\dim_R A(f, 0) = 2 \Rightarrow Rk_0 f = n-1$. Indeed, if $Rk_0 f = k < n$, we may suppose that the matrix $\left(\frac{\partial f_i}{\partial x_j}(0)\right)_{1 \leqslant i,j \leqslant k}$ has maximum rank. Then $x_i' = f_i$, $1 \leqslant i \leqslant k$, $x_{k+j}' = x_{k+j}$, $1 \leqslant j \leqslant n-k$, is a smooth local change of coordinates such that in the new system f acts as follows:

$$y_i = x_i, \quad 1 \leqslant i \leqslant k, \quad y_{k+j} = g_{k+j}(x')$$

with

$$g_{k+j}(0) = \frac{\partial g_{k+j}}{\partial x_{k+h}'}(0) = 0, \quad 1 \leqslant j,\ k \leqslant n-k.$$

If $k < n-1$, we obtain $\dim_R A(f, 0) > 2$, contrary to hypothesis (a). The equalities $Rk_0 f = n-1, \dim_R A(f, 0) = 2$ imply that f is of the form $y_i = x_i$, $1 \leqslant i \leqslant n-1$, $y_n = g_n(x)$, with $g(0) = \frac{\partial g}{\partial x_n}(0) = 0$ and $\frac{\partial^2 g}{\partial x_n^2}(0) \neq 0$, and this is a condition necessary for 0 to be a simple fold point for f_0 (see [3], p. 74).

Lemma 1.2. *Let* $f\colon \boldsymbol{R}_x^n - \boldsymbol{R}_y^n$ *be defined by*

$$y_1 = x_1^2, \quad y_i = x_i, \quad 2 \leqslant i \leqslant n$$

(R^n_x (resp. R^n_y) denotes R^n with the coordinates $\{x_1, \ldots, x_n\}$ (resp. $\{y_1, \ldots, y_n\}$)).

Then a smooth vector field w on R^n_y is f-liftable if and only if w is tangent to $\Delta_r = \{y_1 = 0\}$.

Proof (see [1], p. 562).

Let U be an open subset of R^n, $\varphi\colon U \to R$ a smooth function and $V(\varphi) = \{x \in U\colon \varphi(x) = 0\}$.

DEFINITION 1.3. We say that φ is a *reduced function* on U if and only if for every smooth function g on U ($g \in C^\infty(U)$) which is zero on $V(\varphi)$, there exists $h \in C^\infty(U)$ such that $g = \varphi \cdot h$. We say that a germ at x of a smooth function is *reduced* if and only if it is the germ at x of a reduced function.

THEOREM 1.1. *Let X, Y be smooth manifolds of dimension n, and let $f\colon X \to Y$ be a smooth map. If we suppose:*

(i) $\Delta_r \subset \{y \in \Delta | \forall x \in f^{-1}(y), \dim A(f, x) \leqslant 2\}$;

(ii) *$f^{-1}(\Delta_r) \cap \Sigma$ is dense in Σ;*

(iii) *$\forall x \in X, \exists$ a coordinate neighbourhood U such that $\det J(f)|_U$ is reduced on U;*

then a smooth vector field w on U is f-liftable if and only if w is tangent to Δ_r.

Proof. If the vector field w is f-liftable, it is tangent to Δ_r (Lemma 1.2). For the "if" part, suppose that w is tangent to Δ_r.

(1) Note that there exists a uniquely determined f-lifted vector field v of w on $X - \Sigma$, because $Rk_{x_0} f = n$ for every $x_0 \in X - \Sigma$.

More clearly, if we choose suitable neighbourhoods of x_0 and $y_0 = f(x_0)$, as in Remark 1.1, the vector field v on U is defined by

$$v(x) = (Jf)_x^{-1} w(f(x)) \, \forall \, x \in U.$$

We are going to show that it is possible to extend the vector field v, which is defined only on $X - \Sigma$, to all of X.

(2) We extend v to $f^{-1}(\Delta_r)$.

In view of Lemmas 1.1, 1.2, v can be extended to a neighbourhood of x_0, for every $x_0 \in f^{-1}(\Delta_r) \cap \Sigma$. Moreover, for every $x_1, x_2 \in f^{-1}(\Delta_r) \cap \Sigma$ the two extensions of v, respectively to a neighbourhood U_1 of x_1 and U_2 of x_2, are equal on $(U_1 \cap U_2 - \Sigma)$, which is dense in $U_1 \cap U_2$. This means that v can be extended to $U_1 \cup U_2$.

Thus v can be extended to $f^{-1}(\Delta_r) \cap \Sigma$.

Now observe that, for every x such that $v(x)$ is defined, we have

$$w(f(x)) = f_{*x} v(x)$$

($f_{*x}\colon T_x X \to T_{f(x)} Y$ is the map f restricted to the fibre on x).

If we take suitable coordinate neighbourhoods, as in Remark 1.1, and denote by B_x the transpose of the matrix of cofactors of $(Jf)_x$, and by I the identity matrix, we obtain

$$(\alpha) \qquad B_x w(f(x)) = \det(Jf)_x \cdot Iv(x)$$

for every x such that $v(x)$ is defined.

Thus $\det(Jf)_x = 0$ yields $B_x w(f(x)) = 0$.

(3) We extend v to $\Sigma - (f^{-1}(\Delta_r) \cap \Sigma)$. According to hypothesis (ii), for every $x_0 \in \Sigma - (f^{-1}(\Delta_r) \cap \Sigma)$ there exists a sequence $\{x_n\}_{n \in N}$ of points of $f^{-1}(\Delta_r) \cap \Sigma$ such that $x_n \to x_0$.

Thus $B_{x_n} w(f(x_n)) = 0$ for every $n \in N$ implies $B_{x_0} w(f(x_0)) = 0$.

Then, by (iii), $\det(Jf)_x$ divides every component of the vector field $B_x w(f(x))$ on a neighbourhood of the point x_0.

This means that the vector field v can be extended to such a neighbourhood of x_0.

As in step (2), we easily realize that any two such extensions of v coincide on some neighbourhood of x_0. This means that v can be extended to all of $\Sigma - (f^{-1}(\Delta_r) \cap \Sigma)$.

Consequences:

COROLLARY 1.1. *Let $X \cong R^n$, $Y \cong R^n$, $f: X \to Y$ smooth, defined by*

$$y_1 = x_1^{n+1} + x_2 x_1^{n-1} + \ldots + x_n x_1, \quad y_i = x_i, \quad 2 \leqslant i \leqslant n,$$

where $\{x_1, \ldots, x_n\}$ (respectively $\{y_1, \ldots, y_n\}$) is a system of coordinates on X (resp. on Y) (such a map is usually called the Whitney map in n variables). Then a smooth vector field w on Y is f-liftable if and only if w is tangent to Δ_r.

COROLLARY 1.2. *Let $X \cong R^n$, $Y = R^n$. Let $\{x_1, \ldots, x_{n+1}\}$: $\sum_{i=1}^{n+1} x_i = 0$ be a system of coordinates on X and $\{y_1, \ldots, y_n\}$ be a system of coordinates on Y. Let $f: X \to Y$ be the smooth map defined by*

$$y_i = (-1)^{i+1} \sigma_{i+1}(x), \quad 1 \leqslant i \leqslant n,$$

where $\sigma_1(x) = \sum_{i=1}^{n+1} x_i$, $\sigma_2(x) = \sum_{i<j} x_i x_j, \ldots$, are the elementary symmetric functions of $x_1, \ldots, x_{n+1}$ (such a map is usually called the Vieta map in n variables).

Then a smooth vector field w on Y is f-liftable if and only if it is tangent to Δ_r.

Proof. The Whitney and Vieta maps satisfy the hypotheses of Theorem 1.1.

2. Reduced functions. Let U be an open subset of R^n and $f: U \to R$ a smooth function. Let $V = \{x \in U: f(x) = 0\}$ and f_x the germ of f at the point x.

It is difficult to recognize those functions $f \in C^\infty(U)$ which satisfy the following condition:

$$\forall g \in C^\infty(U):\ g_{|V} \equiv 0 \Rightarrow \exists h \in C^\infty(U):\ g = f \cdot h.$$

However, if f is an analytic function on U, we can give a divisibility criterion, as the following facts indicate:

Let $f\colon U \to \mathbf{R}$ be analytic. We call:

$\mathscr{A}_U$ ($\mathscr{A}_x$) the sheaf (the stalk at x) of germs of analytic functions on U;

$\langle f \rangle$ ($\langle f_x \rangle$) the sheaf of ideals (the stalk at x) of germs of analytic functions on U generated by f (f_x);

$I(V)$ ($I_x(V)$) the sheaf of ideals (the stalk at x) of germs of analytic functions which are zero on V.

Remark 2.1. Let f_x be a germ of a real analytic function and let $f\colon U \to \mathbf{R}$ be a representative on a neighbourhood U of x.

By definition we have

$$\text{(a)} \qquad I_x(V) = \langle f_x \rangle$$

if and only if for every analytic germ g_x, with a representative $g\colon U_1 \to \mathbf{R}$ such that $g_{|V \cap U_1} \equiv 0$, there exists $h_x \in \mathscr{A}_x$ such that $g_x = f_x \cdot h_x$.

Note that if $g_x \in I_x(V) \backslash \langle f_x \rangle$, then there does not exist a smooth germ h_x such that $g_x = f_x \cdot h_x$. Indeed, the quotient of two analytic functions is meromorphic, and if a meromorphic function is C^∞ at every point, then it is also analytic (see [4], p. 361).

Condition (a) is necessary but not sufficient if we choose $g \in C^\infty(U)$, as the following proposition shows:

PROPOSITION 2.1. *Let f_x be a germ of an analytic function and let $f\colon U \to \mathbf{R}$ be a representative.*

Then, for every smooth germ g_x, with a representative $g\colon U_1 \to \mathbf{R}$ such that $g_{|V \cap U_1} \equiv 0$, there exists a smoth germ h_x such that $g_x = f_x \cdot h_x$ if and only if there exists a neighbourhood X of x such that

$$\text{(b)} \qquad I(V) = \langle f \rangle \quad \textit{on all } W.$$

Before proving Proposition 2.1 we prepare the following lemma:

LEMMA 2.1. *Let U be an open subset of $\mathbf{R}^n$ and V be a non-coherent analytic subset of U. Let $0 \in V$ be a non-coherent point of V. Let $g_1, \ldots, g_p$ be analytic functions on an open subset W of U such that the germs at the origin $g_{1,0}, \ldots, g_{p,0}$ generate the ideal $I_0(V)$. Then*

$$\exists f \in C^\infty(U):\ f_{|V} \equiv 0 \quad \textit{and} \quad f_0 \notin \langle g_{1,0}, \ldots, g_{p,0} \rangle_{\varepsilon(n)}$$

$\langle g_{i,0} \rangle_{\varepsilon(n)}$ denotes the ideal generated by $g_{i,0}$ in the ring $\varepsilon(n)$ of smooth germs: $(\mathbf{R}^n, 0) - \mathbf{R}$.

Proof of Lemma 2.1. As 0 is a non-coherent point of V, there exists a sequence $\{x_n\}_{n\in N}$ of distinct points $x_n \in V \cap W$, $x_n \neq 0$, and a sequence of analytic functions $\{f_n\}_{n\in N}$, each defined on a neighbourhood of the ball $D(x_n, \varepsilon_n)$ with centre x_n and radius $\varepsilon_n < \frac{1}{2} \cdot \inf_m d(x_n, x_m)$, such that

$$f_{n,x_n} \in I_{x_n}(V) \quad \text{and} \quad f_{n,x_n} \notin \langle g_{1,x_n}, \ldots, g_{p,x_n} \rangle .$$

Let φ: $R^+ \to R$ be a smooth function defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } \alpha \geqslant 1, \\ 1 & \text{if } \alpha \leqslant \frac{1}{2}. \end{cases}$$

We now consider the series of functions

$$\sum_{n=0}^{\infty} f_n(x)\varphi(t_n \|x - x_n\|) = f(x), \tag{1}$$

where $\{t_n\}_{n\in N}$ is a sequence of real positive numbers such that $1/t_n < \min(\varepsilon_n, M_n^{-1} 2^{-n})$, M_n being the constant given by

$$M_n = \max_n \left| D^\alpha (f_n(x) \cdot \varphi(t_n \|x - x_x\|) t_n) \right| .$$

Series (1) defines a smooth function at every point $x \neq 0$, because $x \notin D(x_n, \varepsilon_n) \to f(x) = 0$; $x \in D(x_n, \varepsilon_n)$ for some $n \to f(x) = f_n(x) \times \times \varphi(t_n \|x - x_n\|)$. To verify that (1) defines a smooth function also at 0, it is sufficient to prove that for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in N$:

$$\sum_{n=0}^{\infty} D^\alpha [f_n(x)\varphi(t_n \|x - x_n\|)] \tag{2}$$

converges uniformly on a neighbourhood of the origin.

To see this we note that (2) may be written as the sum of two series:

$$\sum_{i=0}^{n-1} D^\alpha [f_i(x)\varphi(t_i \|x - x_i\|)] + \sum_{i=n}^{\infty} \frac{1}{t^i} \cdot D^\alpha [f_i(x)(t_i \|x - x_i\|) t_i]$$

and that the absolute value of the k-th term of the second series is dominated by 2^{-k}. This means that series (2) converges uniformly on U, for every α.

The function $f(x)$ satisfies $f_{|V} \equiv 0$ by construction, but it is not possible to find smooth functions $b_1, \ldots, b_p$ on a neighbourhood W' of the origin such that the germ f_0 should satisfy:

$$f_0 = \sum_{i=1}^{p} b_{i,0}\, g_{i,0} . \tag{3}$$

Indeed, (3) means that

$$f(x) = \sum_{i=1}^{p} b_i(x) g_i(x) \;\; \forall\, x \in W' \cap W$$

and hence

$$\hat{f}_x = \sum_{i=1}^{p} \hat{b}_{i,x}\hat{g}_{i,x} \ \forall\, x \in W' \cap W$$

($\hat{f}_x$ denotes the Taylor series of f at the point x).

But, however, we choose W', it is possible to find $x_n \in W' \cap W$ such that:

$$f_{x_n} = f_{n,x_n} \notin \langle g_{1,x_n}, \ldots, g_{p,x_n} \rangle . \tag{4}$$

As the ring of formal power series $\boldsymbol{R}[[x-x_n]]$ is faithfully flat over the ring of convergent power series $\boldsymbol{R}[[x-x_n]]$ (this means, in particular, that an ideal $I \subset \boldsymbol{R}[\{x-x_n\}]$ satisfies $I \cdot \boldsymbol{R}[[x-x_n]] \cap \boldsymbol{R}\{x-x_n\} = I$), f_{x_n} may be written as a combination of the g_{i,x_n} with analytic germ coefficients, contradicting (4).

Proof of Proposition 2.1. Suppose that, for any choice of a neighbourhood W of the origin, (b) is false on W. Then there exists a sequence of open neighbourhoods of the origin $\{W_n\}_{n\in N}$, $W_n \subset W_{n-1}\ \forall\, n$, such that $Y_n = \{y \in W_n | I_y(V) \neq \langle f_y \rangle \neq \emptyset \forall\, n \in N$. Two possibilities can occur:

1° $x \in \bigcap_{n\in N} Y_n$ or

2° $x \notin \bigcap_{n\in N} Y_n$ but $x \in \bigcap_{n\in N} \bar{Y}_n$.

In the first case the germ f_x does not divide the analytic germs at x.

In the second case, by Lemma 2.1, it is possible to find a smooth germ g_x which is not divided by f_x.

On the other hand, if (b) is true on a neighbourhood of the point x, then x is a coherent point of V.

Then (see [5], p. 123) there exist a neighbourhood W of x in U and real functions $f_1, \ldots, f_p$ on W, which are zero on $V \cap W$, and real smooth functions $g_1, \ldots, g_p$ on W, such that

$$g(x) = \sum_{i=1}^{p} g_i(x) f_i(x) \ \forall\, x \in W.$$

But $I(V) = \langle f \rangle$ on W, so there exist real analytic functions $h_1, \ldots, h_p$ on W such that $f_i(x) = h_i(x) f(x)$, $1 \leqslant i \leqslant p, \forall\, x \in W$.

Then $g(x) = h(x) f(x) \forall x \in W$, where $h = \sum_{i=1}^{p} g_i h_i$, $h \in C^\infty(W)$.

Note that the equality: $I(V) = \langle f \rangle$ on a neighbourhood of the point x is a condition stronger than coherence at the point x.

Now we wish to know if an analytic reduced germ: $(\boldsymbol{R}^n, 0) \to \boldsymbol{R}$ is reduced as a germ: $(\boldsymbol{R}^{n+k}, 0) \to \boldsymbol{R}$.

The answer is yes, as the following proposition proves:

Proposition 2.3. *Let* φ_0: $(\boldsymbol{R}^n, 0) \to \boldsymbol{R}$ *be a germ of an analytic reduced function. Then* φ_0 *is reduced as a germ*: $(\boldsymbol{R}^{n+k}, 0) \to \boldsymbol{R}$ $\forall\, k \in \boldsymbol{N}$.

Proof. This follows from the following remarks.

Remark I. Let φ_0: $(\boldsymbol{R}^n, 0) \to \boldsymbol{R}$ be a germ of an analytic reduced function. Then we may suppose that φ_0 is the germ at 0 of a distinguished polynomial $P(x, t) = t^p + a_1(x)t^{p-1} + \ldots + a_p(x)$, where the a_i are analytic functions on a neighbourhood of the origin and $a_i(0) = 0$.

Indeed, if $\varphi_0 \not\equiv 0$, there exists $p \in \boldsymbol{N}$ and coordinates $\{x_1, \ldots, x_{n-1}, t\}$ on a neighbourhood U of the origin such that on U

$$\varphi(x, t) = q(x, t) \cdot P(x, t), \quad \text{where } q \text{ is analytic on } U, q(0, 0) \neq 0.$$

It is clear that φ_0 is reduced if and only if P_0 is reduced.

Remark I'. Also in the case φ_0 is a germ of a smooth reduced function, we may suppose that φ_0 is the germ at the origin of a distinguished polynomial $P(x, t)$, and this time the a_i will be smooth functions on $\boldsymbol{R}^{n-1}$, which are zero at the origin.

Indeed, $\varphi_0 \in \varepsilon(n)$ reduced implies $\varphi_0 \notin m(n)^\infty$ (here $m(n)^\infty$ is the ideal of smooth germs at the origin whose derivatives of every order are zero at the origin).

Otherwise, $\varphi_0 \in m(n)^\infty \Rightarrow \varphi/\|x\|$ is a smooth function on a neighbourhood of the origin with the same zero locus as φ, and φ does not divide $\varphi/(\|x\|)$.

Remark II. Let $P(x, t) = t^p + \sum_{j=1}^{p} a_j(x)t^{p-j}$ be a distinguished polynomial, where the $a_j(x)$ belong to the ring of formal power series $\boldsymbol{R}[[x_1, \ldots \ldots, x_{n-1}]]$ and $a_j(0) = 0$, $1 \leqslant j \leqslant p$.

Then, for any polynomial

$$R(x, t) = \sum_{j=1}^{p} r_j(x)t^{p-j}, \quad \text{where } r_j(x) \in \boldsymbol{R}[[x_1, \ldots, x_{n-1}]],$$

we have:

$$R \in \langle P \rangle_{\boldsymbol{R}[[x,t]]} \leftrightarrow R \equiv 0$$

($\langle P \rangle_{\boldsymbol{R}[[x,t]]}$ denotes the ideal generated by P in the ring $\boldsymbol{R}[[x, t]]$).

Proof. If there exists $B(x, t) = \sum_{j=1}^{\infty} b_j(x)t^j, b_j \in \boldsymbol{R}[[x, t]]$, such that $R = P \cdot B$, all the terms of the series containing t to power $\geqslant p$ must be zero, i.e.

$$b_j + a_1 b_{j+1} + \ldots + a_p b_{j+p} = 0, \quad j = 0, 1, 2, \ldots,$$

and this implies that $\hat{b}_j \in \hat{m}(n)^k$ $\forall\, k \in \boldsymbol{N}$, as we can easily prove by induction ($\hat{m}(n)^k$ is the ideal in $\boldsymbol{R}[[x, t]]$ generated by the monomials of degree $= k$).

Remark II′. If $a_j(x), r_k(x)$ are analytic functions on a connected neighbourhood of the origin, then

$$R \in \langle P \rangle \leftrightarrow R \equiv 0.$$

If $a_j(x), r_k(x)$ are smooth functions, we may only say using this argument that $r_{j,0} \in m(n-1)^\infty$, $1 \leqslant j \leqslant p$.

Let P_0: $(\boldsymbol{R}^n, 0) \to \boldsymbol{R}$ be a germ of a distinguished analytic polynomial such that $I_0(V(P)) = \langle P_0 \rangle$.

Let P: $U \to \boldsymbol{R}$ be a representative defined on an open subset U of $\boldsymbol{R}^n$ by $P(x, t) = t^p + \sum_{j=1}^{p} a_j(x) t^j$, where a_j is analytic, $a_j(0) = 0$, $1 \leqslant j \leqslant p$.

Remark III. Let P_0 be as above. Then for every $k > 0$ and for every analytic germ g_0: $(\boldsymbol{R}^n \times \boldsymbol{R}^k, 0) \to \boldsymbol{R}$ such that $g_{|V(P) \times \boldsymbol{R}^k} \equiv 0$, there exists an analytic germ h_0: $(\boldsymbol{R}^n \times \boldsymbol{R}^k, 0) \to \boldsymbol{R}$ such that $g_0 = P_0 \cdot h_0$.

Proof. There exist analytic functions $H(x, t, u), R(x, t, u)$, defined on a neighbourhood of the origin in $\boldsymbol{R}^{n+k}$ such that on it

$$g(x, t, u) = P(x, t) \cdot H(x, t, u) + R(x, t, u).$$

For every u, $g(\cdot, \cdot, u)$ is zero on $V(P) \cap W$, where W is a neighbourhood of the origin in $\boldsymbol{R}^n$. Hence P divides $g(\cdot, \cdot, u)$ on W. Then P divides $R(\cdot, \cdot, u)$ on $W \leftrightarrow R(\cdot, \cdot, u) = 0$; hence $g_0 = P_0 \cdot H_0$.

Remark IV. Let P_0: $(\boldsymbol{R}^n, 0) \to \boldsymbol{R}$ be a germ of an analytic reduced distinguished polynomial.

Then P_0: $(\boldsymbol{R}^{n+k}, 0) \to \boldsymbol{R}$ is reduced, for every $k \in N$.

Proof. Recall that P_0 is reduced if and only if $I(V(P)) = \langle P \rangle$ on a neighbourhood W of the origin. This implies that $V(P) \cap W$ is a coherent analytic subset of W and therefore $(V(P) \cap W) \times \boldsymbol{R}^k$ is a coherent analytic subset of $W \times \boldsymbol{R}^k$.

Moreover, as P_0 generates the ideal of the germs at 0 of the analytic functions which are zero on $V(P) \times \boldsymbol{R}^k$ (by Remark III), $I(V(P)) = \langle P \rangle$ is true on a neighbourhood of the origin in $\boldsymbol{R}^{n+k}$, and this proves that P_0: $(\boldsymbol{R}^{n+k}, 0) \to \boldsymbol{R}$ is reduced for every $k \in N$.

3. Lifting of diffeomorphisms. We study in this section a problem strictly tied to the lifting of vector fields.

We use the same notation as in Section 1; in particular, Δ denotes the set of critical values of f.

Definition 3.1. We say that a diffeomorphism g: $Y \to Y$ and a diffeomorphism h: $X \to X$ are *f-agreeing* if and only if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

We shall also say that g is *f-liftable* and that h is an *f-lifting of g*.

DEFINITION 3.2. Two diffeomorphisms $g_0, g_1: Y \to Y$ are said to be *isotopic* mod Δ if there exists a smooth map $g: Y \times [0, 1] \to Y$ such that

1° $g(y, 0) = g_0(y) \; \forall \, y \in Y$;

2° $g(y, 1) = g_1(y) \; \forall \, y \in Y$;

3° $g_{|Y \times \{t\}}$ is a diffeomorphism preserving Δ, for every $t \in [0, 1]$;

in particular, g_0, g_1 must preserve Δ.

We have the following:

THEOREM 3.1. *Let X, Y be smooth manifolds, X compact, and let $f: X \to Y$ be a smooth map satisfying the hypotheses of Theorem* 1.1.

Then every diffeomorphisms $g: Y \to Y$ which is isotopic to the identity mod Δ *admits an f-lifting $h: X \to X$ which is isotopic to the identity. (In fact, any isotopy between g and* id_Y *can be lifted to an isotopy between a diffeomorphism $h: X \to X$ and* id_X.)

Proof. Let $g: Y \times \boldsymbol{R} \to Y$ be a smooth map satisfying:

1° $g(y, 0) = y \; \forall \, y \in Y$;

2° $g_{|Y \times \{t\}} = g_t$ is a diffeomorphism preserving Δ for every $t \in \boldsymbol{R}$.

Consider the map $G: Y \times \boldsymbol{R}^2 \to Y \times \boldsymbol{R}$ defined by:

$$G(y, \tau, t) = G_t(y, \tau) = \left(g_{t+\tau} g_\tau^{-1}(y), t+\tau\right).$$

We easily see that the family $\{G_t\}_{t \in \boldsymbol{R}}$ is a one-parameter group of diffeomorphisms of $Y \times \boldsymbol{R}$.

Let us prove for example that $G_{t_1+t_2} = G_{t_1} \cdot G_{t_2}$: indeed,

$$\begin{aligned} G_{t_1} \cdot G_{t_2}(y, \tau) &= G_{t_1}\left(g_{t_2+\tau} g_\tau^{-1}(y, \tau), t_2+\tau\right) \\ &= \left(g_{t_1+t_2+\tau} g_\tau^{-1}(y), t_1+t_2+\tau\right) = G_{t_1+t_2}(y, \tau). \end{aligned}$$

To the family $\{G_t\}_{t \in \boldsymbol{R}}$ there corresponds a vector field

$$W(y, \tau) = \left.\frac{d}{dt}\right|_{t=0} G_t(y, \tau).$$

Consider the map $F: X \times \boldsymbol{R} \to Y \times \boldsymbol{R}$ defined by $F(x, \tau) = \left(f(x), \tau\right)$.

Observe that F satisfies the hypotheses of Theorem 1.1 ((iii) follows from Proposition 2.2) and that the vector field W is tangent to the set of critical values of F (the latter is $\Delta \times \boldsymbol{R}$, where Δ is the set of critical values of f), since

$$W(y, \tau) = \left.\frac{d}{dt}\right|_{t=0} G_t(y, \tau) = \left(\left.\frac{d}{dt}\right|_{t=0} g_{t+\tau} g_\tau^{-1}(y), 1\right)$$

and g_t preserves Δ by hypothesis.

Hence there exists a vector field V on $X \times \boldsymbol{R}$, which is an f-lifting of W, $V(x, \tau) = \left(v(x, \tau), 1\right)$.

As X is a compact manifold, the vector field $v(x, \tau)$ is the derivative vector field of a one-parameter group $\{h_t\}_{t\in R}$ of diffeomorphisms: $X \to X$ (see [2], p. 246).

Write $H_t(y, \tau) = (h_t(x), \tau)$. $\{H_t\}_{t\in R}$ is the flow on $X \times R$ induced by the vector field V; therefore, as the vector fields V, W are f-agreeing, for every t the following diagram commutes:

$$\begin{array}{ccc} X \times R & \xrightarrow{H_t} & X \times R \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ Y \times R & \xrightarrow[G_t]{} & Y \times R \end{array}$$

It follows that for $\tau = 0$ we have $g_t \cdot f(x) = f \cdot h_t(x)$.

This means that g_t and h_t are f-agreeing.

Remark 3.1. Theorem 3.1 is true if we replace the hypothesis "X is compact" by "f is proper".

Indeed, also in this case the vector field $v(x, \tau)$ is the derivative vector field of a one-parameter group of diffeomorphisms of X, because, f being a proper map, we may indefinitely continue every solution (which locally exists and is unique) of the equation $\dot{x} = v(x, \tau)$.

More clearly, as the vector field $v(x, \tau)$ on X and the vector field $w(y, \tau) = \left.\frac{d}{dt}\right|_{t=0} g_{t+\tau} g_\tau^{-1}(y)$ on Y are f-agreeing, f maps integral curves of v into integral curves of w; if there exists $x_0 \in X$ such that the integral curve through x_0 does not continue indefinitely, then, f being a proper map, the f-image of this curve does not continue indefinitely.

This is false because of the construction of W.

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