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ON THE LOCAL OPTIMALITY OF SOME REGULAR SAMPLING PATTERNS IN THE PLANE (II)

1. Introduction. In [3], I have discussed the local optimality of some regular sampling patterns in the plane from the point of view of estimating the mean value of a stationary isotropic stochastic process. I have proved there for the stationary isotropic stochastic process with exponential correlation function $\exp\{-\lambda d\}$, where λ is a positive parameter and d stands for the distance between points, that the pattern T composed the vertices of equilateral triangles of area 1/2 covering the plane is locally optimum with respect to the family of area preserving affine transformations, if $\lambda \geqslant 3\sqrt[4]{12}/2$. In this note, I want to remove this last restriction by proving that the pattern T is locally optimum in the above sense for any positive value of parameter λ. I will also study the local optimality of T with respect to a family of transformations wider than that of affine transformations. These results make it plausible that the problem about the global optimality of T, put in [2] (problem 4.2, p. 141), will have an affirmative answer. I wish to express my thanks to Professor S. Zubrzycki, whose assistance and supervision where invaluable in the preparation of this paper.

2. Definitions and results. Let $\eta(p)$ be a plane stationary isotropic and continuous stochastic process with expected value $E\eta(p) = \mu$, variance $D^2\eta(p) = 1$, and correlation function $R[\eta(p), \eta(q)] = f(|p-q|)$. Every countable set N of points in the plane which has no concentration points will be called a *net*. The net will be called *regular* if for every two of its points p' and p'' there exists an isometric transformation of the plane which transforms p' into p'' and N onto N. The limit $g(N) = \lim_{R\to\infty} n_R/\pi R^2$, if it exists, will be called the *density* of the net; here n_R denotes the number of points of the net in the circle $K(0, R)$ with centre 0 and radius R. The efficiency of the given regular net N from the point of view of the estimation of the expected value μ of the process $\eta(p)$ will be characterized by means of the limiting variance $s_\infty^2 = s_\infty^2(N)$

defined by relation

(1) $$s^2_\infty(N) = \lim_{R\to\infty} n_R D^2 \bar{\eta}_R,$$

where

(2) $$\bar{\eta}_R = \frac{1}{n_R}[\eta(p_{1,R}) + \ldots + \eta(p_{n_R,R})]$$

and $p_{1,R}, \ldots, p_{n_R,R}$ are all points of the net N contained in the circle $K(0, R)$.

Consider now three affine transformations of the plane which preserve the area and thus do not change the density of a regular net of points. The first of them is the extension r_δ transforming the point (x, y) into $(\delta x, y/\delta)$; in symbols:

(3) $$r_\delta\colon (x, y) \to (\delta x, y/\delta);$$

the second is the shearing s_ε which transforms the point (x, y) into $(x, y+\varepsilon x)$; in symbols:

(4) $$s_\varepsilon\colon (x, y) \to (x, y+\varepsilon x);$$

and the third is the translation $t_{a,\beta}$ which transforms the point (x, y) into $(x+a, y+\beta)$; in symbols:

(5) $$t_{a,\beta}\colon (x, y) \to (x+a, y+\beta).$$

We shall consider regular nets with density $g = 1$, namely the net of squares composed of all points with integer coordinates:

(6) $$S = \{(i, j)\colon i, j - \text{integer}\},$$

the net of squares defined by

(7) $$W = \{(i/\sqrt{2}, j/\sqrt{2})\colon i, j - \text{integer},\ i+j = 0 \pmod 2\},$$

the net of regular triangles defined by

(8) $$T = \{(i/\sqrt[4]{12}, j\sqrt[4]{3/4})\colon i, j - \text{integer},\ i+j = 0 \pmod 2\},$$

and two nets of squares with density $g = 1/2$

(9) $$W_1 = \{(i\sqrt{2}, j\sqrt{2})\colon i, j - \text{integer}\},$$

(10) $$W_2 = \{(i\sqrt{2}+1/\sqrt{2}, j\sqrt{2}+1/\sqrt{2})\colon i, j - \text{integer}\}.$$

Define now the four parameter family of regular nets by

(11) $$N_{a,\beta,\varepsilon,\delta} = r_\delta s^\varepsilon (W_1 \cup t_{a,\beta} W_2),$$

which have density $g = 1$ as long as either $\alpha \neq \sqrt{2}/2 \pmod{\sqrt{2}}$ or $\beta \neq \sqrt{2}/2 \pmod{\sqrt{2}}$. This family contains the net of regular triangles, and its affine deformations for we have $T = N_{0,0,0,1/\sqrt[4]{3}}$ and $T_{\varepsilon,\delta} = N_{0,0,\varepsilon/\sqrt[4]{3},\delta/\sqrt[4]{3}}$ and also other nets (see Fig. 1).

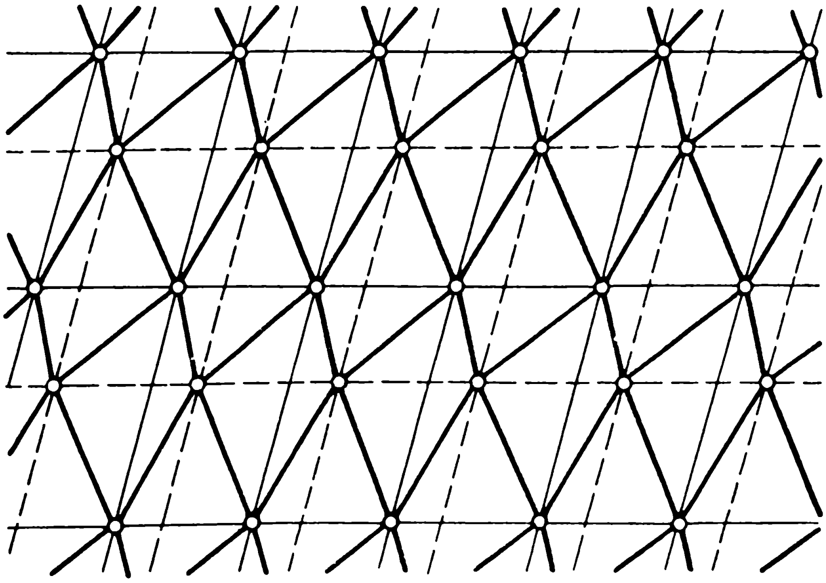

Fig. 1

The efficiency of a given regular net N from the point of view of the estimation of the expected value μ of a stationary isotropic process $\eta(p)$ will be characterized by means of the limiting variance $s^2_\infty(N)$ defined by relation (1). In view of lemma 1 in [3] we have

$$s^2_\infty(N) = \sum_{p \in N} f(|p - p_0|), \qquad p_0 \in N. \tag{12}$$

For a given process $\eta(p)$ and the net $N_{\alpha,\beta,\varepsilon,\delta}$ the limiting variance $s^2_\infty(N_{\alpha,\beta,\varepsilon,\delta})$ becomes a function of variables $\alpha, \beta, \varepsilon$ and δ:

$$s^2_\infty(N_{\alpha,\beta,\varepsilon,\delta}) = n(\alpha, \beta, \varepsilon, \delta). \tag{13}$$

The net $N_{\alpha_0,\beta_0,\varepsilon_0,\delta_0}$ is called *locally optimum*, if the function $n(\alpha, \beta, \varepsilon, \delta)$ has a local minimum at the point $(\alpha, \beta, \varepsilon, \delta) = (\alpha_0, \beta_0, \varepsilon_0, \delta_0)$ with respect to parameters $\alpha, \beta, \varepsilon$ and δ. The net $N_{\alpha_0,\beta_0,\varepsilon_0,\delta_0}$ is called *locally optimum with respect to affine deformations* if the function $n(\alpha_0, \beta_0, \varepsilon, \delta)$ has a local minimum with respect to parameters ε and δ at the point $(\alpha, \beta, \varepsilon, \delta) = (\alpha_0, \beta_0, \varepsilon_0, \delta_0)$. We shall prove

THEOREM 1. *If $\eta(p)$ is a stationary isotropic stochastic process with an exponential correlation function*

$$f(|p - q|) = e^{-\lambda d}, \tag{14}$$

where d stands for the distance of points p and q, then

(a) *the net* $T = N_{0,0,0,1/\sqrt[4]{3}}$ *of regular triangles is locally optimum with respect to affine deformations,*

(b) *if* $\lambda > \sqrt[4]{27}$, *then the net* $T = N_{0,0,0,1/\sqrt[4]{3}}$ *of regular triangles is locally optimum.*

THEOREM 2. *If* $\eta(p)$ *is a stationary isotropic stochastic process with completely monotonic correlation function*

$$\varrho(|p-q|) = \int_0^\infty e^{-\lambda d} dF(\lambda), \tag{15}$$

where $F(\lambda)$ *is a distribution function for which* $F(0+) = 0$ *and* $dF(\lambda) \leqslant \leqslant M\lambda^{1+a}$, $M > 0$, $a > 0$, *then the net* $T = N_{0,0,0,1/\sqrt[4]{3}}$ *of regular triangles is locally optimum with respect to affine deformations.*

For the proof of theorem 1 we need the following

LEMMA 1. *If* $\eta(p)$ *is a stationary isotropic stochastic process with correlation function* $F(x, y)$ *and spectral density*

$$G(u, v) = \int\int e^{-i(ux+vy)} F(x, y)\, dx dy,$$

then

$$s^2_\infty(S_{\varepsilon,\delta}) = 4\pi^2 \sum_{(i,j)\epsilon S} G\big(2\pi(i-\varepsilon j)/\delta,\, 2\pi\delta j\big), \tag{16}$$

where $S_{\varepsilon,\delta} = r_\delta s_\varepsilon(S)$.

Proof. We introduce a new process by the equation

$$\eta^*(p) = \eta^*(x, y) = \eta\left(\delta x, \frac{y+\varepsilon x}{\delta}\right). \tag{17}$$

The correlation function of this process is

$$F^*(x, y) = R[\eta^*(x_1, y_1), \eta^*(x_1+x, y_1+y)] = F\left(\delta x, \frac{y+\varepsilon x}{\delta}\right),$$

and hence the spectral density of this process is given by

$$\begin{aligned} G^*(u, v) &= \int\int F^*(x, y) e^{-i(ux+vy)} dx dy \\ &= \int\int F\left(\delta x, \frac{y+\varepsilon x}{\delta}\right) e^{-i(ux+vy)} dx dy \\ &= \int\int F(x, y) e^{-i\left(\frac{u-\varepsilon v}{\delta} x + v\delta y\right)} dx dy = G\left(\frac{u-\varepsilon v}{\delta}, \delta v\right). \end{aligned} \tag{18}$$

It is easily seen from definition (17) that the net $S_{\varepsilon,\delta}$ applied to the process $\eta(p)$ has the same limiting variance as the net S when applied to the process $\eta^*(p)$. In symbols:

$$s^2_\infty(S_{\varepsilon,\delta}) = s^2_\infty\big(S_{\varepsilon,\delta}|\eta(p)\big) = s^2_\infty\big(S|\eta^*(p)\big).$$

Hence

(19) $$s^2_\infty(S_{\varepsilon,\delta}) = \sum_{(i,j)\epsilon S}' F^*(i,j).$$

By Poisson's formula (see [1], p. 203) we get

(19a) $$\sum_{(i,j)\epsilon S} F^*(i,j) = 4\pi^2 \sum_{(i,j)\epsilon S}' G^*(2\pi i, 2\pi j).$$

From (19a), (18) and (19) follows lemma 1.

Proof of theorem 1. The limiting variance $s^2_\infty(N_{a,\beta,\varepsilon,\delta})$, corresponding to the process $\eta(p)$ with correlation function $f(|p-q|) = h(d^2)$ and the net $N_{a,\beta,\varepsilon,\delta}$ defined by (11), is given by

(20) $$s^2_\infty(N_{a,\beta,\varepsilon,\delta}) = n(a,\beta,\varepsilon,\delta) = \sum_{(u,v)\epsilon W_1} h\big(\delta^2u^2+(v+\varepsilon u)^2/\delta^2\big)+ \\ + \sum_{(u,v)\epsilon W_2} h\big[\delta^2(u+a)^2+\big(v+\beta+\varepsilon(u+a)\big)^2/\delta^2\big].$$

Denote by h' and h'' the first and second derivatives with respect to t of the function $h(t)$ and let $(.)$ and A stand for $(a,\beta,\varepsilon,\delta/\sqrt[4]{3})$ and $(u^2+3v^2)/\sqrt{3}$, respectively. We then have

(i) $$\frac{\partial}{\partial a}n(.)\Big|_{(0,0,0,1)} = \frac{2}{\sqrt{3}}\sum_{(u,v)\epsilon W_2}' h'(A)u = 0,$$

(ii) $$\frac{\partial}{\partial \beta}n(.)\Big|_{(0,0,0,1)} = 2\sqrt{3}\sum_{(u,v)\epsilon W_2} h'(A)v = 0,$$

(iii) $$\frac{\partial}{\partial \varepsilon}n(.)\Big|_{(0,0,0,1)} = 2\sqrt{3}\sum_{(u,v)\epsilon W} h'(A)uv = 0,$$

(iv) $$\frac{\partial}{\partial \delta}n(.)\Big|_{(0,0,0,1)} = \frac{2}{\sqrt{3}}\sum_{(u,v)\epsilon W}' h'(A)(u^2-3v^2) \\ = \frac{1}{\sqrt{3}}\sum_{\substack{i+j=0(\mathrm{mod}\,2)\\ i,j-\mathrm{integer}}}' h'\left(\frac{i^2+3j^2}{\sqrt{12}}\right)(i^2-3j^2) = 0,$$

(v) $$\frac{\partial^2}{\partial a\partial\beta}n(.)\Big|_{(0,0,0,1)} = 4\sum_{(u,v)\epsilon W_2} h''(A)uv = 0,$$

(vi) $$\frac{\partial^2}{\partial\alpha\partial\varepsilon} n(.)\Big|_{(0,0,0,1)} = 2\sum_{(u,v)\in W_2} [h''(A)2u^2+h'(A)\sqrt{3}]v = 0,$$

(vii) $$\frac{\partial^2}{\partial\alpha\partial\delta} n(.)\Big|_{(0,0,0,1)} = \frac{4}{\sqrt{3}}\sum_{(u,v)\in W_2} [h''(A)(u^2-3v^2)/\sqrt{3}+h'(A)]u = 0,$$

(viii) $$\frac{\partial^2}{\partial\beta\partial\varepsilon} n(.)\Big|_{(0,0,0,1)} = 2\sum_{(u,v)\in W_2} [h''(A)6v^2+h'(A)\sqrt{3}]u = 0,$$

(ix) $$\frac{\partial^2}{\partial\beta\partial\delta} n(.)\Big|_{(0,0,0,1)} = 4\sqrt{3}\sum_{(u,v)\in W_2} [h''(A)(u^2-3v^2)-h'(A)\sqrt{3}]v = 0,$$

(x) $$\frac{\partial^2}{\partial\varepsilon\partial\delta} n(.)\Big|_{(0,0,0,1)} = 4\sum_{(u,v)\in W} [h''(A)(u^2-3v^2)-h'(A)\sqrt{3}]uv = 0.$$

The derivatives (i), (iii), (v), (vii), (viii) and (x) vanish because the sum terms calculated for (u, v) and $(-u, v)$ reduce, the derivatives (ii), (vi), and (ix) vanish because the sum terms calculated for (u, v) and $(u, -v)$ reduce. Case (iv) was considered in [3] (lemma 2). Relations (i)-(iv) show that the necessary condition for an extremum of the function $n(\alpha, \beta, \varepsilon, \delta)$ at the point $(\alpha, \beta, \varepsilon, \delta) = (0, 0, 0, 1/\sqrt[4]{3})$ is fulfilled. Because of relations (v)-(x) we have only to examine the existence of an extremum for each of the four variables separately.

For $\lambda \geqslant 3\sqrt[4]{12}/2 \approx 2{,}79$, the statement (a) of theorm 1 follows from the statement (a) of theorem 1 in [3]. Now, for small λ we shall use in our proof the spectral density of the correlation function. If the process $\eta(p)$ is stationary and isotropic, then the spectral density may be represented in the form

$$G(u, v) = \iint e^{-i(ux+vy)} f(\sqrt{x^2+y^2})\,dx\,dy = g(u^2+v^2).$$

It is easy to verify the relations

(20) $$W = W_1 \cup W_2 = r_{1/\sqrt{2}} s_{1/2}(S),$$

(21) $$T = r_{1/\sqrt[4]{12}} s_{1/2}(S),$$

(22) $$T_{\varepsilon,\delta} = r_\delta s_\varepsilon(T) = r_{\delta_1} s_{\varepsilon_1}(S),$$

(23) $$T_{\varepsilon,\delta} = r_{\delta_2} s_{\varepsilon_2}(W),$$

where $\delta_1 = \delta/\sqrt[4]{12}$, $\varepsilon_1 = 1/2+\varepsilon/\sqrt{12}$, $\delta_2 = \delta/\sqrt[4]{3}$, $\varepsilon_2 = \varepsilon/\sqrt{3}$.

Now from (22) and lemma 1 we get

$$s_\infty^2(T_{\varepsilon,\delta})/4\pi^2 = t(\varepsilon, \delta) = \sum_{(i,j)\in S} g\left[4\pi^2\left(\frac{(i-\varepsilon_1 j)^2}{\delta_1^2} + \delta_1^2 j^2\right)\right].$$

Hence by (22)

$$t(\varepsilon,\delta) = \sum_{(u,v)\in S_{\varepsilon_1,\delta_1}} g[4\pi^2(u^2+v^2)] = \sum_{(i,j)\in W} g\left[4\pi^2\left(\delta_2^2 i^2 + \frac{(j+\varepsilon_2 i)^2}{\delta_2^2}\right)\right]$$

follows.

To complete the proof of theorem 1 one has to show that the function

$$m(\varepsilon,\delta) = t(\varepsilon\sqrt{3},\delta) = \sum_{(i,j)\in W} g\left[\frac{4}{\sqrt{3}}\pi^2\left(\delta^2 i^2 + 3\,\frac{(j+\varepsilon i)^2}{\delta^2}\right)\right]$$

has a local minimum at $(\varepsilon,\delta) = (0,1)$. The first derivatives and the second mixed derivative of the function $m(\varepsilon,\delta)$ vanish for the same reason as the derivatives of the function $n(0,0,\varepsilon,\delta/\sqrt[4]{3})$. Let us calculate the second derivatives. We have

$$(24)\quad \frac{\partial^2}{\partial\varepsilon^2} m(\varepsilon,\delta)|_{(0,1)} = 6a \sum_{(i,j)\in W} \big[6ag''[a(i^2+3j^2)]j^2 + g'[a(i^2+3j^2)]\big] i^2,$$

$$(25)\quad \frac{\partial^2}{\partial\delta^2} m(\varepsilon,\delta)|_{(0,1)} = 2a \sum_{(i,j)\in W} \big[2ag''[a(i^2+3j^2)](i^2-3j^2)^2 + g'[a(i^2+3j^2)](i^2+9j^2)\big],$$

where $a = 4\pi^2/\sqrt{3}$. If in the series (24) and (25) we add up the terms for which $i^2+3j^3 = \text{const}$, we get

$$(26)\quad \frac{\partial^2}{\partial\varepsilon^2} m(\varepsilon,\delta)|_{(0,1)} = \frac{4}{3}\frac{\partial^2}{\partial\delta^2} m(\varepsilon,\delta)|_{(0,1)}$$

$$= 9a \sum_{\substack{(i,j)\in W\\ 0\leqslant 3j\leqslant i}} \big[ag''[a(i^2+3j^2)](i^2+3j^2) + 2g'[a(i^2+3j^2)]\big](i^2+3j^2)\, b_{ij},$$

where

$$b_{ij} = \begin{cases} 2 & \text{if}\quad 0 < 3j < i,\\ 1 & \text{if}\quad j = 0 \quad\text{or}\quad i = 3j.\end{cases}$$

The spectral density corresponding to the exponential correlation function (14) is

$$(27)\quad g(u^2+v^2) = \frac{\lambda}{2\pi}(\lambda^2+u^2+v^2)^{-3/2}.$$

Hence

$$(28)\quad \frac{\partial^2}{\partial\varepsilon^2} m(\varepsilon,\delta)|_{(0,1)} = 18\sqrt{3}\pi\lambda \sum_{\substack{(i,j)\in W\\ 0\leqslant 3j\leqslant i}} \frac{\left[\dfrac{\pi^2}{\sqrt{3}}(i^2+3j^2) - \lambda^2\right](i^2+3j^2)\, b_{ij}}{\left(\lambda^2 + \dfrac{4\pi^2}{\sqrt{3}}(i^2+3j^2)\right)^{7/2}}.$$

If $\lambda < \pi\sqrt[4]{\frac{4}{3}} \approx 3{,}36$, then the series (28) is positive since all its terms are positive. Therefore, for fixed $\lambda < \pi\sqrt[4]{\frac{4}{3}}$, the function $s^2_\infty\ (T_{\varepsilon,\delta})$ has a local minimum at the point $(\varepsilon, \delta) = (0, 1)$. This completes the proof of (a). To prove (b) we shall show that if $\lambda > \sqrt[4]{27} \approx 2.28$, then the derivatives

$$(29)\quad \frac{d^2}{d\alpha^2} n(\alpha, 0, 0, 1/\sqrt[4]{3})|_{\alpha=0} = U = \lambda' \sum_{(i,j)\in W_2} \frac{\lambda' i^2 - 3j^2/\sqrt{i^2+3j^2}}{(i^2+3j^2)\exp(\lambda'\sqrt{i^2+3j^2})},$$

$$(30)\quad \frac{d^2}{d\beta^2} n(0, \beta, 0, 1/\sqrt[4]{3})|_{\beta=0} = V = 3\lambda' \sum_{(i,j)\in W_2} \frac{3\lambda' j^2 - i^2/\sqrt{i^2+3j^2}}{(i^2+3j^2)\exp(\lambda'\sqrt{i^2+3j^2})},$$

where $\lambda' = \lambda/\sqrt[4]{3}$, are positive.

Indeed, if $(i, j)\in W_2$, then $i^2+3j^2 \geqslant 1$ and

$$(31)\qquad U > \lambda' \sum_{(i,j)\in W_{2,0}} \frac{\lambda' i^2 - 3j^2}{(i^2+3j^2)^{3/2}\exp(\lambda'\sqrt{i^2+3j^2})} c_{ij},$$

$$(32)\qquad V > \lambda' \sum_{(i,j)\in W_{2,0}} \frac{3\lambda' j^2 - i^2}{(i^2+3j^2)^{3/2}\exp(\lambda'\sqrt{i^2+3j^2})} c_{ij},$$

where

$$c_{ij} = \begin{cases} 4 & \text{if} \quad i > 0 \quad \text{and} \quad j > 0, \\ 2 & \text{if} \quad i = 0 \quad \text{or} \quad j = 0, \end{cases}$$

and

$$W_{2,0} = \left\{\left(\frac{2i+1}{\sqrt{2}}, \frac{2j+1}{\sqrt{2}}\right):\ i, j\text{—integer},\ i \geqslant 0,\ j \geqslant 0,\ i^2+j^2 > 0\right\}.$$

Let us now split the set $W_{2,0}$ into eight disjoint subsets

$$A_1 = W_{2,0} \cap \{(x, y):\ y < x/3,\ x+y = 0 (\mathrm{mod}\ 2\sqrt{2})\},$$
$$A_2 = W_{2,0} \cap \{(x, y):\ y < x/3,\ x+y = \sqrt{2} (\mathrm{mod}\ 2\sqrt{2})\},$$
$$B = W_{2,0} \cap \{(x, y):\ y = x/3\},$$
$$C_1 = W_{2,0} \cap \{(x, y):\ x/3 < y < x,\ x+y = 0 (\mathrm{mod}\ 2\sqrt{2})\},$$
$$C_2 = W_{2,0} \cap \{(x, y):\ x/3 < y < x,\ x+y = \sqrt{2} (\mathrm{mod}\ 2\sqrt{2})\},$$
$$D = W_{2,0} \cap \{(x, y):\ x = y\},$$
$$E_1 = W_{2,0} \cap \{(x, y):\ y > x,\ x+y = 0 (\mathrm{mod}\ 2\sqrt{2})\},$$
$$E_2 = W_{2,0} \cap \{(x, y):\ y > x,\ x+y = \sqrt{2} (\mathrm{mod}\ 2\sqrt{2})\},$$

(see Fig. 2, the points of the sets A_1, B, C_1, and E_1 on it are blackened), and consider the one-to-one transformations

$$a_1: \quad (i,j) \to \left(\frac{i+3j}{2}, \frac{i-j}{2}\right),$$

$$a_2: \quad (i,j) \to \left(\frac{-i+3j}{2}, \frac{i+j}{2}\right),$$

$$a_3 = a_2(a_1): \quad (i,j) \to \left(\frac{i-3j}{2}, \frac{i+j}{2}\right).$$

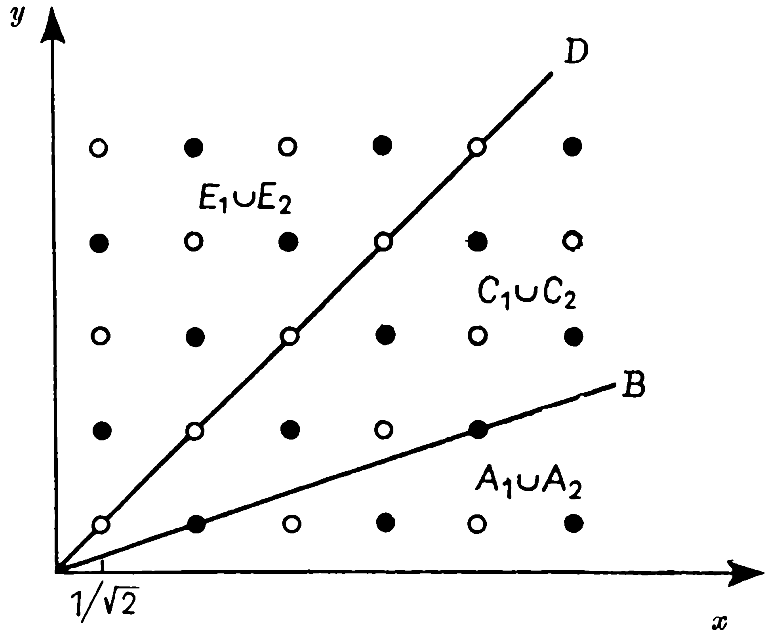

Fig. 2

It is easily seen that

$$a_1(A_1) = C_1, \quad a_3(A_2) = E_1, \quad a_2(C_2) = E_2.$$

Consider the ellipses

$$i^2+3j^2 = \text{const} \tag{33}$$

and denote

$$p(i,j) = \lambda' i^2 - 3j^2, \quad q(i,j) = 3\lambda' j^2 - i^2.$$

It is easy to prove that if point $p = (i,j)$ is on the ellipse (33), then, also the points $a_1(p), a_2(p)$ and $a_3(p)$ are on it. Hence if $\lambda' \geqslant 3$, then

(i) if $(i,j) \in W_{2,0} - (E_1 \cap E_2)$, then $p(i,j) \geqslant 0$,

(ii) if $(i,j) \in E_1$, then $a_3^{-1}(i,j) \in A_2$ and we have $p(i,j)+p\big(a_3^{-1}(i,j)\big) \geqslant 0$,

(iii) if $(i,j) \in E_2$, then $a_2^{-1}(i,j) \in C_2$ and we have $p(i,j)+p\big(a_2^{-1}(i,j)\big) \geqslant 0$,

(iv) if $(i,j) \in W_{2,0} - (A_1 \cap A_2)$, then $q(i,j) \geqslant 0$,

(v) if $(i,j)\epsilon A_1$, then $a_1(i,j)\epsilon C_1$ and we have $q(i,j)+q(a_1(i,j)) \geqslant 0$,

(vi) if $(i,j)\epsilon A_2$, then $a_3(i,j)\epsilon E_1$ and we have $q(i,j)+q(a_3(i,j)) \geqslant 0$.

If $\lambda > 3/\sqrt[4]{3}$, relations (i)-(iii) imply that the derivative (29) is positive and relations (iv)-(vi) imply that the derivative (30) is positive. The proof of theorem 1 is thus complete.

Proof of theorem 2. For a process with correlation function $f(|p-q|) = \varrho(d)$ and the net $T_{\varepsilon,\delta}$ the limiting variance $s_{\infty}^2(T_{\varepsilon,\delta}) = t^*(\varepsilon,\delta)$ is given by

$$t^*(\varepsilon, \delta) = \sum_{(u,v)\epsilon T} \int_0^\infty \exp\{-\lambda\sqrt{u^2\delta^2+(v+\varepsilon u)^2/\delta^2}\}dF(\lambda) = \int_0^\infty t_\lambda(\varepsilon, \delta)dF(\lambda),$$

where $t_\lambda(\varepsilon, \delta)$ is the limiting variance corresponding to the process with exponential correlation function (14). It is easily seen that the function $t_\lambda(\varepsilon, \delta)$ and its first and second derivatives may be dominated by the function P^2/λ^2 summable with respect to $F(\lambda)$. In view of theorem 1 (a) we obtain theorem 2 by differentiating under the integral sign.

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O LOKALNEJ OPTYMALNOŚCI REGULARNYCH SIECI PRÓB NA PŁASZCZYŹNIE (II)

STRESZCZENIE

W pracy [3] rozważone jest zagadnienie estymacji średniej pewnego stacjonarnego, izotropowego procesu stochastycznego na płaszczyźnie. Dla danej sieci punktów i danego koła na płaszczyźnie estymatorem nieobciążonym średniej procesu jest średnia arytmetyczna obserwacji z tego koła. Jako miarę efektywności sieci przyjęto graniczną wartość wariancji tej średniej mnożoną przez ilość punktów użytych do jej określenia przy promieniu kola rosnącym nieograniczenie. Problemem rozważanym w [3] jest pytanie, które sieci o ustalonej gęstości są najbardziej efektywne. Zasadniczy wynik [3] mówi, że dla procesów stacjonarnych izotropowych i ciągłych z wykładniczą funkcją korelacyjną sieć wierzchołków trójkątów równobocznych jest lokalnie optymalna ze względu na deformacje afiniczne, jeśli gęstość sieci jest dostatecznie mała. W tej pracy uogólniono ten rezultat, dowodząc, że sieć trójkątów równobocznych jest w klasie deformacji afinicznych lokalnie optymalna przy każdej gęstości sieci oraz że może ona być lokalnie optymalna w szerszych klasach deformacji albo w szerszych klasach funkcji korelacyjnych.

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О ЛОКАЛЬНОЙ ОПТИМАЛЬНОСТИ РЕГУЛЯРНЫХ СЕТЕЙ ПРОБ НА ПЛОСКОСТИ (II)

РЕЗЮМЕ

В работе [3] рассматривался вопрос средней оценки некоторого стационарного, изотропного стохастического процесса на плоскости. Для даной сети точек и даного круга на плоскости несмещённой оценкой средней процесса является средняя арифметическая наблюдений из этого круга. В качестве эффективности сети принято предельное значение дисперсии этой средней, умноженное на количество точек, употреблённых при её определении при неограничено возрастающим радиусе круга. В [3] рассматривался вопрос, которые сети с определённой плотностью являются наиболее эффективными. Основной результат [3] заключается в том, что для стационарных, изотропных, непрерывных процессов с экспоненциальной кореляционной функцией, сеть вершин равносторонних треугольников является локально оптимальной относительно аффинных деформаций, если плотность сети достаточно мала. В настоящей работе этот результат обобщается и доказывается, что сеть равносторонних треугольников является в классе аффинных деформаций локально оптимальной при всякой плотности сети и что может она оставаться локально оптимальной в более широких классах деформаций или в более широких классах кореляционных функций.
