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FACTORIZATION THEOREMS FOR EXTENSIONS OF MAPS

BY

W. KULPA (KATOWICE)

Let $f: M \to Y$ be a continuous map from a closed subspace M of a normal space X into a metric space Y of weight less than or equal to τ. In this note there are given some sufficient conditions for the existence of continuous maps $h: X \to Z_f$ and $g: M_f \to Y$ such that $f(x) = gh(x)$ for each $x \in M$, where M_f is a closed subspace of a metric space Z_f of weight less than or equal to τ.

1. Preliminaries. Maps considered in this note are assumed to be (uniformly) continuous. We use the notion of uniformity in the covering sense. Some symbols and notation are taken from [5] and [6].

If X is a completely regular space, then by $\mathscr{U}_X^*$ we denote the greatest uniformity inducing the topology of the space X. A family $\mathscr{U} \subset \mathscr{U}_X^*$ which satisfies all axioms of uniformity except for the axiom of separation is said to be a *pseudouniformity*. Symbols $P \succ Q$ and $P \succ_* Q$ mean that P is a refinement and a star-refinement, respectively.

For each pseudouniformity $\mathscr{U}$ put $\mathrm{dw}\mathscr{U} \leqslant (\gamma, \tau)$ if there exists a base $\mathscr{B} \subset U$ with $\mathrm{card}\mathscr{B} \leqslant \gamma$, consisting of locally finite coverings of cardinality less than or equal to τ (we assume that γ and τ are infinite).

A subspace $M \subset X$ is said to be a *$u(\tau)$-subspace* of X if for each locally finite open in M covering $P \in \mathscr{U}_M^*$ with $\mathrm{card}P \leqslant \tau$ there exists a locally finite open covering $Q \in \mathscr{U}_X^*$ with $\mathrm{card}Q \leqslant \tau$ such that $Q|M = \{u \cap M: u \in Q\}$ is a refinement of P.

2. Factorization theorems.

THEOREM 1. *Let $f: M \to Y$ be a map of a $u(\tau)$-subspace of a completely regular space X into a completely regular space Y, the topology of which is induced by a complete uniformity $\mathscr{V}$ with $\mathrm{dw}\mathscr{V} \leqslant (\gamma, \tau)$. Then there exist maps $h: X \to Z_f$ and $g: M_f \to Y$ such that*

1. *$gh(x) = f(x)$ for each $x \in M$;*

2. *$M_f \subset Z_f$ is a closed subspace of a space Z_f, the topology of which is induced by a complete uniformity $\mathscr{V}_f$ with $\mathrm{dw}\mathscr{V}_f \leqslant (\gamma, \tau)$.*

Proof. A map $f\colon (M, \mathcal{U}^*_M) \to (Y, \mathcal{V})$ is uniform. Let $\mathcal{B} \subset \mathcal{V}$ be a base with $\operatorname{card}\mathcal{B} \leqslant \gamma$, consisting of locally finite open coverings of cardinality less than or equal to τ. Since M is a $u(\tau)$-subspace, we can choose by a countable operation (see Proposition 3 in [5]) a pseudouniformity $\mathcal{U} \subset \mathcal{U}^*_X$ with $\operatorname{dw}\mathcal{U} \leqslant (\gamma, \tau)$ and such that $f^{-1}\mathcal{V} \subset \mathcal{U}|M$.

Put

$$X_{\mathcal{U}} = \{[x]_{\mathcal{U}}\colon x \in X\}, \quad \text{where } [x]_{\mathcal{U}} = \bigcap \{\operatorname{st}(x, P)\colon P \in \mathcal{U}\}.$$

The set $X_{\mathcal{U}}$ is a partition of X into sets $[x]_{\mathcal{U}}$, $x \in X$. Put $h\colon X \to X_{\mathcal{U}}$, $x \mapsto [x]_{\mathcal{U}}$, and define the uniformity $\mathcal{U}^{\#}$ on $X_{\mathcal{U}}$ by

$$\mathcal{U}^{\#} = \{P^{\#}\colon P \in \mathcal{U}\}, \quad \text{where } P^{\#} = \{X_{\mathcal{U}} - h(X-u)\colon u \in P\}.$$

The map $h\colon (X, \mathcal{U}) \to (X_{\mathcal{U}}, \mathcal{U}^{\#})$ is uniform. Since $h^{-1}\mathcal{U}^{\#} = \mathcal{U}$ and $f^{-1}\mathcal{V} \subset \mathcal{U}|M$, there exists a uniform map $\bar{g}\colon \big(h(M), \mathcal{U}^{\#}|h(M)\big) \to (Y, \mathcal{V})$ such that $\bar{g}[x]_{\mathcal{U}} = f(x)$.

Let $(Z_f, \mathcal{V}_f)$ be a completion of the space $(X_{\mathcal{U}}, \mathcal{U}^{\#})$. Then $(M_f, \mathcal{V}_f|M_f)$, where $M_f = \operatorname{cl}_{Z_f} h(M)$ is a completion of the space $\big(h(M), \mathcal{U}^{\#}|h(M)\big)$. Since the space $(Y, \mathcal{V})$ is complete, the unique uniform extension $g\colon (M_f, \mathcal{V}_f) \to (Y, \mathcal{V})$ of the map $\bar{g}$ exists. Thus $gh(x) = f(x)$ for each $x \in M$, M_f is a closed subspace of Z_f, and $\operatorname{dw}\mathcal{V}_f \leqslant (\gamma, \tau)$.

THEOREM 2. *Let $f\colon M \to Y$ be a map of a $u(\tau)$-subspace of a completely regular space X into a completely regular space Y, the topology of which is induced by a uniformity $\mathcal{V}$ with $\operatorname{dw}\mathcal{V} \leqslant (\gamma, \tau)$. If M has a complete pseudouniformity $\mathcal{U}' \subset \mathcal{U}^*_M$ with $\operatorname{dw}\mathcal{U}' \leqslant (\gamma, \tau)$, then there exist maps $h\colon X \to Z_f$ and $g\colon M_f \to Y$ such that*

1. *$gh(x) = f(x)$ for each $x \in M$;*
2. *$M_f \subset Z_f$ is a closed subset of a space Z_f, the topology of which is induced by a uniformity $\mathcal{V}_f$ with $\operatorname{dw}\mathcal{V}_f \leqslant (\gamma, \tau)$.*

Proof. A map $(M, \mathcal{U}^*_M) \to (Y, \mathcal{V})$ is uniform. Since $M \subset X$ is $u(\tau)$-subspace, we can choose by a countable operation a pseudouniformity $\mathcal{U} \subset \mathcal{U}^*_X$ with $\operatorname{dw}\mathcal{U} \leqslant (\gamma, \tau)$ and such that $f^{-1}\mathcal{V} \cup \mathcal{U}' \subset \mathcal{U}|M$. As in the previous proof, put $X_{\mathcal{U}} = \{[x]_{\mathcal{U}}\colon x \in X\}$ and $Z_f = X_{\mathcal{U}}$ with the topology induced by the uniformity

$$\mathcal{V}_f = \mathcal{U}^{\#} = \{P^{\#}\colon P \in \mathcal{U}\}, \quad \text{where } P^{\#} = \{Z_f - h(X-u)\colon u \in P\}.$$

The map $h\colon (X, \mathcal{U}) \to (Z_f, \mathcal{V}_f)$, $x \mapsto [x]_{\mathcal{U}}$, is uniform. Define

$$g\colon \big(h(M), \mathcal{U}^{\#}|h(M)\big) \to (Y, \mathcal{V})$$

by $g[x]_{\mathcal{U}} = f(x)$ for each $x \in M$. Put $M_f = h(M) \subset Z_f$. We have $gh(x) = f(x)$ for each $x \in M$ and $\operatorname{dw}\mathcal{V}_f \leqslant (\gamma, \tau)$. We show that the set $M_f \subset Z_f$ is closed. The pseudouniformity $\mathcal{U}|M$ is complete, since $\mathcal{U}' \subset \mathcal{U}|M$. Suppose that there exists a point $z \in \operatorname{cl}_{Z_f} h(M) - h(M)$. A family

$$\{h^{-1}\operatorname{st}(z, Q) \cap M\colon Q \in \mathcal{U}^{\#}\} = \{\operatorname{st}(x, P) \cap M\colon P \in \mathcal{U}\}, \quad x \in h^{-1}(z),$$

is a Cauchy filter in the complete pseudouniform space $(M, \mathscr{U}|M)$. And since $P' \succ_* P$ implies $\{\mathrm{cl}\, u\colon u \in P'\} \succ P$, there exists a point

$$y \in \bigcap \{\mathrm{cl}_M \mathrm{st}(x, P) \cap M\colon P \in \mathscr{U}\} = \bigcap \{\mathrm{st}(x, P) \cap M\colon P \in \mathscr{U}\}.$$

Thus $h(y) = z$, $y \in M$, a contradiction with $z \notin h(M)$.

THEOREM 3. *Let $f\colon M \to Y$ be a map of a $u(\tau)$-subspace, closed and $\mathscr{G}_\gamma$ in a normal space X, into completely regular space Y, the topology of which is induced by a uniformity $\mathscr{V}$ with $\mathrm{dw}\mathscr{V} \leqslant (\gamma, \tau)$. Then there exist maps $h\colon X \to Z_f$ and $g\colon M_f \to Y$ such that*

1. *$gh(x) = f(x)$ for each $x \in M$;*
2. *$M_f \subset Z_f$ is a closed subset of a space Z_f, the topology of which is induced by a uniformity $\mathscr{V}_f$ with $\mathrm{dw}\mathscr{V}_f \leqslant (\gamma, \tau)$ and $h^{-1}M_f = M$.*

Proof. Let $M = \bigcap \{G\colon G \in \mathscr{G}\}$, where $\mathscr{G}$ with $\mathrm{card}\,\mathscr{G} \leqslant \gamma$ is a family of open sets. Since X is a normal space,

$$\mathscr{U}_{\mathscr{G}} = \big\{\{G, X - M\}\colon G \in \mathscr{G}\big\} \subset \mathscr{U}_X^*.$$

By a countable operation we can find a pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ with $\mathrm{dw}\mathscr{U} \leqslant (\gamma, \tau)$ and such that $\mathscr{U}_{\mathscr{G}} \subset \mathscr{U}$ and $f^{-1}\mathscr{V} \subset \mathscr{U}|M$. Put $Z_f = X_{\mathscr{U}}$ with topology induced by the uniformity $\mathscr{U}^{\#}$, $h(x) = [x]_{\mathscr{U}}$ for $x \in X$, $g[x]_{\mathscr{U}} = f(x)$ for $x \in M$, $M_f = h(M)$. To see that $M_f \subset Z_f$ is a closed subspace, consider $x \in M$. There exists a $G \in \mathscr{G}$ such that $x \notin G$. Let $P = \{G, X - M\}$, $z = h(x)$. Then $\mathrm{st}(z, P^{\#}) \cap h(M) = \emptyset$. Since $P^{\#} \in \mathscr{U}^{\#}$, $h(x) \notin \mathrm{cl}_{Z_f} h(M)$. This implies

$$h(X - M) \cap h(M) = \emptyset \quad \text{and} \quad \mathrm{cl}_{Z_f} h(M) = h(M).$$

A space X is said to be *γ-feathered* if there exists a family $\mathscr{P}$, $\mathrm{card}\,\mathscr{P} \leqslant \gamma$, of coverings of X with open sets in the Čech-Stone compactification βX such that, for each $x \in X$,

$$\bigcap \{\mathrm{st}(x, P)\colon P \in \mathscr{P}\} \subset X.$$

In the case $\gamma = \aleph_0$ the space X is called *feathered* or a *p-space*.

THEOREM 4. *Let $f\colon M \to Y$ be a map from a closed subspace M of a γ-feathered paracompact space X into a completely regular space Y, the topology of which is induced by a uniformity $\mathscr{V}$ with $\mathrm{dw}\mathscr{V} \leqslant (\gamma, \tau)$. Then there exist maps $h\colon X \to Z_f$ and $g\colon M_f \to Y$ such that*

1. *$gh(x) = f(x)$ for each $x \in M$;*
2. *$h\colon X \to Z_f$ is a perfect map into a paracompact space Z_f, the topology of which is induced by a uniformity $\mathscr{V}_f$ with $\mathrm{dw}\mathscr{V}_f \leqslant (\gamma, \tau)$.*

Proof. Let $\mathscr{P}$ with $\mathrm{card}\,\mathscr{P} \leqslant \gamma$ be a family of coverings of X open in βX and such that, for each $x \in X$,

$$\bigcap \{\mathrm{st}(x, P)\colon P \in \mathscr{P}\} \subset X.$$

Since X is paracompact, $\mathscr{P}|X \subset \mathscr{U}_X^*$. For each $P \in \mathscr{P}$ there exists a locally finite open covering $Q \in \mathscr{U}_X^*$ such that $\bar{Q} \succ P$, where

$$\bar{Q} = \{\bar{v}\colon v \in Q\}, \quad \bar{v} = \mathrm{cl}_{\beta X}\bigcup\{u \subset \beta X\colon u \cap X \subset v,\ u \text{ is open in } \beta X\}.$$

Hence, by the compactness of βX, the family $\{\mathrm{st}(x, P)\colon P \in \mathscr{U}\}$ is a base of neighbourhoods of $[x]_{\mathscr{U}}$, and $[x]_{\mathscr{U}}$ is compact for each $x \in X$, where $\mathscr{U} \subset \mathscr{U}_X^*$ is a pseudouniformity with $\mathrm{dw}\,\mathscr{U} \leqslant (\gamma, \tau)$ and such that $\mathscr{P}|X \subset \mathscr{U}$, $f^{-1}\mathscr{V} \subset \mathscr{U}|M$. Thus $h\colon X \to Z_f = X_{\mathscr{U}}$, $x \mapsto [x]_{\mathscr{U}}$, is a perfect map. Put $g[x]_{\mathscr{U}} = f(x)$ for each $x \in M$. The topology of the space Z_f is induced by the uniformity $\mathscr{U}^{\#} = \mathscr{V}_f$ with $\mathrm{dw}\,\mathscr{U}^{\#} = \mathrm{dw}\,\mathscr{U} \leqslant (\gamma, \tau)$. By a theorem of Michael the space Z_f is paracompact.

3. Some remarks. Each metrizable space Y of weight less than or equal to τ has a uniformity $\mathscr{V}$ with $\mathrm{dw}\,\mathscr{V} \leqslant (\aleph_0, \tau)$ which induces the topology and, conversely, if $\mathrm{dw}\,\mathscr{V} \leqslant (\aleph_0, \tau)$, then the uniformity $\mathscr{V}$ induces on Y a metrizable topology of weight less than or equal to τ. Thus in the case $(\aleph_0, \tau)$ the factorization theorems give some answer to the question when for each map $f\colon M \to Y$ from a closed subspace of a normal space X into a metrizable space Y of weight less than or equal to τ there exist a closed subspace $M_f \subset Z_f$ of a metrizable space Z_f with weight less than or equal to τ and maps $h\colon X \to Z_f$, $g\colon M_f \to Y$ such that the diagram

$$(*) \qquad \begin{array}{ccc} X & \xrightarrow{\ h\ } & Z_f \\ \cup & & \cup \\ M & \xrightarrow{\ h|M\ } & M_f \\ & {\scriptstyle f}\searrow \quad Y \quad \swarrow{\scriptstyle g} & \end{array}$$

commutes.

Theorems 1-3 show that if M is a closed $u(\tau)$-subspace of a normal space X and Y is complete metric (Theorem 1) or if M is complete in the Čech sense (Theorem 2) or if M is $\mathscr{G}_\delta$ (Theorem 3), then diagram $(*)$ is satisfied. If X is a p-paracompact space, then diagram $(*)$ is satisfied for each closed set $M \subset X$ (Theorem 4).

Now let us try to explain for which classes of spaces X each closed subset $M \subset X$ is a $u(\tau)$-subspace. For example, if X is collectionwise normal, then each closed subset $M \subset X$ is a $u(\tau)$-subspace for each τ. If X is τ-collectionwise normal, then each closed subset $M \subset X$ is a $u(\tau)$-subspace ([4], see also [10]). Since each normal space X is $\aleph_0$-collectionwise normal [8], each closed set $M \subset X$ is a $u(\aleph_0)$-subspace of X.

Theorem 1 yields the following characterization of collectionwise normal spaces.

THEOREM 5. *A space X is collectionwise normal if and only if, for each map $f\colon M \to Y$ of an arbitrary closed subset $M \subset X$ into a complete metric space Y, there exist maps $h\colon X \to Z_f$ and $g\colon M_f \to Y$ such that*

1. *$gh(x) = f(x)$ for each $x \in M$;*
2. *$M_f \subset Z_f$ is a closed subset of a complete metric space Z_f and*

$$\operatorname{weight} Z_f \leqslant \aleph_0 + \operatorname{weight} Y.$$

By similar theorems one can characterize normal spaces (then Y and Z_f are Polish spaces), τ-collectionwise normal spaces (then Y and Z_f are complete metric spaces of weight less than or equal to τ), and p-paracompact spaces (then h is a perfect map).

Now consider some applications of the factorization theorems. Note that if for a map f: $M \to Y$, $M \subset X$, into a metric space the diagram (*) commutes, then f has an extension over some $\mathscr{G}_\delta$ closed subset $M \subset X$ (see [2]).

If a metric space of weight less than or equal to τ is an AE-space for a class of metric spaces of weight less than or equal to τ, then Y is an AE-space for the class of spaces for which diagram (*) is satisfied. Consider factorization theorems of this kind:

Let h: $X \to Y$ be a map into a metric space Y, weight $Y \leqslant \tau$, *and let X have a property* (c) (*for example,* $\dim X \leqslant n$; see [9], [12], [1], [7], [13], [3]). *Then there exist maps h_1: $X \to Z$ and h_2: $Z \to Y$ such that $h = h_2 h_1$, the space Z is metrizable with weight less than or equal to τ, and Z has the property* (c).

Applying this theorem, one can obtain some stronger versions of Theorems 1-4 in which the space Z_f has some additional properties, the same as the space X has. For example, Pasynkov has proved the following:

For each map h: $X \to Y$ of completely regular space X, $\dim X \leqslant n$, *into a metric space Y,* weight $Y \leqslant \tau$, *there exist maps h_1: $X \to Z$ and h_2: $Z \to Y$ such that $h = h_2 h_1$, Z is a metrizable space of weight less than or equal to τ and* $\dim Z \leqslant n$.

Applying the theorem of Pasynkov to Theorem 1 for $\gamma = \aleph_0$, we can obtain the following corollary:

If f: $M \to Y$ is a map from a closed subspace M of a τ-collectionwise normal space X, $\dim X \leqslant n$, *into a complete metric space Y,* weight $Y \leqslant \tau$, *then there exist maps $\bar{h}$: $X \to \bar{Z}$ and $\bar{g}$: $\bar{M} \to Y$ such that $f(x) = \bar{g}\bar{h}(x)$ for each $x \in \bar{M}$, $\bar{M}$ is a closed subset of a metric space $\bar{Z}$,* $\dim \bar{Z} \leqslant n$ *and* weight $\bar{Z} \leqslant \tau$.

Indeed, it follows from Theorem 1 that there exist maps h: $X \to Z_f$ and g: $M_f \to Y$ such that $gh(x) = f(x)$ for each $x \in M$, and $M_f \subset Z_f$ is a closed subspace of a metrizable space Z_f of weight less than or equal to τ. By Pasynkov's theorem, there exist maps h_1: $X \to \bar{Z}$ and h_2: $\bar{Z} \to Z_f$ such that

$$h = h_2 h_1, \quad \dim \bar{Z} \leqslant \dim X, \quad \operatorname{weight} \bar{Z} \leqslant \operatorname{weight} Z_f$$

and $\bar{Z}$ is metrizable. Put $\bar{h} = h_1$, $\bar{M} = \mathrm{cl}_{\bar{Z}}\bar{h}(M)$ and $\bar{g} = g(h_2|M)$ ($\bar{g}$ is well defined, since $h_2M = h_2\mathrm{cl}\,h_1(M)$, $\mathrm{cl}\,h_2h_1(M) = \mathrm{cl}\,h(M) \subset M_f$).

Now, we can obtain (see [11]) the following theorems of this kind:

If a complete metric space Y, weight$Y \leqslant \tau$, *is an AE-space for the class of metric spaces of weight less than or equal to τ and of dimension less than or equal to n, then Y is an AE-space for the class of τ-collectionwise normal spaces of dimension less than or equal to n.*

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
KATOWICE

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