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ASSOCIATIVE AND IDEMPOTENT ALGEBRAS ARE AT MOST TERNARY

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0. Let $\mathfrak{A} = (X;\ \boldsymbol{F})$ be an algebra. We denote by $\boldsymbol{A}^{(n)}(\boldsymbol{F})$ the set of all n-ary algebraic operations in $\mathfrak{A}$, and by $\boldsymbol{A}(\boldsymbol{F})$ the set of all algebraic operations in $\mathfrak{A}$ (see [1]). For two algebras $\mathfrak{A}$ and $\mathfrak{B}$ we write $\mathfrak{A} = \mathfrak{B}$ if $\mathfrak{A} = (X;\ \boldsymbol{F})$, $\mathfrak{B} = (X;\ \boldsymbol{G})$ and $\boldsymbol{A}(\boldsymbol{F}) = \boldsymbol{A}(\boldsymbol{G})$. An algebraic operation $f(x_1, \ldots, x_n)$ is called *idempotent* if $f(x, \ldots, x) = x$ for any $x \in X$. If any $f \in \boldsymbol{F}$ is idempotent, we say that the algebra $\mathfrak{A}$ is *idempotent*. An algebraic operation $f(x_1, \ldots, x_n)$ is called to be *associative* if for any $x_1, \ldots, x_{2n-1} \in X$ it satisfies

$$f\big(f(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}\big) = f\big(x_1, f(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}\big)$$
$$= \ldots = f\big(x_1, \ldots, x_{n-1}, f(x_n, \ldots, x_{2n-1})\big).$$

We assume that any nullary and unary operation is associative. We say that an algebra $\mathfrak{A} = (A;\ \boldsymbol{F})$ is *associative* if there exists a set $\boldsymbol{G}$ of associative operations such that $\mathfrak{A} = (A;\ \boldsymbol{G})$. In the sequel by saying that $\mathfrak{A} = (X;\ \boldsymbol{F})$ is associative we shall mean that all operations belonging to $\boldsymbol{F}$ are associative. We denote by $\mathfrak{I}(\mathfrak{A})$ the algebra $\big(X;\ \boldsymbol{I}(\boldsymbol{F})\big)$, where $\boldsymbol{I}(\boldsymbol{F})$ is the set of all idempotent algebraic operations in $\mathfrak{A}$. The algebra $\mathfrak{I}(\mathfrak{A})$ is called the *full idempotent reduct* of $\mathfrak{A}$ or, briefly, the *idempotent reduct* of $\mathfrak{A}$.

For any algebra $\mathfrak{A} = (X;\ \boldsymbol{F})$ E. Marczewski has defined a number $\varrho(\mathfrak{A})$, called the *arity* of $\mathfrak{A}$, as follows:

$\varrho(\mathfrak{A}) = \min\big\{n\colon \mathfrak{A} = \big(X;\ \boldsymbol{A}^{(n)}(\boldsymbol{F})\big)\big\}$; if this minimum does not exist, we accept $\varrho(\mathfrak{A}) = -1$.

In this paper we prove (Theorem 1) that if $\mathfrak{A} = (X;\ \boldsymbol{F})$ is an idempotent, associative algebra and $\boldsymbol{F} \neq \emptyset$, then $\mathfrak{A} = \big(X;\ \{t(f)\}_{f\in\boldsymbol{F}}\big)$, where each $t(f)$ belongs to $\boldsymbol{A}(\boldsymbol{F})$ and is at most ternary. From this it follows that $\varrho(\mathfrak{A}) \leqslant 3$.

Further we prove (Theorem 2) that if $\mathfrak{A} = \big(X;\ f(x_1, \ldots, x_n)\big)$ is an associative algebra, $f(x, \ldots, x) \neq x$ and there exist idempotent algebraic operations in $\mathfrak{A}$ different from projections, then

$$\mathfrak{I}(\mathfrak{A}) = \big(X;\ t(x_1, x_2, x_3)\big) \quad \text{for some } t \in I(f).$$

1. LEMMA 1. *If an operation $p(x_1, \ldots, x_n)$ for $n \geqslant 3$ is idempotent and associative, then it is an iteration of the operation*

$$t(x_1, x_2, x_3) = p(x_1, x_2, x_3, \ldots, x_3).$$

Proof. Write

$$t_2(x_1, x_2, x_3) = t(x_1, x_2, x_3)$$

and, for $2 < k < n-1$,

$$t_{k+1}(x_1, x_2, \ldots, x_{k+2}) = t_k\big(x_1, \ldots, x_{k-1}, t(x_k, x_{k+1}, x_{k+2}), x_{k+2}\big).$$

Now it is enough to show that

$$t_k(x_1, \ldots, x_{k+1}) = p(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{k+1}), \tag{1}$$

and then to take

$$t_{n-1}(x_1, \ldots, x_n) = p(x_1, \ldots, x_n).$$

By assumption, (1) holds for $k = 2$. Assume that (1) holds for a fixed k, $2 \leqslant k < n-1$. Then

$$\begin{aligned}
&t_{k+1}(x_1, \ldots, x_{k+2}) \\
&\quad = t_k\big(x_1, \ldots, x_{k-1}, t(x_k, x_{k+1}, x_{k+2}), x_{k+2}\big) \\
&\quad = p\big(x_1, \ldots, x_{k-1}, p(x_k, x_{k+1}, x_{k+2}, \ldots, x_{k+2}), x_{k+2}, \ldots, x_{k+2}\big) \\
&\quad = p\big(x_1, \ldots, x_{k+1}, p(x_{k+2}, \ldots, x_{k+2}), x_{k+2}, \ldots, x_{k+2}\big) \\
&\quad = p(x_1, \ldots, x_{k+1}, x_{k+2}, \ldots, x_{k+2}),
\end{aligned}$$

which completes the proof.

From Lemma 1 it follows

THEOREM 1. *If $\mathfrak{A} = (X;\ F)$ is an idempotent, associative algebra, then $\mathfrak{A} = \big(X;\ \{t(f)\}_{f \in F}\big)$, where $t(f) = f$ if f is less than ternary and $t(f)(x_1, x_2, x_3) = f(x_1, x_2, x_3, \ldots, x_3)$ otherwise.*

Remark. The assumption of associativity in Lemma 1 and Theorem 1 is essential. In fact, take an algebra

$$\mathfrak{B} = \big(\{a_1, \ldots, a_n\}, f(x_1, \ldots, x_n)\big) \qquad (n > 3),$$

and put $f(a_{i_1}, \ldots, a_{i_n}) = a_{i_1}$ if all elements $a_{i_1}, \ldots, a_{i_n}$ are different and $f(a_{i_1}, \ldots, a_{i_n}) = a_{i_n}$ otherwise (see [3]). Then any algebraic operation having less than n variables is a projection.

2. Let $\mathfrak{A} = (X; f(x_1, \ldots, x_n))$ for $n \geqslant 2$ be an algebra with the unique associative fundamental operation. Write

$$f_1(x_1, \ldots, x_{1+1(n-1)}) = f(x_1, \ldots, x_n),$$

$$f_{k+1}(x_1, \ldots, x_{1+(k+1)(n-1)}) = f\big(f_k(x_1, \ldots, x_{1+k(n-1)}), x_{2+k(n-1)}, \ldots, x_{1+(k+1)(n-1)}\big) \quad (k \geqslant 1).$$

In view of the associativity of the operation f we have

(i) Any algebraic operation in $\mathfrak{A}$ is a projection or is equal to some operation of the form $f_k(x_{i_1}, \ldots, x_{i_{1+k(n-1)}})$, where $k \geqslant 1$.

In particular, any idempotent algebraic operation not being a projection is of the form f_k in some variables.

If idempotent algebraic operations not being projections exist in $\mathfrak{A}$, denote by k_0 the smallest integer for which $f_{k_0}(x_1, \ldots, x_{1+k_0(n-1)})$ is idempotent.

LEMMA 2. *Any idempotent algebraic operation in $\mathfrak{A}$ not being a projection is generated by the operation $f_{k_0}(x_1, \ldots, x_{1+k_0(n-1)})$.*

Proof. Observe first that any of the operations f_{sk_0} for $s = 2, 3, 4, \ldots$ is an iteration of f_{k_0}, since

$$f_{2k_0}(x_1, \ldots, x_{1+2k_0(n-1)}) = f_{k_0}\big(f_{k_0}(x_1, \ldots, x_{1+k_0(n-1)}), x_{2+k_0(n-1)}, \ldots, x_{1+2k_0(n-1)}\big),$$

and so on. Thus any f_{sk_0} is idempotent. Further, we can easily see that an operation $f_k(x_{i_1}, \ldots, x_{i_{1+k(n-1)}})$ (perhaps with repetitions of variables) is idempotent iff so is the operation $f_k(x_1, \ldots, x_{1+k(n-1)})$; moreover, the second one generates the first one by identifying or changing variables. So, to prove our lemma, it is enough, by (i), to show that if an operation $f_m(x_1, \ldots, x_{1+m(n-1)})$ is idempotent, then $m = sk_0$ for some natural s. Assume, to the contrary, that $m = s_0k_0 + r$, where $s_0 = \max\{s: sk_0 < m\}$, and $0 < r < k_0$. Then we have

$$f_r(x, \ldots, x) = f_r\big(f_{s_0k_0}(x, \ldots, x), x, \ldots, x\big) = f_m(x, \ldots, x) = x,$$

since $f_{s_0k_0}$ and f_m are idempotent by assumption. Thus f_r is idempotent which contradicts the definition of the number k_0.

THEOREM 2. *For any algebra $\mathfrak{A} = (X; f(x_1, \ldots, x_n))$ with the unique associative fundamental operation, one of the four possibilities holds:*

(a) $\mathfrak{I}(\mathfrak{A}) = (X; \emptyset)$;

(b) $n = 2$ *and* $\mathfrak{A}$ *is an idempotent semigroup*;

(c) $n \geqslant 3$, f *is idempotent, and* $\mathfrak{A} = (X; f(x_1, x_2, x_3, \ldots, x_3))$;

(d) $n \geqslant 2$, f *is not idempotent, in* $\mathfrak{A}$ *there exist idempotent algebraic operations not being projections, and*

$$\mathfrak{I}(\mathfrak{A}) = (X; f_{k_0}(x_1, x_2, x_3, \ldots, x_3)).$$

Proof. If the projections are the only idempotent algebraic operations in $\mathfrak{A}$, then we have (a). If $n = 2$ and f is idempotent, then we have (b). If $n \geqslant 3$ and f is idempotent, then we have (c) by Lemma 1. If f is not idempotent and in $\mathfrak{A}$ there exist idempotent algebraic operations different from projections, then $n \geqslant 2$. So, by Lemma 2, $f_{k_0}(x_1, \ldots, x_{1+k_0(n-1)})$ generates all idempotent operations in $\mathfrak{A}$. It is easy to observe that $f_{k_0}(x_1, \ldots, x_{1+k_0(n-1)})$ is associative, since f is associative and all variables in f_{k_0} are different. Thus, by Lemma 1, we have (d).

COROLLARY 1. *For any semigroup* $\mathfrak{S} = (X; x\cdot y)$ *we have three possibilities*:

(e) $x\cdot x = x$ *for all* $x \in X$;

(f) $\mathfrak{S}$ *is not idempotent and* $\mathfrak{J}(\mathfrak{S}) = (X; \emptyset)$;

(g) $\mathfrak{S}$ *is not idempotent and* $\mathfrak{J}(\mathfrak{S}) = (X;\ x_1\cdot x_2\cdot x_3^{k_0-1})$, *where* $k_0 = \min\{k\colon x^{k+1} = x\}$.

COROLLARY 2. *For any group* $\mathfrak{G} = (G; x\cdot y)$ *with the exponent* m *we have* $\mathfrak{J}(\mathfrak{G}) = (G;\ x_1\cdot x_2\cdot x_3^{m-1})$ (cf. [2]).

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