ANNALES
POLONICI MATHEMATICI
XLVIII (1988)

# Siciak's extremal function of convex sets in $C^N$

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**Abstract.** In this paper we give formulas for Siciak's extremal function for a class of convex sets in $C^N$ that contains, in particular, closed balls in $C^N$ and $R^N$, as well as standard simplexes in $R^N$.

**Introduction.** Let $E$ be a compact set in $C^N$. The Siciak extremal function of $E$ is defined by

$$\Phi(z, E) = \sup\{|p(z)|^{1/\deg p}:\ p \in \mathscr{P}(C^N),\ \|p\|_E \leqslant 1,\ \deg p \geqslant 1\},$$

for $z \in C^N$, where $\mathscr{P}(C^N)$ denotes the space of all polynomials of $N$ complex variables and $\|p\|_E = \sup|p|(E)$ (see [7]). It is known ([9], see also [8]) that

$$\Phi(z, E) = \sup\{u(z):\ u \in \exp L,\ u|_E \leqslant 1\},$$

where $L$ is the set of all plurisubharmonic functions $f$ in $C^N$ with $\sup\{f(z) - \log(1+\|z\|)\} < +\infty$ ($\|z\| = (|z|^2 + \ldots + |z_N|^2)^{1/2}$). If $E$ is a compact set in $C$ with positive logarithmic capacity, then $\log\Phi(z, E)$ is equal to the generalized Green function of the unbounded component of $C\setminus E$ with pole at infinity.

The extremal function has many applications in the polynomial approximation theory and in other sections of complex analysis of one or several variables. Therefore, an important (and in general difficult) problem is that of giving an explicit formula for the extremal function. Recently Lundin [4] found a representation for the extremal function of a convex symmetric set in $R^N$. In particular, he gave an explicit formula for the extremal function of the unit ball in $R^N$. In this paper we give a similar representation for a class of convex symmetric sets in $C^N$. This class contains in particular convex symmetric subsets of $R^N$. In our considerations we use the main idea of Lundin's proof from [4], but we work out this idea in a different way. This permits us to give effective formulas for the extremal function of convex symmetric polyhedra in $R^N$.

I wish to thank W. Pleśniak for his encouragement and help in writing this paper.

**1. Basic lemma.** In our considerations the crucial role is played by the following lemma.

**1.1.** LEMMA. *Let $E$ be a compact set in $C^N$ and let $z_0 \in C^N \setminus E$. Assume that there exist a domain $D \subset C$ and a continuous mapping $f\colon \bar{D} \to C^N$, holomorphic in $D$, which satisfies the following conditions*:

($\alpha$) $z_0 \in f(D)$,

($\beta$) $f(\partial D) \subset E$.

(Here $\bar{D}$ and $\partial D$ denote the closure and the boundary of $D$ in $C$, respectively.) *Let a continuous function $w\colon f(\bar{D}) \to [1, +\infty)$ satisfy*:

(i) $w(z) = 1$ *for* $z \in f(\partial D)$,

(ii) $\log(w \circ f)$ *is harmonic function in* $D$,

(iii) $\|f(\zeta)\| \leqslant Mw(f(\zeta))$, $\zeta \in \bar{D}$, *where $M$ is a constant.*

*Then for every $z \in f(D)$, $\Phi(z, E) \leqslant w(z)$. In particular, $\Phi(z_0, E) \leqslant w(z_0)$.*

Proof. Fix $u \in L$, $u|_E \leqslant 0$. Consider the function $v\colon \bar{D} \to [-\infty, +\infty)$, $v(\zeta) := u(f(\zeta)) - \log w(f(\zeta))$. This is a subharmonic function on $D$ and it is bounded above in $D$. If $\zeta_0 \in \partial D$ then, $u$ being upper-semicontinuous, we have

$$\limsup_{\zeta \to \zeta_0, \zeta \in D} v(\zeta) = \limsup_{\zeta \to \zeta_0, \zeta \in D} u(f(\zeta)) \leqslant \limsup_{z \to f(\zeta_0)} u(z) \leqslant u(f(\zeta_0)) \leqslant 0.$$

Using the Ascoli maximum principle for subharmonic functions (see [2]) we see that $v \leqslant 0$ on $D$. Therefore $u(z) \leqslant \log w(z)$ for $z \in f(D)$, whence $\Phi(z, E) \leqslant w(z)$ for $z \in f(D)$.

**2. Some properties of a function of Zhukovskiĭ type.** Let $\alpha, \beta \in C$ be fixed, $\alpha \neq 0$, $|\beta| \leqslant |\alpha|$. Let $g(\zeta, \alpha, \beta) = \alpha\zeta + \beta\zeta^{-1}$, $|\zeta| \geqslant 1$, $E_0 = E_0(\alpha, \beta) = C \setminus g(C \setminus \bar{B})$, where $\bar{B} = \{|\zeta| \leqslant 1\}$. The function $g$ is holomorphic and univalent on $|\zeta| > 1$ and $E_0$ is a symmetric set limited by an ellipse (if $|\beta| < |\alpha|$) or it is a line segment (if $|\beta| = |\alpha|$). Let $h(\zeta) = h(\zeta, \alpha, \beta) := g^{-1}(\zeta)$: $C \setminus E_0 \to C \setminus \bar{B}$, where $g(\zeta) = g(\zeta, \alpha, \beta)$. The function $h$ has the form $h(\zeta) = (\zeta + (\zeta^2 - 4\alpha\beta)^{1/2})/2\alpha$ if we choose an appropriate branch of square root; $|h|$ extends to a continuous function on $C$, which is equal to 1 on $E_0$. We denote the extended function also by $|h|$. It is easy to verify the following properties of the functions $g$ and $h$.

**2.1.** PROPOSITION. (1) $\bar{\alpha}\bar{\zeta}g(\zeta) - \beta\zeta^{-1}\overline{g(\zeta)} = |\alpha|^2|\zeta|^2 - |\beta|^2|\zeta|^{-2}$, $|\zeta| \geqslant 1$;

(2) $|h(\zeta)| = r > 1$ *if and only if $\zeta$ is a point of the ellipse*

$$\begin{aligned} e_r &= \{|\zeta|^2 + |\zeta^2 - 4\alpha\beta| = 2g(r^2, |\alpha|^2, |\beta|^2)\} \\ &= \{|\zeta - 2(\alpha\beta)^{1/2}| + |\zeta + 2(\alpha\beta)^{1/2}| = 2g(r, |\alpha|, |\beta|)\}; \end{aligned}$$

(3) $\zeta \in E_0$ *if and only if* $|\zeta|^2 + |\zeta^2 - 4\alpha\beta| \leqslant 2(|\alpha|^2 + |\beta|)$;

(4) $|h(\zeta)| \leqslant r$, $r > 1$, *if and only if* $\zeta \in \operatorname{conv} e_r = \{|r\bar{\alpha}\zeta - r^{-1}\beta\bar{\zeta}| \leqslant |\alpha|^2 r^2 - |\beta|^2 r^{-2}\}$;

(5) $\Phi(\zeta, E_0) = |h(\zeta)|$, $\zeta \in C$.

**3. The extremal function for convex symmetric sets.** For a fixed set $E_0 = E_0(\alpha, \beta)$ and a compact set $K \subset C^N$ ($K \subset R^N$, if $\beta \neq 0$) which contains $N$ linearly independent points, we define a set $E = E(\alpha, \beta, K)$ as follows:

$$E = \{z \in C^N\colon \ \langle z, y\rangle \in E_0 \text{ for every } y \in K\},$$

where $\langle \cdot \rangle$ denotes the scalar product in $C^N$. Then $E$ is a compact, convex and symmetric subset of $C^N$. The main result of this paper is the following:

3.1. THEOREM. (a) $\Phi(z, E) = \max\{\Phi(\langle z, y\rangle, E_0)\colon \ y \in K\} = \max\{|h(\langle z, y\rangle, \alpha, \beta)|\colon \ y \in K\}$, $z \in C^N$.

(b) *If* $z_0 \in C^N \setminus E$ *and* $\Phi(z_0, E) = |h(\langle z_0, y_0\rangle)|$, *then*

$$\log \Phi(\alpha\zeta c + \beta\zeta^{-1}\bar{c}, E) = \log|\zeta|, \quad |\zeta| \geqslant 1,$$

*where* $c \in C^N$ *is the vector given by the condition*

$$\alpha\zeta_0 c + \beta\zeta_0^{-1}\bar{c} = z_0 \quad \textit{with} \quad \zeta_0 = h(\langle z_0, y_0\rangle). \tag{6}$$

(c) $E_R := \{z\colon \ \Phi(z, E) \leqslant R\} = E(\alpha R, \beta/R, K)$, $R > 1$.

Proof. If $y \in K$ is fixed, then $u(z) = \log|h(\langle z, y\rangle)| \in L$ and $u|_E = 0$. Hence

$$\max\{|h(\langle z, y\rangle)|\colon \ y \in K\} \leqslant \Phi(z, E).$$

In order to prove the opposite inequality and assertion (b) of the theorem, let us fix a point $z_0 \in C^n \setminus E$. Then

$$\max|h(\langle z_0, y\rangle)| = |h(\langle z_0, y_0\rangle)| > 1.$$

Define $f(\zeta) := \alpha\zeta c + \beta\zeta^{-1}\bar{c}$, $|\zeta| \geqslant 1$, where $c$ is given by (6). Then

$$c = (|\alpha|^2 r^2 - |\beta|^2 r^{-2})^{-1}(\bar{\alpha}\bar{\zeta}_0 z_0 - \beta\zeta_0^{-1}\bar{z}_0), \quad \text{where } r = |\zeta_0| = |h(\langle z_0, y_0\rangle)|.$$

We have $f(\zeta_0) = z_0$ and it is obvious that $f$ satisfies assumption ($\alpha$) of Lemma 1.1 for $D = C\setminus\bar{B}$. Next, observe that $f(\partial D) \subset E$ is equivalent to

$$|\langle f(e^{i\theta}), y\rangle|^2 + |\langle f(e^{i\theta}), y\rangle^2 - 4\alpha\beta| \leqslant 2(|\alpha|^2 + |\beta|^2) \quad \text{for every } y \in K,$$

$\theta \in R$. The last inequality is equivalent to

$$|\langle c, y\rangle| = |\bar{\alpha}r\langle z_0, y\rangle - \beta r^{-1}\overline{\langle z_0, y\rangle}|(|\alpha|^2 r^2 - |\beta|^2 r^{-2})^{-1} \leqslant 1 \tag{7}$$

for every $y \in K$, which is satisfied in view of (4). Thus, condition ($\beta$) of Lemma 1.1 is also fulfilled. We have $\langle z_0, y_0\rangle = g(h(\langle z_0, y_0\rangle))$ and, by (1), $\langle c, y_0\rangle = 1$. Therefore $\langle f(\zeta), y_0\rangle = g(\zeta)$. Put

$$w(z) := |h(\langle z, y_0\rangle)| \quad \text{for } z \in f(\bar{D}).$$

Then

$$w(f(\zeta)) = |h(\langle f(\zeta), y_0\rangle)| = |\zeta|, \quad \zeta \in \bar{D},$$

and the function $w$ satisfies conditions (i) and (ii). Moreover,

$$\|f(\zeta)\| \leqslant 2|\alpha| \, \|c\| \, |\zeta|, \quad \zeta \in \bar{D},$$

whence we get (iii). Now, applying Lemma 1.1, we end the proof of (a) and (b). Assertion (c) is obtained from (a) by use of Proposition 2.1.

**4. Remarks and applications.** We now give some corollaries to Theorem 3.1. Let $E$ be a compact, convex and symmetric subset of $\boldsymbol{R}^N$ with $\operatorname{Int} E \neq \emptyset$ (in $\boldsymbol{R}^N$). (We regard here $\boldsymbol{R}^N$ as a subset of $\boldsymbol{C}^N$ such that $\boldsymbol{C}^N = \boldsymbol{R}^N + i\boldsymbol{R}^N$.) Then

$$E = \{z \in \boldsymbol{C}^N\colon\ a(y)\,\langle z, y\rangle \in [-1, 1] \text{ for every } y \in K\},$$

where $K$ is a compact subset of the unit sphere $S_{N-1}$ in $\boldsymbol{R}^N$ and $a(y) := 1/\max\{\langle x, y\rangle\colon\ x \in E\}$ is a continuous function on $K$. If we put, in Theorem 3.1, $\alpha = \beta = \frac{1}{2}$, $E_0 = [-1, 1]$ and $K_1 = \{a(y)\,y\colon\ y \in K\}$ instead of $K$, then we get

4.1. Corollary.

$$\begin{aligned}\Phi(z, E) = \max\{\Phi(a(y)\,\langle z, y\rangle, [-1, 1])\colon\ y \in K\} \\ = \max\{|h(a(y)\,\langle z, y\rangle)|\colon\ y \in K\}\end{aligned}$$

*with* $h(\zeta) = \zeta + (\zeta^2 - 1)^{1/2}$.

4.2. Remark. If $K = S_{N-1}$, then Corollary 4.1 reduces to Lundin's result [4]. Another proof of Lundin's theorem has been given by Bedford and Taylor [1].

Let $E$ be a compact, convex and symmetric polyhedron in $\boldsymbol{R}^N$ with $\operatorname{Int} E \neq \emptyset$. Then $E$ can be written in the form

$$E = \bigcap_{k=1}^{n} \{x \in \boldsymbol{R}^N\colon\ -\alpha^{(k)} \leqslant x_1 \beta_1^{(k)} + \ldots + x_N \beta_N^{(k)} \leqslant \alpha^{(k)}\},$$

where $\alpha^{(k)} > 0$ and $\operatorname{lin}(\beta^{(1)}, \ldots, \beta^{(n)}) = \boldsymbol{R}^N$. From Corollary 4.1 (with $K$ finite) we get

4.3. Corollary.

$$\Phi(z, E) = \max_{k=1,\ldots,n} |h(\langle z, \beta^{(k)}\rangle/\alpha^{(k)})|,$$

*where* $h(\zeta) = \zeta + (\zeta^2 - 1)^{1/2}$.

Let us also notice that Corollary 4.1 can be derived from Corollary 4.3 (it suffices to approximate $E$ by symmetric polyhedra $E_n$ and make use of the limit procedure: $\Phi(z, E_n) \nearrow \Phi(z, E)$ if $E_n \supset E_{n+1}$ and $E = \bigcap_{n=1}^{\infty} E_n$, see [8]).

4.4. Lundin's example. Let $\bar{B}$ be the closed unit ball in $\boldsymbol{R}^N$: $\bar{B} = \{x_1^2 + \ldots + x_N^2 \leqslant 1\}$. In this case we have $K = S_{N-1}$ and $a(y) \equiv 1$. From Corollary 4.1 we get

$$\Phi(z, \bar{B}) = \max\{|h(\langle z, y\rangle)|:\ y \in S_{N-1}\}. \tag{8}$$

(Another proof of (8) is given in [6].)

Let $z_0 \in \boldsymbol{C}^N \setminus \bar{B}$, $r = \Phi(z_0, \bar{B}) = |h(\langle z_0, y_0\rangle)|$ and let $c = a+ib$, where $a, b \in \boldsymbol{R}^N$, be given by (6). Then $z_0 = \frac{1}{2}(\zeta_0+\zeta_0^{-1})a + i\frac{1}{2}(\zeta_0-\zeta_0^{-1})b$. We have $\langle a, y_0\rangle = 1$, $\langle b, y_0\rangle = 0$ and from (7) we get $\|a\| \leqslant 1$, $\|b\| \leqslant 1$. Hence we have $a = y_0 \in S_{N-1}$. Now, we can write

$$\begin{aligned}
\|z_0\|^2 &= |\tfrac{1}{2}(\zeta_0+\zeta_0^{-1})|^2 + |\tfrac{1}{2}(\zeta_0-\zeta_0^{-1})|^2\|b\|^2,\\
\langle z_0, \bar{z}_0\rangle - 1 &= (1-\|b\|^2)\big(\tfrac{1}{2}(\zeta_0-\zeta_0^{-1})\big)^2,\\
\|z_0\|^2 + |\langle z_0, \bar{z}_0\rangle - 1| &= \tfrac{1}{2}(r^2+r^{-2}) = g(r^2).
\end{aligned}$$

Thus we get Lundin's formula [5]:

$$\Phi(z_0, \bar{B}) = \big(h(\|z_0\|^2 + |\langle z_0, \bar{z}_0\rangle - 1|)\big)^{1/2}.$$

In some cases we are able to give a representation formula for the extremal function of a set $E$, which is non-symmetric, by using Corollary 4.1 and the following result of Klimek [3].

4.5. Theorem ([3]). *Let $f = (f_1, \ldots, f_N)\colon \boldsymbol{C}^N \to \boldsymbol{C}^N$ be a polynomial mapping with $\deg f_1 = \ldots = \deg f_N = d \geqslant 1$, such that $\hat{f}^{-1}(0) = (\hat{f}_1, \ldots, \hat{f}_N)^{-1}(0) = \{0\}$, where $\hat{f}_k$ denotes the homogeneous part of $f_k$ of degree $d$. If $E$ is a compact set in $\boldsymbol{C}^N$, then*

$$\Phi(z, f^{-1}(E)) = \big(\Phi(f(z), E)\big)^{1/d}.$$

Let $E$ be a compact, convex polyhedron in $\boldsymbol{R}^N$ with $\operatorname{Int} E \neq \emptyset$. Suppose that $0 \in E$ and that $E$ has the representation

$$E = \{z \in \boldsymbol{C}^N:\ 2\langle z, y^{(k)}\rangle + b_k \in [-1, 1],\ k = 1, \ldots, n\},$$

where $n > N$, $b_1, \ldots, b_n \in \boldsymbol{R}$, $y^{(1)}, \ldots, y^{(n)} \in \boldsymbol{R}^N$, $\operatorname{lin}(y^{(1)}, \ldots, y^{(N)}) = \boldsymbol{R}^N$. Assume that there exists $a \in E$ such that $2\langle a, y^{(k)}\rangle + b_k = -1$ for $k = 1, \ldots, N$ and suppose that $y^{(k)} = \alpha_1^{(k)} y^{(1)} + \ldots + \alpha_N^{(k)} y^{(N)}$ with $\alpha_1^{(k)}, \ldots, \alpha_N^{(k)} \geqslant 0$ for $k = N+1, \ldots, n$.

Let us choose $w^{(1)}, \ldots, w^{(N)} \in \boldsymbol{R}^N$ so that $\langle w^{(k)}, y^{(l)}\rangle = \delta_{kl}$, $k, l = 1, \ldots, N$. Under the above assumptions we define

$$f(z) = z_1^2 w^{(1)} + \ldots + z_N^2 w^{(N)} + a.$$

We have $\hat{f}^{-1}(0) = \{0\}$ and

$$F := f^{-1}(E) = I_N \cap \bigcap_{k=N+1}^{n} \{\alpha_1^{(k)} x_1^2 + \ldots + \alpha_N^{(k)} x_N^2 \leqslant \tfrac{1}{2}(1-\tilde{b}_k)\}$$

where $I_N = [-1, 1]^N$ and $\tilde{b}_k = b_k + 2\langle a, y^{(k)}\rangle$. Then $F$ is a compact, convex and symmetric subset of $\boldsymbol{R}^N$ with $\operatorname{Int} F \neq \emptyset$ and we can apply Corollary 4.1

to $F$. By Theorem 4.5 we get

$$\Phi(z, E) = \big(\Phi\big((\langle z-a, y^{(1)}\rangle)^{1/2}, \ldots, (\langle z-a, y^{(N)}\rangle)^{1/2}, F\big)\big)^2.$$

**4.6.** EXAMPLE. Let $E$ be the standard simplex in $\boldsymbol{R}^N$, i.e., $E = \operatorname{conv}(0, e_1, \ldots, e_N)$, where $\{e_1, \ldots, e_N\}$ is the standard orthonormal basis in $\boldsymbol{R}^N$. We can write $E$ in the form

$$E = \{z \in \boldsymbol{C}^N\colon\ 2z_1 - 1, \ldots, 2z_N - 1 \in [-1, 1] \text{ and } 2(z_1 + \ldots + z_N) - 1 \in [-1, 1]\}.$$

Let $f(z) = (z_1^2, \ldots, z_N^2)$. Then $f^{-1}(0) = \{0\}$ and $f^{-1}(E) = \bar{B}$, where $\bar{B}$ is the unit ball in $\boldsymbol{R}^N$. We have $\Phi(z, \bar{B}) = (\Phi(z_1^2, \ldots, z_N^2, E))^{1/2}$ and using Lundin's formula we get

$$\Phi(z, E) = h(|z_1| + \ldots + |z_N| + |z_1 + \ldots + z_N - 1|).$$

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*Reçu par la Rédaction le 1985.12.31*