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Newton polygons and the Łojasiewicz exponent of a holomorphic mapping of C^2

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Abstract. Let $(f, g)\colon (C^2, 0) \to (C^2, 0)$ be a germ of a holomorphic mapping. We give an estimate of the Łojasiewicz exponent

$$l_0(f, g) = \inf\{\Theta > 0\colon \max\{|f(z)|, |g(z)|\} \geqslant C|z|^\Theta \text{ for } z \in C^2 \text{ near } 0\}$$

in terms of the Newton polygons of f and g.

1. Estimation of the Łojasiewicz exponent. For any convergent power series $f(X, Y) = \sum C_{p,q} X^p Y^q \in C\{X, Y\}$, we use ord$f$, resp. in$f$, to denote the order, resp. the initial form of f. We call f *convenient* if $f(0, 0) = 0$ and $f(X, 0) f(0, Y) \neq 0$ in $C\{X, Y\}$. For any convenient f, we denote by $\mathfrak{N}_f$ the set of all segments of the Newton polygon (cf. [1] and [10] for the detailed description of the Newton polygon). If $S \in \mathfrak{N}_f$ then we let in(f, S) = the sum of all monomials $C_{p,q} X^p Y^q$ such that $(p, q) \in S$. For any segment S in the plane (p, q), we denote by S_1, S_2 the projections of S on the axes, and by $|S_1|$, $|S_2|$ their lengths. Moreover, we put $\|S\| = \min\{|S_1|, |S_2|\}$ and $[S, T] = \min\{|S_1|\,|T_2|, |S_2|\,|T_1|\}$ for two segments S, T. Any pair f, g of power series without constant term induces a germ of a holomorphic mapping $(f, g)\colon C^2 \to C^2$ which we denote briefly by (f, g).

DEFINITION 1.1. The germ (f, g) determined by convenient power series f, g is *non-degenerate* if for $S \in \mathfrak{N}_f$ and $T \in \mathfrak{N}_g$ one has the following:

(a) either S and T are not parallel, i.e., $|S_1|\,|T_2| \neq |S_2|\,|T_1|$, or

(b) the segments S and T are parallel and the system of equations in$(f, S)(X, Y) = 0$, in$(g, T)(X, Y) = 0$ has no solutions in $(C\setminus\{0\}) \times (C\setminus\{0\})$.

One can check that the nondegeneracy condition is generic in the sense of Kouchnirenko (cf. [4]). For any germ $(f, g)\colon C^2 \to C^2$, we define the multiplicity: $m_0(f, g)$ = C-codimension of the ideal generated by f, g in $C\{X, Y\}$, and the Łojasiewicz exponent: $l_0(f, g)$ = the greatest lower bound of the set of all $\Theta > 0$ which satisfy the condition: there exist positive constants C, R such that $\max\{|f(x, y)|, |g(x, y)|\} \geqslant C(\max\{|x|, |y|\})^{\Theta}$ for all $(x, y) \in \mathbf{C}^2$ such that $\max\{|x|, |y|\} < R$ (cf. [2], [6], [8]). If the germ (f, g) has an isolated zero at $0 \in \mathbf{C}^2$, then both $m_0(f, g)$ and $l_0(f, g)$ are finite.

Let us recall the following well-known result.

THEOREM 1.2. *Suppose that $f, g \in \mathbf{C}\{X, Y\}$ are convenient. Then* $m_0(f, g) \geqslant \sum_{S \in \mathfrak{N}_f} \sum_{T \in \mathfrak{N}_g} [S, T]$, *the equality holding if the germ (f, g) is non-degenerate.*

For the sake of completeness we give the proof of Theorem 1.2 in Section 2.

The main result of this note is

THEOREM 1.3. *Suppose that $f, g \in \mathbf{C}\{X, Y\}$ are convenient. Then*

$$l_0(f, g) \geqslant \max\left\{\max_{S \in \mathfrak{N}_f}\left\{\frac{1}{\|S\|}\sum_{T \in \mathfrak{N}_g}[S, T]\right\}, \max_{T \in \mathfrak{N}_g}\left\{\frac{1}{\|T\|}\sum_{S \in \mathfrak{N}_f}[S, T]\right\}\right\},$$

equality holding if the germ (f, g) is non-degenerate.

The proof of Theorem 1.3 will be given in Section 3. Note here that Lichtin in [7] gave an estimation of $l_0(\partial h/\partial X, \partial h/\partial Y)$ in terms of the Newton polygon of the series $h \in \mathbf{C}\{X, Y\}$ satisfying a non-degeneracy condition. Theorem 1.3 and Lichtin's result are independent. If $\mathfrak{N}_f = \mathfrak{N}_g = \mathfrak{N}$ then the right-hand side of the inequality in Theorem 1.2 equals the double area between the polygon and the two axes. In this case, Theorem 1.3 has also a simple geometrical meaning.

LEMMA 1.4. *Suppose that $\mathfrak{N}_f = \mathfrak{N}_g = \mathfrak{N}$ and let $(m, 0)$ and $(0, n)$ be the points of $\mathfrak{N}$ which lie on the axes. Then the right-hand side of inequality* (1.3) *is tequal to* $\max(m, n)$.

Proof. We have to check that $\max_{S \in \mathfrak{N}}\{(1/\|S\|)\sum[S, T]\} = \max(m, n)$ ($\sum$ denotes the summation over all $T \in \mathfrak{N}$). Obviously, $\sum|T_1| = m$, $\sum|T_2| = n$; hence $\sum[S, T] \leqslant |S_1|n$, $\sum[S, T] \leqslant |S_2|m$ and we get $(1/\|S\|)\ \sum[S, T] \leqslant \max(m, n)$ for any $S \in \mathfrak{N}$. Let $A, B \in \mathfrak{N}$ be such that $|A_1|/|A_2| \leqslant |S_1|/|S_2| \leqslant |B_1|/|B_2|$ for any $S \in \mathfrak{N}$. We have then $(1/\|A\|)\sum[A, T] = (|A_1|/\|A\|)n$, $(1/\|B\|)\sum[B, T] = (|B_2|/\|B\|)m$ and the lemma follows since

$$\max((|A_1|/\|A\|)n, (|B_2|/\|B\|)m) \geqslant \max(m, n).$$

2. Puiseux expansions. For any Puiseux series $Y(X) = aX^{\alpha} + a'X^{\alpha'} + \ldots \in \mathbf{C}\{X\}^* = \bigcup_{k \geqslant 1} \mathbf{C}\{X^{1/k}\}$ ($\alpha < \alpha' < \ldots$ rational numbers, $a, a', \ldots \in \mathbf{C}\setminus\{0\}$) we put $\operatorname{ord} Y(X) = \alpha$, in $Y(X) = aX^{\alpha}$. Let $f(X, Y) \in \mathbf{C}\{X, Y\}$. The solution of $f(X, Y) = 0$ in $\mathbf{C}\{X\}^*$ is a Puiseux series $Y(X) \in \mathbf{C}\{X\}^*$ such that $f(X, Y(X)) = 0$ in $\mathbf{C}\{X\}^*$. The smallest integer m such that

$\partial^m f/\partial Y^m(X, Y(X)) \neq 0$ in $C\{X\}^*$ is called the *multiplicity of the solution* $Y(X)$. *In what follows the solutions are always counted with multiplicities.* We have the following classical result.

THEOREM 2.1 (Newton–Puiseux, see [1], [10]). *Let* $f(X, Y) \in C\{X, Y\}$ *be a convenient power series.*

I. *The equation* $f(X, Y) = 0$ *has* $\sum_{S \in \mathfrak{N}_f} |S_2| = \operatorname{ord} f(0, Y)$ *solutions in* $C\{X\}^*$. *For each* $S \in \mathfrak{N}_f$ *there correspond* $|S_2|$ *solutions of order* $|S_1|/|S_2|$. *Let* $a \in C\setminus\{0\}$ *and let* α *be a rational number. Then* in $Y(X) = aX^\alpha$ *for a solution corresponding to* S *if and only if* $\operatorname{in}(f, S)(X, aX^\alpha) = 0$ *in* $C\{X\}^*$.

II. *The equation* $f(X, Y) = 0$ *has* $\sum_{S \in \mathfrak{N}_f} |S_1| = \operatorname{ord} f(X, 0)$ *solutions in* $C\{Y\}^*$. *To each* $S \in \mathfrak{N}_f$ *there correspond* $|S_1|$ *solutions of order* $|S_2|/|S_1|$. *Let* $b \in C\setminus\{0\}$ *and let* β *be a rational number. Then* $\operatorname{in} X(Y) = bY^\beta$ *for a solution* $X(Y) \in C\{Y\}^*$ *corresponding to* S *if and only if* $\operatorname{in}(f, S)(bY^\beta, Y) = 0$ *in* $C\{Y\}^*$.

The Newton polygons of the power series $f(X, Y)$ and $f(Y, X)$ are symmetrical with respect to the diagonal $p = q$, therefore, part II of the theorem follows from part I. As an application of Theorem 2.1 we get a characterization of the non-degeneracy condition given in Section 1.

LEMMA 2.2. *Let* $f, g \in C\{X, Y\}$ *be convenient power series. The following three conditions are equivalent*:

(i) *The germ* $(f, g)\colon (C^2, 0) \to (C^2, 0)$ *is non-degenerate.*

(ii) *For every solution* $Y(X) \in C\{X\}^*$ *of* $f(X, Y) = 0$ *and every solution* $\bar{Y}(X) \in C\{X\}^*$ *of* $g(X, Y) = 0$ *one has* $\operatorname{ord}(Y(X) - \bar{Y}(X)) = \min\{\operatorname{ord} Y(X), \operatorname{ord} \bar{Y}(X)\}$.

(iii) *For every solution* $X(Y) \in C\{Y\}^*$ *of* $f(X, Y) = 0$ *and every solution* $\bar{X}(Y) \in C\{Y\}^*$ *of* $g(X, Y) = 0$ *one has* $\operatorname{ord}(X(Y) - \bar{X}(Y)) = \min\{\operatorname{ord} X(Y), \operatorname{ord} \bar{X}(Y)\}$.

Proof. It is easy to see that $\operatorname{ord}(Y(X) - \bar{Y}(X)) = \min\{\operatorname{ord} Y(X), \operatorname{ord} \bar{Y}(X)\}$ if and only if $\operatorname{in} Y(X) \neq \operatorname{in} \bar{Y}(X)$. Thus the equivalence (i)$\Leftrightarrow$(ii) follows from part I of the Newton–Puiseux Theorem and the definition of the non-degeneracy condition. Analogously, we check (i)$\Leftrightarrow$(iii).

The proof of Theorem 1.2 uses Theorem 2.1 and the proposition given below.

PROPOSITION 2.3 (Zeuten's Rule). *Let* $f(X, Y), g(X, Y) \in C\{X, Y\}$ *be such that* $f(0, 0) = g(0, 0) = 0$ *and* $f(0, Y)g(0, Y) \neq 0$ *in* $C\{Y\}$. *Let* $(Y_i(X))$, *resp.* $(\bar{Y}_j(X))$, *be the sequence of all solutions in* $C\{X\}^*$ *(counted with multiplicities) of* $f(X, Y) = 0$, *resp.* $g(X, Y) = 0$. *Then* $m_0(f, g) = \sum_i \sum_j \operatorname{ord}(Y_i(X) - \bar{Y}_j(X))$.

Proof. By the Weierstrass Preparation Theorem we may assume that f, g are Y-distinguished polynomials. Let $R_{f,g}(X)$ be the Y-resultant of f, g. One can prove directly that $m_0(f, g) = \operatorname{ord} R_{f,g}(X)$ (cf. [3], p. 21). On the other hand, the expression of the resultant in terms of roots yields $\operatorname{ord} R_{f,g}(X) = \sum_i \sum_j \operatorname{ord}(Y_i(X) - \bar{Y}_j(X))$ and Proposition 2.3 follows.

Now, we can prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that $f, g \in C\{X, Y\}$ are convenient. Using Zeuten's Rule, we get $m_0(f, g) = \sum_i \sum_j \operatorname{ord}(Y_j(X) - \bar{Y}_j(X)) \geqslant \sum_i \sum_j \min\{\operatorname{ord} Y_i(X), \operatorname{ord} \bar{Y}_j(X)\}$. According to Lemma 2.2 equality holds if (and only if) the germ (f, g) is non-degenerate. To get the result it suffices to note that part I of the Newton–Puiseux Theorem gives

$$\sum_i \sum_j \min\{\operatorname{ord} Y_i(X), \operatorname{ord} \bar{Y}_j(X)\} = \sum_{S \in \mathfrak{N}_f} \sum_{T \in \mathfrak{N}_g} |S_2||T_2| \min\{|S_1|/|S_2|, |T_1|/|T_2|\}$$
$$= \sum_{S \in \mathfrak{N}_f} \sum_{T \in \mathfrak{N}_g} [S, T].$$

3. Computation of $l_0(f, g)$. For any power series $f(X, Y)$, $g(X, Y)$ without constant term $l_0(f, g, X)$ is defined to be

$$\inf\{\Theta > 0 \colon \max\{|f(x, y)|, |g(x, y)|\} \geqslant C|x|^{\Theta} \text{ for } (x, y) \text{ near } 0 \in C^2\}.$$

Analogously, we define $l_0(f, g, Y)$. Thus, we have $l_0(f, g) = \max\{l_0(f, g, X), l_0(f, g, Y)\}$. In the sequel we will need the following lemma.

LEMMA 3.1. *Let $F(Y), G(Y) \in C[Y]$ be non-constant polynomials. Then we have for each $y \in C$:*

$$\max\{|F(y), |G(y)|\} \geqslant 2^{-\max(\deg F, \deg G)} \min\{\min_{y \in G^{-1}(0)} |F(y)|, \min_{y \in F^{-1}(0)} |G(y)|\}.$$

Proof. Write $F(Y) = a \prod_{i=1}^{k} (Y - y_i)$, $G(Y) = b \prod_{j=1}^{l} (Y - \bar{y}_j)$ in $C[Y]$. Fix $y \in C$.

First case. $\min_{i=1}^{k} |y - y_i| \geqslant \min_{j=1}^{l} |y - \bar{y}_j|$, then $|y - y_i| \geqslant |y - \bar{y}_{j_0}|$ for all $i = 1, \ldots, k$ and a $j_0 \in \{1, \ldots, l\}$. Hence $2|y - y_i| \geqslant |y - y_i| + |y - \bar{y}_{j_0}| \geqslant |y_i - \bar{y}_{j_0}|$ for $i = 1, \ldots, k$ and consequently

$$2^{\deg F}|F(y)| = 2^k |a| \prod_i |y - y_i| \geqslant |a| \prod_i |y_i - \bar{y}_{j_0}| = |F(\bar{y}_{j_0})| \geqslant \min_{y \in G^{-1}(0)} |F(y)|.$$

Second case. $\min_{i=1}^{k} |y - y_i| \leqslant \min_{j=1}^{l} |y - \bar{y}_j|$, a similar calculation as above shows that $2^{\deg G}|G(y)| \geqslant \min_{y \in F^{-1}(0)} |G(y)|$.

Combining the two cases we get the lemma.

PROPOSITION 3.2 (cf. [2]). *Let* $f(X, Y)$, $g(X, Y) \in C\{X, Y\}$ *be such that* $f(0, 0) = g(0, 0) = 0$ *and* $f(0, Y)g(0, Y) \neq 0$ *in* $C\{Y\}$. *Let* $(Y_i(X))$, *resp.* $(\bar{Y}_j(X))$, *be the sequence of all solutions in* $C\{X\}^*$ *(counted with multiplicities) of* $f(X, Y) = 0$, *resp.* $g(X, Y) = 0$. *Then*

$$l_0(f, g, X) = \max\{\max_j \{\sum_i \operatorname{ord}(\bar{Y}_j(X) - Y_i(X))\}, \max_i \{\sum_j \operatorname{ord}(Y_i(X) - \bar{Y}_j(X))\}\}.$$

Proposition 3.2 is a modification of a result due to Chądzyński and Krasiński (cf. [2] and Appendix to this note). Their proof is based on the "horn neighbourhoods" method used by Kuo and Lu in [5]. Before proceeding to the proof of Proposition 3.2, let us note that for any $z(T) \in C\{T\}$ (T one variable) there are $C_1, C_2 > 0$ such that $C_1|t|^q \leqslant |z(t)| \leqslant C_2|t|^q$ with $q = \operatorname{ord} z(T)$ for $t \in C$ near $0 \in C$.

P r o o f o f P r o p o s i t i o n 3.2. Let l^* be the right-hand side of the equality stated in Proposition 3.2. Choose an integer $d \geqslant 1$ such that $Y_i(T^d)$, $\bar{Y}_j(T^d) \in C\{T\}$. Using the Weierstrass Preparation Theorem, we may assume that f, g are Y-distinguished, so $f(T^d, Y) = \prod_{i=1}^{k} (Y - Y_i(T^d))$, $g(T^d, Y) = \prod_{j=1}^{l} (Y - \bar{Y}_j(T^d))$. Fix $t \in C$ sufficiently small. Applying Lemma 3.1 to the polynomials $f(t^d, Y)$, $g(t^d, Y) \in C[Y]$, we get

$$\begin{aligned} &\max\{|f(t^d, y)|, |g(t^d, y)|\} \\ &\geqslant 2^{-\max(k,l)} \min\{\min_{j=1}^{l}\{\prod_{i=1}^{k} |Y_i(t^d) - \bar{Y}_j(t^d)|\}, \min_{i=1}^{k}\{\prod_{j=1}^{l} |Y_i(t^d) - \bar{Y}_j(t^d)|\}\} \\ &\geqslant C|t|^{dl^*} = C|t^d|^{l^*} \quad \text{for some } C > 0. \end{aligned}$$

Hence $\max\{|f(x, y)|, |g(x, y)|\} \geqslant C|x|^{l^*}$ for $x \in C$ near 0, and consequently, $l_0(f, g, X) \leqslant l^*$. Now, let $l > 0$ be such that $\max\{|f(x, y)|, |g(x, y)|\} \geqslant C|x|^l$ for small x. Hence $|g(t^d, Y_i(t^d))| \geqslant C|t^d|^l$, $|f(t^d, \bar{Y}_j(t^d))| \geqslant C|t^d|^l$ and we get

$$\operatorname{ord} \prod_{i=1}^{k} (\bar{Y}_j(X) - Y_i(X)) = \operatorname{ord} f(X, \bar{Y}_j(X)) = (1/d) \operatorname{ord} f(T^d, \bar{Y}_j(T^d)) \leqslant l;$$

similarly,

$$\operatorname{ord} \prod_{j=1}^{l} (Y_j(X) - \bar{Y}_j(X)) = \operatorname{ord} g(X, Y_j(X)) = (1/d) \operatorname{ord} g(T^d, Y_i(T^d)) \leqslant l.$$

This shows that $l^* \leqslant l$, so $l_0(f, g, X) \geqslant l^*$. Therefore, we get the desired equality $l_0(f, g, X) = l^*$.

We are in a position to prove Theorem 1.3. The proof is based on Proposition 3.2 and the Newton–Puiseux Theorem.

Proof of Theorem 1.3. We assume that $f, g \in C\{X, Y\}$ are convenient and use the notation introduced above. If $Y_i(X) \in C\{X\}^*$ is a solution of $f(X, Y) = 0$ corresponding to the segment $S \in \mathfrak{N}_f$, then

$$\sum_j \operatorname{ord}(Y_j(X) - \bar{Y}_j(X)) \geqslant \sum_j \min\{\operatorname{ord} Y_j(X), \operatorname{ord} \bar{Y}_j(X)\}$$
$$= \sum_{T \in \mathfrak{N}_g} \min\{|S_1|/|S_2|, |T_1|/|T_2|\}\,|T_2| = (1/|S_2|) \sum_{T \in \mathfrak{N}_g} [S, T]$$

with equality for non-degenerate germs.

If $\bar{Y}_j(X) \in C\{X\}^*$ is a solution of $g(X, Y) = 0$ corresponding to the segment $T \in \mathfrak{N}_g$, then

$$\sum_i \operatorname{ord}(\bar{Y}_j(X) - Y_i(X)) \geqslant \sum_i \min\{\operatorname{ord} \bar{Y}_j(X), \operatorname{ord} Y_i(X)\}$$
$$= \sum_{S \in \mathfrak{N}_f} \min\{|T_1|/|T_2|, |S_1|/|S_2|\}\,|S_2| = (1/|T_2|) \sum_{S \in \mathfrak{N}_f} [S, T]$$

with equality for non-degenerate germs.

According to Proposition 3.2, we get

$$(3.3) \qquad l_0(f, g, X) \geqslant \max\Big\{\max_S \Big\{(1/|S_2|) \sum_{T \in \mathfrak{N}_g} [S, T]\Big\}, \max_T \Big\{(1/|T_2|) \sum_{S \in \mathfrak{N}_f} [S, T]\Big\}\Big\}$$

with equality if (f, g) is non-degenerate.

Let $(X_r(Y))$, resp. $(\bar{X}_s(Y))$, be the sequence of all solutions in $C\{Y\}^*$ of $f(X, Y) = 0$, resp. $g(X, Y) = 0$. Proposition 3.2 yields

$$l_0(f, g, Y) = \max\Big\{\max_s \Big\{\sum_r \operatorname{ord}(\bar{X}_s(Y) - X_r(Y))\Big\}, \max_r \Big\{\sum_s \operatorname{ord}(X_r(Y) - \bar{X}_s(Y))\Big\}\Big\}.$$

Using this formula and part II of Theorem 2.1, we get

$$(3.4) \qquad l_0(f, g, Y) \geqslant \max\Big\{\max_S \Big\{(1/|S_1|) \sum_{T \in \mathfrak{N}_g} [S, T]\Big\}, \max_T \Big\{(1/|T_1|) \sum_{S \in \mathfrak{N}_f} [S, T]\Big\}\Big\}$$

with equality if (f, g) is non-degenerate.

Now, we obtain the theorem from (3.3) and (3.4) since $l_0(f, g) = \max\{l_0(f, g, X), l_0(f, g, Y)\}$.

Appendix. We prove here the formula for $l_0(f, g)$ due to Chądzyński and Krasiński (cf. [2], Main Theorem, and [9] where a special case is given). Assume that the germ (f, g): $(C^2, 0) \to (C^2, 0)$ has an isolated zero at the origin. Let $f = \prod_u f_u$, $g = \prod_v g_v$ be factorizations of f and g into irreducible factors in $C\{X, Y\}$.

THEOREM (cf. [2], [9]). *With the above notation,*

$$l_0(f, g) = \max\Big\{\max_r \{m_0(f_u, g)/\operatorname{ord} f_u\}, \max_v \{m_0(f, g_v)/\operatorname{ord} g_v\}\Big\}.$$

Proof. Put l_* = the right-hand side of the above equality. Both sides of the equality being invariant under linear changes of coordinates X, Y, we may assume that $\operatorname{ord} f(X, 0) = \operatorname{ord} f(0, Y) = \operatorname{ord} f$, $\operatorname{ord} g(X, 0) = \operatorname{ord} g(0, Y) = \operatorname{ord} g$. Using Zeuten's Rule, we get from Proposition 3.2

$$l_0(f, g, X) = \max\{\max_u \{m_0(f_u, g)/m_0(f_u, X)\}, \max_v \{m_0(f, g_v)/m_0(g_v, X)\}\}.$$

Hence, $l_0(f, g, X) = l_*$ since $m_0(f_u, X) = \operatorname{ord} f_u(0, Y) = \operatorname{ord} f_u$ and $m_0(g_v, X) = \operatorname{ord} g_v(0, Y) = \operatorname{ord} g_v$. Similarly we check that $l_0(f, g, Y) = l_*$ and the theorem follows.

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