COVERINGS OF STABLY PARALLELIZABLE MANIFOLDS

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Let $M$, $\tilde{M}$ be smooth, compact, stably parallelizable manifolds such that $\tilde{M} \to M$ is a finite covering. One may ask how framings on $M$, their invariants and properties are related to those of $\tilde{M}$. In this note we consider the property of being the boundary of a parallelizable manifold.

We investigate the following construction. Given a $d$-fold covering $\tilde{N} \to N$ with $N$ stably parallelizable and a homotopy sphere $\Sigma$, we have the covering

$$\tilde{N} \# \Sigma \# \ldots \# \Sigma \to N \# \Sigma,$$

$d$ times

Under some conditions on $\tilde{N}$ and $\Sigma$, the manifold $\tilde{N} \# \Sigma \# \ldots \# \Sigma$ bounds a parallelizable manifold but $N \# \Sigma$ does not. This provides a series of examples of coverings such that the covering manifold admits a framing which is a framed boundary, but the covered manifold does not. Our construction shows that, given a covering $\tilde{M} \to M$, a mere change of differential structure often leads to an example of this type.

The following problem is especially interesting for compact Lie groups:

(*) If $\tilde{M} \to M$ is a finite covering, is $M$ the boundary of a parallelizable manifold provided that $\tilde{M}$ is such a boundary?

Since Guest and Pastor [3] proved that most of 1-connected Lie groups are boundaries of parallelizable manifolds, an affirmative answer to (*) for Lie groups would imply that all compact Lie groups are such boundaries. Note that [4] implies that for coverings of even degree of Lie groups the answer is "yes".

We are also motivated by a question investigated in [2], namely the relation between the $e$-invariant of $M$ (which is a compact Lie group $G$) and of $\tilde{M}$.

We assume that all manifolds considered are compact, connected and oriented. First we construct a family of $(2m)$-fold coverings of stably parallelizable manifolds where $\tilde{M}$ admits a framing which gives the zero element in $\Omega^r$ but no framing of $M$ bounds a framed manifold.
To prove our first theorem we use the generalized arithmetic genus

$$\alpha: \Omega_n^{\text{Spin}} \to \mathcal{KO}(S^n)$$

(cf. [8]). For a framing $\varphi$ on $M$ let $\alpha(M, \varphi)$ denote the $\alpha$-genus of $M$ with the spin structure determined by $\varphi$.

**Theorem 1.** Let $n = 8k + 1, 8k + 2$ and $\tilde{N}^n \to N^n$ be a $d$-fold covering of $n$-dimensional, stably parallelizable manifolds and $d = \text{ord} \pi_n^S$. Assume that $\tilde{N}^n$ bounds a parallelizable manifold and $N^n$ admits a Riemannian metric of positive scalar curvature. Let $\Sigma^n$ be an exotic sphere with non-zero $\alpha$-genus. Then in the covering

$$\tilde{N}^n \# \Sigma^n \# \ldots \# \Sigma^n \to N^n \# \Sigma^n$$

d times

the manifold $\tilde{N}^n \# d \cdot \Sigma^n$ bounds a parallelizable manifold but $N^n \# \Sigma^n$ does not.

**Proof.** Let $\varphi$ be a framing $M^n = N^n \# \Sigma^n$. Then there exist a framing $\varphi_1$ on $\Sigma^n$ and $\varphi + (-\varphi_1)$ on $N^n$ such that

$$(M^n, \varphi) = (N^n, \varphi + (-\varphi_1)) + (\Sigma^n, \varphi_1) \quad \text{in } \Omega_n^{\text{fr}}.$$

The above fact is obtained by a standard construction ([6], Section 4) of the framed connected sum of two framed manifolds that corresponds to the sum in $\pi_n^S$. We get the equality

$$N^n = N^n \# S^n = N^n \# \Sigma^n \# (-\Sigma^n) = M^n \# (-\Sigma^n),$$

and thus we can write

$$\varphi = [\varphi + (-\varphi_1)] + \varphi_1.$$

Since $N^n$ admits a Riemannian metric of positive scalar curvature, we have $\alpha(N^n, \varphi - \varphi_1) = 0$ by the Hitchin–Lichnerowicz theorem ([5], [7]). Hence $\alpha(M^n, \varphi) \neq 0$, so $(M^n, \varphi)$ is not a framed boundary.

On the other hand, by assumption there exists a framing $\tilde{\psi}$ on $\tilde{N}^n$ such that $(\tilde{N}^n, \tilde{\psi}) = 0$ in $\Omega_n^{\text{fr}}$. Now, for any framing $\psi$ on $\Sigma^n$ we have

$$(\tilde{N}^n \# \Sigma^n \# \ldots \# \Sigma^n, \tilde{\psi} \# \psi \# \ldots \# \psi) = (\tilde{N}^n, \tilde{\psi}) + d \cdot (\Sigma^n, \psi) = 0 \quad \text{in } \pi_n^S.$$

d times

d times

**Example.** It is shown in [8] and [1] that for $n = 8k + 1, 8k + 2$ there exists an exotic sphere $\Sigma^n$ such that $\alpha(\Sigma^n) \neq 0$. Now the following gives examples of manifolds satisfying the assumptions of Theorem 1:

1. $(n = 9)$: $N^9 = L(\text{ord } \pi_3^S; q_1, q_2, q_3, q_4) \times S^2$, where $L(p; q_1, q_2, q_3, q_4)$ denotes the $(2n-1)$-dimensional lens space with $\pi_1 = \mathbb{Z}_p$ and $N^9 = S^7 \times S^2$ is the standard $(\text{ord } \pi_3^S)$-fold covering of $N^9$;

2. $(n = 10)$: $N^{10} = L(\text{ord } \pi_{10}^S; q_1, q_2, q_3, q_4) \times S^3$, and so on.
Now we will consider coverings of odd degree.

**Theorem 2.** Let \( n \not\equiv 0 \pmod{4} \), \( \tilde{N}^n \rightarrow N^n \) be a \( p \)-fold covering, where \( \tilde{N}^n \), \( N^n \) are stably parallelizable manifolds and \( N^n \) bounds a parallelizable manifold. Assume that a prime \( p > 2 \) divides \( \text{ord } \pi_8^n, |\text{im } J| \not\equiv 0 \pmod{p} \) and that

\[
\prod_{i=1}^{n-1} \text{ord } H^i(N^n, \pi_i SO) < p-1.
\]

Then there exists an exotic sphere \( \Sigma^n \) such that in the covering

\[
\tilde{N}^n \not\equiv p \cdot \Sigma^n \rightarrow N^n \not\equiv \Sigma^n
\]

the covering manifold \( \tilde{N}^n \not\equiv p \cdot \Sigma^n \) bounds a parallelizable manifold but \( N^n \not\equiv \Sigma^n \) does not.

**Proof.** Choose a cyclic subgroup \( G < \pi_8^n \) of order \( p \). Since \( p \) does not divide \( \text{ord } (\text{im } J) \), \( G \) projects isomorphically onto a cyclic subgroup of \( \pi_8^n/\text{im } J \) (which will be denoted also by \( G \)).

The orbit of the cobordism class \([\varphi_0]\) under the action of \([N^n, SO]\) consists of all framed cobordism classes on \( N^n \). The obstructions to extend a homotopy between two maps \( N^n \rightarrow SO \) belong to \( H^i(N^n, \pi_i SO) \). If we choose a cell decomposition of \( N^n \) with only one \( n \)-cell, then for any two maps \( N^n \rightarrow SO \) which are equal on the \((n-1)\)-skeleton we can find a map \( f \colon S^n \rightarrow SO \) such that one of these maps is given by twisting the other on the \( n \)-cell by \( f \). Hence there are framings \( \varphi_1, \ldots, \varphi_l \) of \( N \), where

\[
l \leq \prod_{i=1}^{n-1} \text{ord } H^i(N^n, \pi_i SO),
\]

such that any framing on \( N \) is of the form

\[
(N^n, \varphi_i) \not\equiv (S^n, f),
\]

where \( i = 1, 2, \ldots \) or \( l \) and \((S^n, f)\) is the image of \([f]\in \pi_n SO\) by the \( J \)-homomorphism.

Now we see that there exists \( g \in G \subset \pi_8^n/\text{im } J \) such that its sum with any element \((N^n, \varphi) \in \pi_8^n/\text{im } J\) is non-zero.

Since the Wall group of surgery obstruction in dimensions considered is 0 or \( Z_2 \), \( g \) is represented by some exotic sphere \( \Sigma^n \). The above calculations show that \( N^n \not\equiv \Sigma^n \) with arbitrary framing is non-zero in \( \Omega_n^n \). But \( \tilde{N}^n \not\equiv p \cdot \Sigma^n \) bounds a parallelizable manifold, because \( N^n \) does and \( \text{ord } (g) = p \). Thus

\[
\tilde{N}^n \not\equiv p \cdot \Sigma^n \rightarrow N^n \not\equiv \Sigma^n
\]

satisfies the assertion of the theorem.

**Example.** The smallest dimension possible is \( n = 38 \). Then there exists an element \( \beta_1 \) of order \( p = 5 \) in \( \pi_8^{38} \) (the notation \( \beta_s \) for \( s \not\equiv 0 \pmod{p} \) is from
Thus \( \tilde{N}^{38} = S^1 \times S^{37} \) is a 5-fold covering of itself,

\[
\prod_{i=1}^{37} \text{ord} H^i(N^{38}, \pi_i \text{SO}) = 2 < p - 1
\]

and there exists \( \Sigma^{38} \) such that for

\[
\tilde{N}^{38} \# 5 \cdot \Sigma^{38} \to N^{38} \# \Sigma^{38}
\]

the covering manifold bounds a parallelizable manifold but the base manifold does not. Another series can be obtained by using elements \( \lambda' \in \pi_n^{5} \) (see [9] and [10]), where

\[
n = 2(2p^2 + 1)(p - 1) - 5 \equiv 1 \pmod{4}.
\]

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