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A CHARACTERIZATION OF GROUP-VALUED MEASURES SATISFYING THE COUNTABLE CHAIN CONDITION

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Let $\mathcal{S}$ be a σ-ring of sets and let (G, τ) be an Abelian Hausdorff topological group. Given a measure $\mu\colon \mathcal{S} \to G$, we denote by $\mathcal{N}(\mu)$ the family of all $E \in \mathcal{S}$ such that $\mu(F) = 0$ for any $F \subset E$, $F \in \mathcal{S}$. Following Drewnowski [1] and Musiał [3], we say that μ satisfies the *countable chain condition* (shortly CCC) if any family of pairwise disjoint members of $\mathcal{S} \setminus \mathcal{N}(\mu)$ is (at most) countable. As easily seen, when the topology τ is metrizable, any measure $\mu\colon \mathcal{S} \to G$ satisfies CCC. We shall establish in the sequel a result* (Theorem 1) which can be regarded as a partial converse of this assertion. However, first let us remark that the full converse fails to be true in general. Indeed, take for G a Banach space and let τ be the weak topology of G. Then any measure $\mu\colon \mathcal{S} \to G$ satisfies CCC by the Orlicz-Pettis theorem, but τ is non-metrizable unless G is finite-dimensional.

We begin with the following

LEMMA. *Let $\mu\colon \mathcal{S} \to G$ be a measure satisfying CCC. Then there exists a pseudo-metrizable group topology $\tau_0 \subset \tau$ on G such that $\mathcal{N}(\mu) = \{E \in \mathcal{S}\colon \mu(F) \in \overline{\{0\}}^{\tau_0}$ for any $F \subset E, F \in \mathcal{S}\}$.*

Proof. The reasoning proceeds in three steps.

Step I. Suppose $\{V_n\}$ is a sequence of closed neighbourhoods of 0 in G with $V_{n+1} + V_{n+1} \subset V_n$ and $S \in \mathcal{S}$ is such that

$$\mu(S) \notin \bigcap_{n=1}^{\infty} V_n.$$

Then there is an $\tilde{S} \in \mathcal{S}$ with the following properties: $\tilde{S} \subset S$, $\mu(\tilde{S}) \notin \bigcap_{n=1}^{\infty} V_n$ and $E \in \mathcal{N}(\mu)$ whenever $E \subset \tilde{S}$, $E \in \mathcal{S}$ and $\mu(E) \in \bigcap_{n=1}^{\infty} V_n$ (cf. (2°) of [2]).

* Results of this note were presented at the Summer School on Subadditive Measures, Integrals and Operators, held at Modra-Piesky, Czechoslovakia, in September 1973.

Assuming the contrary, we shall be able to construct by induction a family $\{E_a\}_{a<\omega_1}$ of pairwise disjoint members of $\mathscr{S}\setminus\mathscr{N}(\mu)$ such that $E_a \subset S$ and $\mu(E_a)\in \bigcap_{n=1}^{\infty} V_n$. However, the latter is evidently impossible if μ satisfies CCC. Suppose, for $a < a_0 < \omega_1$, E_a with the listed properties are already defined. Since the set $\bigcap_{n=1}^{\infty} V_n$ is closed and

$$\Big(\bigcap_{n=1}^{\infty} V_n\Big)+\Big(\bigcap_{n=1}^{\infty} V_n\Big) \subset \bigcap_{n=1}^{\infty} V_n$$

by our assumptions on V_n, it follows from the countable additivity of μ and the inductive hypothesis that

$$\mu\Big(\bigcup_{a<a_0} E_a\Big)\in \bigcap_{n=1}^{\infty} V_n.$$

Setting

$$\tilde{S} = S\setminus \bigcup_{a<a_0} E_a,$$

we have $\tilde{S}\in \mathscr{S}$ and $\mu(\tilde{S})\notin \bigcap_{n=1}^{\infty} V_n$. What we have assumed to the contrary implies the existence of a set $E_{a_0}\in\mathscr{S}\setminus\mathscr{N}(\mu)$ with $E_{a_0} \subset \tilde{S}$ and $\mu(E_{a_0})\in \bigcap_{n=1}^{\infty} V_n$.

Step II. There exists a double sequence $\{V_n^i\}$ of closed symmetric neighbourhoods of 0 in G and a sequence $\{S_i\}$ of pairwise disjoint members of $\mathscr{S}$ with the following properties:

(a) $V_{n+1}^i + V_{n+1}^i \subset V_n^i$.

(b) If $E \subset S_i$, $E\in \mathscr{S}$ and $\mu(E)\in \bigcap_{n=1}^{\infty} V_n^i$, then $E\in\mathscr{N}(\mu)$.

(c) If $F\cap \bigcup_i S_i \neq \emptyset$ and $F\in\mathscr{S}$, then $F\in\mathscr{N}(\mu)$.

It is clearly sufficient to prove this assertion under the additional assumption that μ is non-trivial, i. e., $\mathscr{N}(\mu) \neq \mathscr{S}$. Consider the class $\boldsymbol{M}$ consisting of all families $\mathfrak{F}$ of pairwise disjoint members of $\mathscr{S}\setminus\mathscr{N}(\mu)$ such that for any $S\in\mathfrak{F}$ there is a sequence $\{V_n^S\}$ of closed symmetric neighbourhoods of 0 in G with $V_{n+1}^S + V_{n+1}^S \subset V_n^S$ and $E\in\mathscr{N}(\mu)$ whenever $E \subset S$, $E\in\mathscr{S}$ and $\mu(E)\in \bigcap_{n=1}^{\infty} V_n^S$. In view ot the additional assumption, $\boldsymbol{M}$ is non-empty by Step I. The Kuratowski-Zorn lemma gives a maximal (with respect to set inclusion) element $\mathfrak{F}_0$ of $\boldsymbol{M}$. Since μ satisfies CCC, $\mathfrak{F}_0$ is countable. Let $\{S_i\}$ be an enumeration of members of $\mathfrak{F}_0$ and let $\{V_n^i\}$ be the corresponding double sequence of neighbourhoods of 0. The definition of $\boldsymbol{M}$ shows that $\{S_i\}$ and $\{V_n^i\}$ satisfy (a) and (b). Property (c) is a consequence of Step I and the maximality of $\mathfrak{F}_0$.

Step III. Construction of τ_0.

Put $V_n = V_n^1 \cap V_n^2 \cap \ldots \cap V_n^n$, where V_n^i are the same as in Step II. Let τ_0 be the group topology for which $\{V_n\}$ forms a neighbourhood base at 0. Clearly, $\tau_0 \subset \tau$. Moreover, by the well-known theorem of Kakutani, τ_0 is pseudo-metrizable. Suppose $E \in \mathscr{S}$ and $\mu(F) \in \overline{\{0\}}^{\tau_0}$ for all $F \subset E$, $F \in \mathscr{S}$. It follows from (b) and (c), respectively, that $E \cap S_i \in \mathscr{N}(\mu)$ and $E \setminus \bigcup_i S_i \in \mathscr{N}(\mu)$. Since $\mathscr{N}(\mu)$ is a σ-ideal, we get $E \in \mathscr{N}(\mu)$. Thus τ_0 has all the desired properties.

Note. If (G, τ) of the Lemma is additionally assumed to be a linear topological (resp., locally convex) space, then τ_0 can be chosen a linear space (resp., locally convex) topology on G.

THEOREM 1. *Suppose G is an Abelian Hausdorff topological group and $\mathscr{S}$ is a σ-ring of sets. A measure $\mu\colon \mathscr{S} \to G$ satisfies CCC if and only if there exists a metrizable group H and a continuous homomorphism $h\colon G \to H$ such that $\mathscr{N}(\mu) = \mathscr{N}(h \circ \mu)$.*

Proof. If μ satisfies CCC, then there exists a topology τ_0 with the properties listed in the Lemma. Hence, in particular, $G_0 = \overline{\{0\}}^{\tau_0}$ is a closed subgroup of (G, τ_0) and the quotient group H of (G, τ_0) modulo G_0 is metrizable. Let h be the natural homomorphism of G onto H. As well known, h is continuous, so that $h \circ \mu$ is a measure. Since $G_0 = h^{-1}(0)$, it follows from the Lemma that $\mathscr{N}(\mu) = \mathscr{N}(h \circ \mu)$.

Let us note another important consequence of the Lemma, namely a result stating that the two notions of absolute continuity coincide for measures satisfying CCC.

THEOREM 2. *Let G and H be Abelian Hausdorff topological groups and let $\mu\colon \mathscr{S} \to G$ and $\nu\colon \mathscr{S} \to H$ be measures. If μ satisfies CCC and $\mathscr{N}(\mu) \subset \mathscr{N}(\nu)$, then to every neighbourhood W of 0 in H there exists a neighbourhood V of 0 in G such that $\nu(E) \in W$ whenever $E \in \mathscr{S}$ and $\mu(F) \in V$ for all $F \subset E$, $F \in \mathscr{S}$.*

Proof. Let τ_0 be the topology which existence is stated in the Lemma. Denote by $\{V_n\}$ a neighbourhood base at 0 of τ_0 such that V_n are closed and $V_{n+1} + V_{n+1} \subset V_n$. If the conclusion of the Theorem is false, there exists a neighbourhood W of 0 in H and a sequence $\{E_n\} \subset \mathscr{S}$ such that $\mu(F) \in V_n$ for $F \subset E_n$, $F \in \mathscr{S}$ and $\nu(E_n) \notin W$. Put

$$E_0 = \bigcap_{p=1}^{\infty} \bigcup_{m=p}^{\infty} E_m.$$

Then $\mu(F) \in V_n$ for all $F \subset \bigcup_{m=n+1}^{\infty} E_m$, so that we obtain $E_0 \in \mathscr{N}(\mu)$. Consequently, $E_0 \in \mathscr{N}(\nu)$. Since

$$\bigcup_{m=n}^{\infty} (E_m \setminus E_0) \downarrow \emptyset,$$

there is an n_0 such that $\nu(F) \in W$ for all $F \subset \bigcup_{m=n_0}^{\infty} (E_m \setminus E_0)$, $F \in \mathcal{S}$ (see, e. g., [4], Lemma 1). In particular, $\nu(E_{n_0} \setminus E_0) \in W$. It now follows that $\nu(E_{n_0}) \in W$ which is a contradiction.

Added in proof. A slightly weaker version of Theorem 2 has been recently published by Traynor [5].

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