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Remarks on differential concomitants of densities

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Introduction. Let X^n be an n-dimensional manifold. If the transformation of the coordinate system has the form

$$(1)\qquad \tilde{x}^i = \tilde{x}^i(x^k), \quad i, k = 1, 2, \ldots, n,$$

then we put

$$(2)\quad \begin{aligned} &\text{(a)} & A^i_k &= \frac{\partial \tilde{x}^i}{\partial x^k}, \\ &\text{(b)} & B^i_k &= \frac{\partial x^i}{\partial \tilde{x}^k}, \\ &\text{(c)} & J &= \operatorname{Det}\|A^i_k\| \neq 0, \\ &\text{(d)} & A^i_{j_1\ldots j_s} &= \frac{\partial^s \tilde{x}^i}{\partial x^{j_s}\ldots \partial x^{j_1}}, \\ &\text{(e)} & B^i_{j_1\ldots j_s} &= \frac{\partial^s x^i}{\partial \tilde{x}^{j_s}\ldots \partial \tilde{x}^{j_1}}. \end{aligned}$$

The partial derivative of a function U with respect to x^l will be denoted by $U_{,l}$. We consider a non-zero density field of weight $-p \neq 0$ on X^n (q is a W- or G-density).

We say that a geometric object Ω is a *differential concomitant of order s of the density* q if for every coordinate systems (x^i) we have

$$(3)\qquad \Omega = \Omega(q, q_{,l}, \ldots, q_{,l_1\ldots l_s}).$$

Let q be a G-density of weight -1. We put

$$(4)\qquad \Omega_k \overset{\mathrm{df}}{=} \frac{\partial}{\partial x^k}\Psi(q),$$

where Ψ denotes a real differentiable function of the variable u. S. Gołąb has proved that (4) is a geometric object if and only if

$$\Psi(u) = \Gamma \ln|u| + \Delta_i, \quad \begin{cases} \Delta_1, & u > 0, \\ \Delta_{-1}, & u < 0, \end{cases}$$

i.e.

$$\Omega_k = \Gamma(\ln|q|)_{,k}$$

(cf. [1], p. 144).

The object Ω_k is a differential concomitant of the first order of the density q.

In the present paper we consider differential concomitants of the first and second order of q, where q is a W— or G-density of weight $-p \neq 0$.

In section 1 we shall determine the general form of differential concomitants of the first order of q which are purely differential geometric objects of the class $r \leqslant 2$.

In section 2 we show that differential concomitants of the second order which are purely differential object of the first class do not exist in the sense that every such concomitant is an algebraic concomitant of the density q.

However, every differential concomitant of the second order which is a purely differential geometric object of the second class is an algebraic concomitant of the density q and of the object

$$\Gamma_i = -(\ln|q|)_{,i}.$$

1. Let ω be a purely differential geometric object of the class r. After a change of the coordinate system (1) the components ω of these objects are transformed by the rule

$$\tilde{\omega} = F(\omega, L), \tag{1.1}$$

where

$$L = \{A^i_k, A^i_{k_1k_2}, \ldots, A^i_{k_1\ldots k_r}\} \in L^n_r,$$

and the function F satisfies the following equations:

$$(1.2) \qquad \begin{aligned} &\text{(a)} \qquad F[F(\omega, L_1), L_2] = F(\omega, L_2L_1), \\ &\text{(b)} \qquad F(\omega, E) = \omega, \end{aligned}$$

L_1, L_2 denote here arbitrary elements of the differential group L^n_r and E is the unit element of L^n_r.

If we denote by $\tilde{q}$ the component of the density q in the system $(\tilde{x}^i)$, then the transformation formula has the form

$$\tilde{q} = \varphi(J)q, \tag{1.3}$$

where

$$\varphi(J) = \begin{cases} |J|^p & \text{for } W\text{-density}, \\ \operatorname{sgn} J\,|J|^p & \text{for } G\text{-density}. \end{cases} \tag{1.4}$$

We prove the following theorem:

THEOREM 1. *If a purely differential geometric object of the first class is a differential concomitant of the first order of a non-zero $W(G)$ density q of weight $-p \neq 0$, then this object is an algebraic concomitant of the density q.*

Proof. If an object ω is a differential concomitant of the first order of the density q, then it must satisfy the equation

$$\omega(\tilde{q}, \tilde{q}_{,k}) = F[\omega(q, q_{,k}), A], \tag{1.5}$$

where

$$\tilde{q}_{,k} = \varphi(J)[\ln|\varphi(J)|_{,k}q + B^s_k q_{,s}] = \varphi(J)[B^s_k q_{,s} - pA^i_j B^j_{ik} q]. \tag{1.6}$$

Putting $A = \|A^i_k\| = B = \|B^i_k\| = E$ in relations (1.3) and (1.6), we get

$$\tilde{q} = q, \tag{1.7}$$

$$\tilde{q}_{,k} = q_{,k} - pB^i_{ik} q. \tag{1.8}$$

We put

$$B^i_{jk} = \begin{cases} \dfrac{q_{,k}}{pq} = \dfrac{1}{p}\ln|q|_{,k}, & i = j = k, \\ 0, \quad i \neq j \text{ or } j \neq k. \end{cases} \tag{1.9}$$

Then we have

$$\tilde{q}_{,k} = 0. \tag{1.10}$$

The values $\Gamma_k \stackrel{\mathrm{df}}{=} -\dfrac{1}{p}\ln|q|_{,k}$ are a geometric object which has the following transformation rule:

$$\tilde{\Gamma}_k = B^s_k \Gamma_s - (\ln|J|)_{,k}.$$

Inserting $A = B = E$ and (1.9) into equation (1.5), we get by (1.7), (1.10) and (1.2b)

$$\omega(q, 0) = \omega(q, q_{,k}).$$

Therefore

$$\omega(q, q_{,k}) = f(q). \tag{1.11}$$

This completes the proof.

THEOREM 2. *If a purely differential geometric object of the class $r = 2$ is a differential concomitant of the first order of a non-zero $W(G)$-density q of weight $-p \neq 0$, then this object is an algebraic concomitant of the density q and of the object $\Gamma_k = -\dfrac{1}{p}(\ln|q|)_{,k}$.*

Proof. If a purely differential geometric object ω of the class $r = 2$ is a differential concomitant of the first order of the density q, then it must satisfy the equation

$$\omega(\tilde{q}, \tilde{q}_{,k}) = F[\omega(q, q_{,k}), L], \tag{1.12}$$

where

$$L = \{A^i_j, A^i_{jk}\}.$$

From (1.12), (1.2a) and (1.2b) it follows that

$$\omega(q, q_{,k}) = F[\omega(\tilde{q}, \tilde{q}_{,k}), L^{-1}], \tag{1.13}$$

where

$$L^{-1} = \{B^i_k, B^i_{k_1k_2}\}.$$

Putting $A = \|A^i_k\| = B = \|B^i_k\| = E$ and (1.9) into (1.13), we obtain by (1.7) and (1.10)

$$\omega(q, q_{,k}) = F[\omega(q, 0), \delta^i_k, -\Gamma_k];$$

therefore

$$\omega(q, q_{,k}) = h(q, \Gamma_k).$$

This completes the proof.

COROLLARY 1. *The covariant derivative of the density q with respect to connection such that $\Gamma_k = -\dfrac{1}{p}(\ln|q|)_{,k}$ is equal to zero.*

COROLLARY 2. *If the geometric object Λ_k of the second class with the transformation formula*

$$\Lambda_k = B^s_k \Lambda_s - (\ln|J|)_{,k}$$

is the differential concomitant of the first order of the non-zero $W(G)$-density q, then

$$\Lambda_k = \Gamma_k.$$

Proof. We consider the following equation:

$$\Lambda_k(\tilde{q}, \tilde{q}_{,i}) = B^s_k \Lambda_s(q, q_{,i}) + A^l_s B^s_{kl}. \tag{1.14}$$

Inserting $A = B = E$ and (1.9) into equation (1.14), we obtain

$$\Lambda_k(q, 0) = \Lambda_k(q, q_{,i}) - \Gamma_k. \tag{1.15}$$

We put $T_k = \Lambda_k(q, 0)$.

From (1.15) it follows that T_k is a covariant vector and T_k is the algebraic concomitant of the density q; therefore

$$T_k = 0.$$

This completes the proof.

2. Let ω be a purely differential geometric object of class r. We assume that this object is a differential concomitant of the second order of the density q with transformation formula (1.4), i.e. it must satisfy the equation

$$\omega(\tilde{q}, \tilde{q}_{,k}, \tilde{q}_{,kl}) = F[\omega(q, q_{,k}, q_{,kl}), L], \tag{2.1}$$

where

$$\begin{aligned}\tilde{q}_{,kl} = {} & \varphi(J)_{,l}(B^s_k q_{,s} - pA^i_s B^s_{ik} q) + \\ & + \varphi(J)(B^s_{kl} q_{,s} + B^s_k B^t_l q_{,st} + pA^j_s A^i_t B^t_{jl} B^s_{ik} q - \\ & - pA^i_s B^s_{ikl} q - pA^i_s B^s_{ik} B^t_l q_{,t}).\end{aligned} \tag{2.2}$$

We prove the following theorem:

Theorem 3. *If a purely differential geometric object of the first class is a differential concomitant of the second order of a non-zero $W(G)$ density q of weight $-p \neq 0$, then this object is an algebraic concomitant of the density q.*

Proof. After substituting $A = \|A^i_k\| = E$ and (1.9) in (2.2), we obtain

$$\tilde{q}_{,kl} = \begin{cases} -\Gamma_k q_{,k} + q_{,kk} + \Gamma^2_k pq - pB^i_{ikk} q + \Gamma_k p \cdot q_{,k}, & k = l, \\ q_{,kl} - B^i_{ilk} p \cdot q + \Gamma_k p \cdot q_{,l}, & l \neq k. \end{cases} \tag{2.3}$$

If we put

$$B^i_{jkl} = \begin{cases} (2-p)\Gamma^2_k + \dfrac{q_{,kk}}{pq}, & i = j = k = l; \\ 0, & i \neq j \vee i \neq k \vee i \neq l, i \geqslant 2; \\ 0, & j = k = l \geqslant 2, i = 1; \\ -p\Gamma_k \Gamma_l + \dfrac{q_{,kl}}{pq} & \text{for all permutation of } j, k, l, \\ & j \neq k \vee j \neq l \vee k \neq l, i = l, \end{cases} \tag{2.4}$$

then we have

$$\tilde{q}_{,kl} = 0. \tag{2.5}$$

Inserting $A = B = E$, (1.9) and (2.4) into equation (2.1), where $L = \{A^i_j\}$, we obtain by (1.7), (1.10), (2.5) and (1.2b)

$$\omega(q, q_{,k}, q_{,kl}) = \omega(q, 0, 0); \tag{2.6}$$

therefore

$$\omega(q, q_{,k}, q_{,kl}) = f(q). \tag{2.7}$$

This completes the proof.

THEOREM 4. *If a purely differential geometric object of the second class is a differential concomitant of the second order of a non-zero $W(G)$-density q of weight $-p \neq 0$, then this object is an algebraic concomitant of the density q and of the object $\Gamma_k = -\frac{1}{p}(\ln|q|)_{,k}$.*

Proof. If we substitute $A = B = E$, (1.9) and (2.4) in the inverse equation (2.1), then we obtain by (1.7), (1.10), (2.5) and (1.13)

$$\omega(q, q_{,k}, q_{,kl}) = F[\omega(q, 0, 0), \delta_k^i, \Gamma_k] \stackrel{\mathrm{df}}{=} f(q, \Gamma_k). \tag{2.8}$$

This complets the proof.

We find the partial derivative of the Γ_k:

$$\Gamma_{k,l} = -\frac{q_{,kl}}{pq} - p\Gamma_k\Gamma_l.$$

Therefore

$$q_{,kl} = -pq\Gamma_{k,l} - p^2 q\Gamma_k\Gamma_l. \tag{2.9}$$

The general theorem is true.

THEOREM 5. *If a purely differential geometric object of the class $r = 3$ is a differential concomitant of the second order of a non-zero $W(G)$ density q of weight $-p \neq 0$, then this object is an algebraic concomitant of the density q and a differential concomitant of the first order of the object*

$$\Gamma_k = -\frac{1}{p}(\ln|q|)_{,k}.$$

Proof. The proof follows instantly from (1.7), (1.10), (2.5), (1.13) and (2.9).

References

[1] J. Aczél und S. Gołąb, *Funktionalgleichungen der Theorie der geometrischen Objekte*, Warszawa 1960.

[2] S. Gołąb, *Rachunek tensorowy*, Warszawa 1966.

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