COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

A SEMI-GROUP OF PROBABILITY MEASURES WITH NON-SMOOTH DIFFERENTIABLE DENSITIES ON A LIE GROUP

BY

PAWEŁ GŁOWACKI (WROCŁAW) AND ANDRZEJ HULANICKI (WROCŁAW)

DEDICATED
TO PROFESSOR STANISŁAW HARTMAN
IN APPRECIATION
FOR HIS TEACHINGS AND HIS COURAGE

The theory of convolution semi-groups of probability measures on a Lie group, started with the paper of G. Hunt [6] in 1956, is now a well established theory, cf. e.g. [4]. One of the major facts which contributed to the development of the theory is that there is a deep analogy of the behavior of the semi-groups of probability measures on, in general, non-commutative Lie groups and the behavior of such semi-groups on $\boldsymbol{R}^n$. This is already reflected in the paper of Hunt where the structure of the infinitesimal generator and its domain are proved to be very similar to the ones of a semi-group on $\boldsymbol{R}^n$. The analogy is so striking that many properties formulated for the semi-groups of probability measures on a Lie group G, if hold true in case $G = \boldsymbol{R}^n$, are conjectured for the general case.

In this paper we show that there are limits to this belief.

Let $\{\mu_t\}_{t>0}$ be a strongly continuous semi-group of probability measures on a Lie group G. Let A be the infinitesimal generator of it. The domain of A contains $C_c^\infty(G)$.

Suppose $\{\mu_t\}_{t>0}$ has the following properties:

(i) $\mu_t = \mu_t^*$;

(ii) $d\mu_t(x) = p_t(x)\,dx$ and $p_t \in L^2(G)$;

(iii) If $X_1, \ldots, X_n$ is a basis of the Lie algebra of G, then for a constant c and $\varepsilon > 0$ $\|(X_j^* X_j)^\varepsilon\|_{L^2} \leqslant c(\|Af\|_{L^2} + \|f\|_{L^2})$ for $f \in C_c^\infty(G)$, for all $j = 1, \ldots, n$, where $(X^* X)^\varepsilon$ is a fractional power of the non-negative operator $X^* X$.

Since (ii) implies

$$(*) \qquad p_t \in \bigcap_{k=1}^{\infty} D(A^k),$$

cf. e.g. (1.13) below, (iii) means that p_t allow some differentiability in all the directions with the derivatives in $L^2(G)$.

Question. Do all the derivatives of p_t exist and belong to $L^2(G)$?

It is not difficult to see that if G is Abelian, the answer to this question is "yes". In fact, since $X_j^* X_j$ and A are essentially self-adjoint on $C_c^\infty(G)$, by the spectral theorem,

$$X_j^* X_j f = \int_0^\infty \lambda \, dE_j(\lambda) f,$$

$$-Af = \int_0^\infty \lambda \, dE(\lambda) f$$

for $f \in C_c^\infty(G)$. Moreover, since both $X_j^* X_j$ and A are convolution operators and the group is commutative, $E_j(M)$ and $E(N)$ commute, whence (iii) implies that for every k there is an l such that

$$\|(\sum_{j=1}^{n} X_j^* X_j)^k f\| \leqslant C(\|A^l f\| + \|f\|),$$

whence, by $(*)$, the answer follows.

The aim of this paper is to show that in the simplest case of a non-commutative Lie group — the Heisenberg group — there is a convolution semi-group of probability measures $\{\mu_t\}_{t>0}$, which is stable in the sense of [2], satisfies (i)–(iii) but, by far, not all the derivatives of p_t are in L^2.

Let G be the Heisenberg group, i.e. $G = \boldsymbol{R}^3$ as a manifold, the multiplication being defined by

$$(x, y, z)(x', y', z') = (x+x', y+y', z+z'+xy').$$

Let X, Y, Z be the basis of the left-invariant Lie algebra of G corresponding to the one-parameter subgroups

$$\boldsymbol{R} \ni x \to (x, 0, 0) \in G,$$

$$\boldsymbol{R} \ni y \to (0, y, 0) \in G,$$

$$\boldsymbol{R} \ni z \to (0, 0, z) \in G.$$

Then

$$(**) \qquad \begin{aligned} X &= D_1, \\ Y &= D_2 + xD_3, \\ Z &= D_3, \end{aligned}$$

where $D_1 = \frac{\partial}{\partial x}$, $D_2 = \frac{\partial}{\partial y}$, $D_3 = \frac{\partial}{\partial z}$. Let

$$A = -X^2 + |Y|,$$

where $|Y| = (Y^* Y)^{1/2}$.

It is easy to verify that $-A$ is the infinitesimal generator of a semi-group of probability measures $\{\mu_t\}_{t>0}$ on G, and that $\mu_t = \mu_t^*$, cf. e.g. [5]. J. Cygan and A. Hulanicki observed that the measures μ_t have densities p_t in $L^2(G)$ and later P. Głowacki [2] proved that for some $\varepsilon > 0$ (iii) holds with $X_1 = X$, $X_2 = Y$, $X_3 = Z$. Thus μ_t satisfy (i)–(iii) and following the analogy of the behavior of the semi-groups of probability measures on a Lie group and on $\boldsymbol{R}^n$ mentioned above, it has been conjectured that p_t's are infinitely many times differentiable and all the derivatives belong to $L^2(G)$. Since it is also true that the semi-groups generated by $-(X^* X)^a - (Y^* Y)^b$, $0 < a, b \leqslant 2$, have properties (i)–(iii), an attempt has been made to prove that they form Poisson-like approximate commutative identities which are homogeneous for irrational a/b, which among other things means that they are C^∞ and their derivatives decay fast at infinity. In fact, such commutative approximate identities have been recently found in another way, cf. [3].

In this paper we show that, although the semi-group generated by $-A$ satisfies (i)–(iii), it is true that already $X^4 p_t$ does not belong to $L^2(G)$ which, of course, also buries all the hope to produce Poisson-like commutative approximate identities using semi-groups generated by $-(X^* X)^a - (Y^* Y)^b$.

As we have mentioned, the fact that the semi-group generated by $-A$ satisfies (i)–(iii) is known. However we reprove it here presenting an elementary new proof.

Preliminaries. For $t > 0$ let

$$\delta_t(x, y, z) = (t^{1/2} x, ty, t^{3/2} z).$$

δ_t is an automorphism of G and

$$X(f \circ \delta_t) = t^{1/2} Xf \circ \delta_t,$$
$$Y(f \circ \delta_t) = tYf \circ \delta_t,$$
$$Z(f \circ \delta_t) = t^{3/2} Zf \circ \delta_t.$$

Consequently,

(0.1) $$A(f \circ \delta_t) = tAf \circ \delta_t \quad \text{for every } f \in C_c^\infty(G).$$

For $(x, y, z) \in G$ we define unitary operators on $L^2(G)$ by

(0.2) $$\pi^{\pm 1}_{(x,y,z)} \phi(u) = e^{\pm i(z+uy)} \phi(x+u)$$

and for $\lambda \in \boldsymbol{R}\setminus\{0\}$ we let

$$
\text{(0.3)} \qquad \begin{aligned} \pi^\lambda_{(x,y,z)} &= \pi^1_{\delta_{|\lambda|}(x,y,z)} \quad &\text{if } \lambda > 0,\\ \pi^\lambda_{(x,y,z)} &= \pi^{-1}_{\delta_{|\lambda|}(x,y,z)} \quad &\text{if } \lambda < 0. \end{aligned}
$$

Of course, π^λ is a unitary representation of G on $L^2(\boldsymbol{R})$.

For an $f \in L^1(G)$ we define

$$\pi^\lambda_f = \int f(x, y, z)\, \pi^\lambda_{(x,y,z)}\, dx\, dy\, dz.$$

For every $\lambda \in \boldsymbol{R}\setminus\{0\}$ and $f \in L^1(G) \cap L^2(G)$ π^λ_f is a Hilbert–Schmidt operator and the following Plancherel formula holds:

$$\text{(0.4)} \qquad \|f\|^2_{L^2} = \int_{-\infty}^{\infty} \|\pi^\lambda_f\|_{\mathrm{HS}}\, |\lambda|\, d\lambda.$$

Then, for f in $L^2(G)$ and almost all λ, π^λ_f is Hilbert–Schmidt with the kernel

$$\text{(0.5)} \quad k_\lambda(x, y) = |\lambda|^{-3} \int f\big(|\lambda|^{-1/2}(y-x),\, |\lambda|^{-1}u,\, |\lambda|^{-3/2}z\big)\, e^{\pm i(ux+z)}\, du\, dz.$$

For a bounded measure μ on G the operator π^λ_μ is defined by

$$\pi^\lambda_\mu = \int \pi^\lambda_{(x,y,z)}\, d\mu(x, y, z).$$

Let $\{\mu_t\}_{t>0}$ be a strongly continuous semi-group of symmetric probability measures on G and let P be the infinitesimal generator of it on $L^2(G)$. For $\lambda \in \boldsymbol{R}\setminus\{0\}$ we define π^λ_P to be the infinitesimal generator of the strongly continuous semi-group of contractions $\{\pi^\lambda_{\mu_t}\}_{t>0}$ on $L^2(\boldsymbol{R})$. Then $-\pi^\lambda_P$ is self-adjoint and positive, and C_c^∞ is its core, cf. e.g. [1].

The following characterization of the domain of P will be useful:

(0.6) *A function f in $L^2(G)$ is in the domain of P if and only if $\pi^\lambda_P \pi^\lambda_f$ is Hilbert–Schmidt for almost every λ and*

$$\int \|\pi^\lambda_P \pi^\lambda_f\|^2_{\mathrm{HS}}\, |\lambda|\, d\lambda < \infty.$$

Then

$$\pi^\lambda_{Pf} = \pi^\lambda_P \pi^\lambda_f.$$

For a closable operator T we write $\bar{T}$ for its closure.

We note that

$$
\begin{aligned}
\pi^1_X \phi(u) &= \frac{d}{du}\phi(u),\\
\pi^1_Y \phi(u) &= iu\phi(u),\\
\pi^1_Z \phi(u) &= \phi(u),
\end{aligned}
$$

consequently,

$$\pi^1_A \phi(u) = -\frac{d^2}{du^2}\phi(u) + |u|\,\phi(u).$$

Elementary estimates.

(1.1) PROPOSITION. *Let M and N be the following operators on $L^2(\mathbf{R})$:*

$$M\phi(u) = 1_{[-1,1]}(u)\phi(u),$$
$$N\phi(u) = (1_{[-1,1]}\hat{\phi})^{\vee}(u)$$

and let $M' = I-M$, $N' = I-N$. There exists a constant c such that

$$((M'+N')\phi, \phi) \geqslant c(\phi, \phi). \tag{1.2}$$

Proof. Suppose (1.2) does not hold, i.e., there exists a sequence $\{\phi_n\}$ of functions in $L^2(\mathbf{R})$ such that $\|\phi_n\| = 1$ and

$$(M'\phi_n, \phi_n)+(N'\phi_n, \phi_n) \to 0.$$

Consequently, since both summands on the left are non-negative, we have $\lim_{n\to\infty} \|M'\phi_n\| = \lim_{n\to\infty} \|N'\phi_n\| = 0$. Thus

$$\phi_n = M\phi_n + M'\phi_n \quad \text{with } \|M'\phi_n\| \to 0$$

and

$$\phi_n = N\phi_n + N'\phi_n \quad \text{with } \|N'\phi_n\| \to 0.$$

This implies

$$\phi_n = MN\phi_n + \varepsilon_n \quad \text{with } \|\varepsilon_n\| \to 0.$$

But MN is a compact operator, so there exists a ϕ with $\|\phi\| = 1$ such that

$$\phi = MN\phi. \tag{1.3}$$

But this is impossible, since M, N are projections and so (1.3) implies that $\phi = M\phi = N\phi$ and, consequently, both ϕ and $\hat{\phi}$ have support in $[-1, 1]$.

(1.4) COROLLARY. *We have*

$$\|\pi_A^1 \phi\| \geqslant c\|\phi\|$$

for some constant $c > 0$.

We notice that (0.1) and (0.2), (0.3) imply that

$$\pi_A^\lambda = |\lambda|\,\pi_A^1. \tag{1.5}$$

(1.6) PROPOSITION. *If $f \in D_{\bar{A}^{3n/2}}$, then $f \in D_{\bar{Z}^n}$ and*

$$\|\bar{Z}^n f\| \leqslant C^n \|\bar{A}^{3n/2} f\|. \tag{1.7}$$

Proof. Let $f \in D_{\bar{A}^{3n/2}}$. Then, by (1.5)

$$\|(\pi_{\bar{A}}^\lambda)^{3n/2}\pi_f^\lambda \phi\| \geqslant c^{3n/2}|\lambda|^{3n/2}\|\pi_f^\lambda \phi\| = c^{3n/2}\|\pi_{\bar{Z}^n}^\lambda \pi_f^\lambda \phi\|$$

for almost every $\lambda \in \mathbf{R}\setminus\{0\}$ and all ϕ in $L^2(\mathbf{R})$, since, of course, $\pi_Z^\lambda \phi = \operatorname{sgn}\lambda\,|\lambda|^{3/2}\phi$.

Consequently, for almost every λ, $\pi^{\lambda}_{\bar{Z}^n}\pi^{\lambda}_f$ is Hibert–Schmidt and

$$\|\pi^{\lambda}_{\bar{Z}^n}\pi^{\lambda}_f\|_{\mathrm{HS}} \leqslant c^{-3n/2}\|\pi^{\lambda}_{\bar{A}^{3n/2}f}\|_{\mathrm{HS}}.$$

This, by (0.6), implies that $f \in D_{\bar{Z}^n}$ and (1.7) holds with $C = c^{-3/2}$.

(1.8) PROPOSITION. *If $f \in D_{\bar{A}}$, then $f \in D_{\bar{X}} \cap D_{\bar{Y}} \cap D_{\bar{X}^2}$ and*

(i) $\|\bar{X}f\| \leqslant \|\bar{A}f\| + \|f\|$,
(ii) $\|\bar{Y}f\| \leqslant C\|\bar{A}f\|$,
(iii) $\|\bar{X}^2 f\| \leqslant \|\bar{A}f\|$.

Proof. $C_c^{\infty}(G)$ is a common core for all the operators under consideration, therefore (1.8) will follow as soon as we prove that the estimates (i)–(iii) hold for $f \in C_c^{\infty}(G)$. We have

$$\|X^2 f\| = -(X^2 f, f) \leqslant (Af, f) \leqslant \|Af\|^2 + \|f\|^2$$

which proves (i). The same estimates give also

$$\|\pi^{\lambda}_X \phi\| \leqslant \|\pi^{\lambda}_A \phi\| + \|\phi\| \tag{1.9}$$

for all $\lambda \in \boldsymbol{R}\setminus\{0\}$ and ϕ in $\mathscr{S}(\boldsymbol{R})$.

Now, to prove (ii) and (iii) it is sufficient, by the Plancherel formula, to show that for almost every $\lambda \neq 0$ we have

$$\|\pi^{\lambda}_{X^2 f}\|^2_{\mathrm{HS}} + \|\pi^{\lambda}_{Yf}\|^2_{\mathrm{HS}} \leqslant C\|\pi^{\lambda}_{Af}\|^2_{\mathrm{HS}} \tag{1.10}$$

for $f \in C_c^{\infty}(G)$ which, in turn, is a consequence of

$$\|(\pi^{\lambda}_X)^2\phi\|^2 + \|\pi^{\lambda}_Y\phi\|^2 \leqslant C\|\pi^{\lambda}_A\phi\|^2, \qquad \phi \in C_c^{\infty}(\boldsymbol{R}). \tag{1.11}$$

Since both sides of (1.11) are symmetric and homogeneous of the same degree with respect to λ, it suffices to prove it for $\lambda = 1$ only. Let us abbreviate writing $\pi = \pi^1$. We have

$$\|\pi_A \phi\|^2 = \|\pi^2_X \phi\|^2 + \|\pi_{|Y|}\phi\|^2 - 2\mathrm{Re}(\pi^2_X\phi, \pi_{|Y|}\phi).$$

If $\phi \in \mathscr{S}(\boldsymbol{R})$, then $\pi_X\phi$ is in the domain of $\pi_{|Y|}$ and $\pi_{|Y|}\phi$ is in the domain of π_X. Moreover, a simple computation shows that

$$\pi_X\pi_{|Y|}\phi(u) = \pi_{|Y|}\pi_X\phi(u) + \operatorname{sgn} u\phi(u).$$

Consequently,

$$\|\pi^2_X\phi\|^2 + \|\pi_Y\phi\|^2 \leqslant \|\pi_A\phi\|^2 + \|\pi_X\phi\|^2 + \|\phi\|^2 \leqslant C\|\pi_A\phi\|^2,$$

the last inequality being a consequence of (1.4) and (1.9), and so (ii) and (iii) are proved.

(1.12) *If $g \in D_{\bar{A}^N}$ for sufficiently large N, then the weak derivatives $D_1^2 g$, $D_2 g$ and $D_3^3 g$ of g are locally square integrable.*

Proof. By (**) this is an immediate consequence of (1.6) and (1.8).

(1.13) PROPOSITION. *Let $\{\mu_t\}_{t>0}$ be a strongly continuous semi-group of symmetric probability measures on a Lie group G. Let P be the infinitesimal generator of it on $L^2(G)$. Then P is self-adjoint and non-positive and for every $t>0$, $n \in N$ and $f \in L^2(G)$ $f * \mu_t$ is in the domain of P^n. If, moreover, all μ_t are absolutely continuous with respect to the Haar measure on G and their densities p_t are square integrable, then p_t's themselves belong to the domain of P^n for every $n \in N$.*

Proof. Being an infinitesimal generator of a strongly continuous semi-group of hermitian contractions on a Hilbert space the operator P is self-adjoint and non-positive, so the first part follows easily from the spectral theorem. Since $p_t = p_{t/2} * p_{t/2}$, the second assertion follows from the first one.

Theorems.

(2.1) *There exist an N in N and a constant $C>0$ such that for every g in $D_{\bar{A}^N}$*

$$|g(0,0,0)| \leqslant C(\|\bar{A}^N g\| + \|g\|). \tag{2.2}$$

Proof. Let f be a fixed function in $C_c^\infty(G)$, such that $f(0,0,0)=1$. Let N be such that (1.12) holds. If $g \in D_{\bar{A}^N}$, then by the ordinary Euclidean Plancherel theorem, we obtain

$$\begin{aligned} |g(0,0,0)| &= |fg(0,0,0)| \\ &\leqslant C'(\|D_1^2(fg)\|^2 + \|D_2(fg)\|^2 + \|D_3^3(fg)\|^2 + \|fg\|^2)^{1/2} \end{aligned}$$

which, by the Leibniz formula and (**) leads to

$$|g(0,0,0)| \leqslant C''(\|\bar{X}^2 g\|^2 + \|\bar{Y} g\|^2 + \|\bar{Z}^3 g\|^2 + \|g\|^2)^{1/2},$$

whence, by (1.8), we obtain (2.2).

(2.3) *Let $\{\mu_t\}_{t>0}$ be the semi-group generated by $-\bar{A}$. Then for every $t>0$ $d\mu_t(x) = p_t(x)\,dx$, where $p_t \in L^2(G)$.*

Proof. Let $f \in L^2(G)$. By (1.13), $g = f * \mu_t$ is a smooth vector for $\bar{A}$. By (2.1),

$$|\langle f, \mu_t\rangle| = |g(0,0,0)| \leqslant C(\|\bar{A}^N(f * \mu_t)\| + \|f * \mu_t\|) \leqslant C'\|f\|.$$

Thus μ_t defines a continuous linear form on $L^2(G)$ and consequently it has a square integrable density.

(2.4) COROLLARY. *We have*

$$\|\bar{X}^2 p_t\| \leqslant C\|\bar{A}p_t\|, \quad \|\bar{Y}p_t\| \leqslant C\|\bar{A}p_t\|, \quad \|(\bar{Z}^*\bar{Z})^{1/3} p_t\| \leqslant C\|\bar{A}p_t\|.$$

(2.5) *Let $\Delta = X^2+Y^2+Z^2$ be a left-invariant Laplace operator on G. Then for $t>0$ $\Delta p_t \in C(G)$. In particular, $p_t \in C^2(G)$.*

Proof. By (1.8) and (2.4)

$$\bar{X}^2 p_t = \bar{X}^2 p_{t/2} * p_{t/2} \in C_0(G),$$
$$\bar{Z}^2 p_t = \bar{Z}^2 p_{t/2} * p_{t/2} \in C_0(G).$$

Let $Y' = D_2$. Then Y' commutes with convolutions on the right, whence

$$\overline{YY'} p_t = \bar{Y} p_{t/2} * \bar{Y}' p_{t/2} \in C_0(G)$$

and, since Z is central,

$$\overline{YZ} p_t = \bar{Y} p_{t/2} * \bar{Z} p_{t/2} \in C_0(G).$$

On the other hand,

$$Y^2 = YY' + XYZ,$$

whence $\bar{Y}^2 p_t \in C(G)$.

(2.6) *If $f, \bar{X}f, \ldots, \bar{X}^{n+1} f \in L^2(G)$, then for each $\phi \in L^2(\boldsymbol{R})$ and almost all $\lambda \in \boldsymbol{R} \setminus \{0\}$ we have $\pi_f^\lambda \phi \in C^n(\boldsymbol{R})$.*

Proof. By (0.4), for almost all λ, π_f^λ is Hilbert–Schmidt with the kernel $k_\lambda(x, y)$. It follows from (0.5) that the kernel of $\pi_{\bar{X}^j f}^\lambda$ is equal to a multiple of the weak derivative

$$\left(\frac{\partial}{\partial x}\right)^j k_\lambda(x, y).$$

By an obvious induction, the proof can be reduced to the case $n = 0$. Therefore what remains to be proved is that if K is a Hilbert–Schmidt operator on $L^2(\boldsymbol{R})$ with a kernel k such that the weak derivative $\frac{\partial}{\partial x} k(x, y)$ is square-integrable, then for every ϕ in $L^2(\boldsymbol{R})$, $K\phi$ is a continuous function which is an exercise in real variable.

(2.7) THEOREM. *It is not true that for some $t > 0$ $\bar{X}^4 p_t \in L^2(G)$.*

Proof. Suppose $\bar{X}^4 p_t \in L^2(G)$. Then, since $\bar{X}^2 p_t \in L^2(G)$, we obtain $\bar{X}^3 p_t \in L^2(G)$. Consequently we have

$$p_t,\ \bar{X} p_t,\ \bar{X}^2 p_t,\ \bar{X}^3 p_t,\ \bar{X}^4 p_t \in L^2(G). \tag{2.8}$$

Let $\lambda \in \boldsymbol{R} \setminus \{0\}$ be such that according to (2.6) $\pi_{p_t}^\lambda \phi \in C^3(\boldsymbol{R})$ for every ϕ in $L^2(\boldsymbol{R})$. Let $\phi_0, \phi_1, \ldots$ be the orthonormal basis in $L^2(\boldsymbol{R})$ consisting of eigenfunctions of the Hilbert–Schmidt, self-adjoint operator $\pi_{p_t}^\lambda$. Then, of course, $\phi_0, \phi_1, \ldots$ are eigenfunctions of

$$|\lambda|^{-1} \pi_A^\lambda = \frac{-d^2}{du^2} + |u| = \pi_A.$$

By (2.8) and (2.6)

$$\phi_0, \phi_1, \ldots \in C^3(\mathbf{R}).$$

On the other hand,

$$\pi_A \phi_n = \lambda_n \phi_n, \quad \lambda_n \neq 0, \quad \text{by (1.4)}.$$

Consequently,

$$-\phi_n''(u) = \lambda_n \phi_n(u) + |u| \phi_n(u).$$

Hence $\phi_n^{(3)} \in C(\mathbf{R})$ only if $\phi_n(0) = 0$. But since, by Sobolev's inequality and (1.8),

$$|\phi(0)| \leqslant C'(\|\phi'\| + \|\phi\|) \leqslant C\|\pi_A \phi\|,$$

for a constant C, the functional F defined on $L^2(\mathbf{R})$ by

$$\langle F, \phi \rangle = (\pi_{\bar{A}}^{-1} \phi)(0)$$

is bounded and

$$\langle F, \phi_n \rangle = \lambda_n^{-1} \phi_n(0).$$

Thus, if $\phi_n(0) = 0$ for all $n = 0, 1, \ldots$, we have $F = 0$ which is impossible since $\pi_{\bar{A}}^{-1}$ maps $L^2(\mathbf{R})$ onto the domain of $\pi_{\bar{A}}$ which contains $\mathscr{S}(\mathbf{R})$.

REFERENCES

[1] M. Duflo, *Représentations de semi-groupes de mesures sur un groupe localement compact*, Annales de l'Institut Fourier, Grenoble 28 (1978), p. 225–249.

[2] P. Głowacki, *Stable semi-groups of measures on the Heisenberg group*, Studia Mathematica 79 (1984), p. 105–138.

[3] —, *On commutative approximate identities on non-graded homogeneous groups*, Communications in Partial Differential Equations 9 (10) (1984), p. 979–1016.

[4] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer-Verlag, Berlin–Heidelberg.

[5] A. Hulanicki, *A class of convolution semi-groups of measures on a Lie group*, Lecture Notes in Mathematics 828 (1980), p. 82–101.

[6] G. Hunt, *Semi-groups of measures on Lie groups*, Transactions of the American Mathematical Society 81 (1956), p. 264–293.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WROCŁAW, POLAND

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
WROCŁAW, POLAND

Reçu par la Rédaction le 25. 04. 1984