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THE MAX NORM IN R^n-ISOMETRIES AND MEASURE

BY

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In this paper* we show that any isometry f between two subsets E and F of R^n, relative to the metric

$$(*)\qquad \varrho(x, y) = \max_{1\leqslant i\leqslant n} |x_i - y_i|,$$

is locally Euclidean, in the sense that E (minus a Lebesgue null set) can be partitioned into countably many pieces (each measurable if E is), on each of which f is an ordinary Euclidean isometry.

It follows easily that if E and F are measurable, then they have the same Lebesgue measure. More interestingly, we also show, for $n = 2$, that if E and F are bounded sets (measurable or not), then they are assigned the same measure by all Banach universal extensions of Lebesgue measure.

1. Introduction and statement of results. Let ϱ be the metric in R^n given by $(*)$. For $E, F \subseteq R^n$, $f\colon E \to F$ is a *ϱ-isometry*, and E and F are *ϱ-isometric*, if f is surjective and

$$\varrho\big(f(x), f(y)\big) = \varrho(x, y) \quad \text{for all } x, y \in E.$$

(Isometries relative to the usual metric will now be called *Euclidean* to distinguish them clearly from ϱ-isometries.)

This paper is a study of ϱ-isometries and their relation to measure.

The ϱ-isometries of the whole space form a (rather small) subgroup of the group of Euclidean isometries (see Lemma 1 below), but ϱ-isometries on smaller sets can be quite unlike Euclidean isometries:

Example. Let $E = [-1/2, 1/2] \times \{0\}$ and define $f\colon E \to R^2$ by

$$(x, 0) \overset{f}{\mapsto} (x, x^2)$$

* By permission of the first-named author, these results first appeared in the Ph. D. Thesis (University of Colorado, 1979) of the second-named author.

(Fig. 1). Since for $-1/2 \leqslant x, y \leqslant 1/2$ we have $|x+y| \leqslant 1$, and hence also

$$\varrho\big((x, x^2), (y, y^2)\big) = \max(|x-y|, |x-y|\,|x+y|) = |x-y|$$
$$= \varrho\big((x, 0), (y, 0)\big),$$

f is a ϱ-isometry.

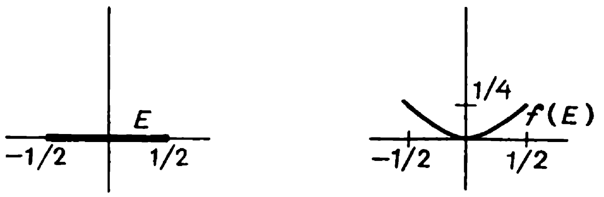

Fig. 1

Our main result is

THEOREM 1. *Let E be any subset of $\boldsymbol{R}^n$ and $\bar{E}$ the closure of E. There exist pairwise disjoint closed polyhedral regions $P_1, P_2, \ldots$ in $\boldsymbol{R}^n$ such that*

(a) *for any ϱ-isometry $f\colon E \to \boldsymbol{R}^n$ and any $i \geqslant 1$, the restriction of f to $E \cap P_i$ is Euclidean;*

(b) *$\bar{E} \setminus \bigcup_{i=1}^{\infty} P_i$ has Lebesgue measure zero.*

Condition (b) implies that the "bad" part $E \setminus \bigcup_{i=1}^{\infty} P_i$ of E has measure zero and is nowhere dense.

Theorem 1 yields easily that Lebesgue measure is *ϱ-invariant*, i.e. whenever E and F are ϱ-isometric measurable sets, they have the same Lebesgue measure. This is a special case of the main theorem of [3], where it is proved that Lebesgue measure is σ-invariant for any "reasonable" translation invariant metric σ in $\boldsymbol{R}^n$ consistent with the usual topology.

A theorem proved by Banach and Tarski in 1924 should be recalled here (see [2]): Two measurable subsets E and F of $\boldsymbol{R}^n$ have the same Lebesgue measure iff E can be partitioned into measurable parts $E_0, E_1, \ldots$ and F into $F_0, F_1, \ldots$ in such a way that E_0 and F_0 have Lebesgue measure 0 and, for each $i \geqslant 1$, E_i is isometric (in the Euclidean sense) to F_i.

If the set E in Theorem 1 is bounded, then the outer measure of $\bigcup_{i=n}^{\infty} E \cap P_i$ tends to 0 as n tends to ∞. Thus this theorem allows us to study ϱ-invariance for some finitely additive measures.

THEOREM 2. *Any Banach measure in the plane, if restricted to bounded sets, is ϱ-invariant. That is, if $E, F \subseteq \boldsymbol{R}^2$ are bounded and ϱ-isometric (but not necessarily measurable), and μ is a universal, finitely additive extension of Lebesgue measure in $\boldsymbol{R}^2$, invariant under Euclidean isometries, then $\mu(E) = \mu(F)$.*

(The first Banach measure was described in [1]. See [4] for a survey.)

By using linear transformations in a routine way we can extend Theorem 1 (and hence Theorem 2) to metrics arising from norms whose unit balls are parallelepipeds (parallelograms for Theorem 2). And it is not hard to adapt the proof of Theorem 1 given below to norms whose unit balls are some other common polyhedral shapes.

CONJECTURE. Theorem 1 holds for any translation invariant metric consistent with the usual topology on $\boldsymbol{R}^n$. (**P 1264**)

We would like to thank Jan Mycielski for criticizing earlier versions of this paper.

2. Sets on which ϱ-isometries are Euclidean. The strategy for proving Theorem 1 is first to find a large class of simple sets on which every ϱ-isometry must be Euclidean, and then to show how to divide up an arbitrary set into countably many such subsets (modulo a very "thin" set).

Definitions. For $x \in \boldsymbol{R}^n$ and $\gamma \geqslant 0$ we define the *γ-star* at x to be the set $\{x\} \cup \{(x_1, \ldots, x_i \pm \gamma, \ldots, x_n) \mid i = 1, \ldots, n\}$ (Fig. 2).

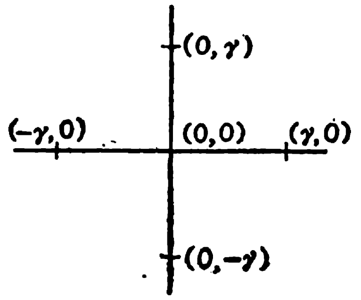

Fig. 2. The γ-star at $(0, 0)$ in $\boldsymbol{R}^2$

The *γ-dual-cube* at x is the convex hull of the γ-star at x.

LEMMA 1. *Let S be a γ-star at $x \in \boldsymbol{R}^n$ and $f\colon S \to \boldsymbol{R}^n$ a ϱ-isometry. Then f is Euclidean and $f(S)$ is a γ-star at $f(x)$.*

Proof. The proof is by induction on n. The lemma is trivial in $\boldsymbol{R}$. In $\boldsymbol{R}^n$, $n > 1$, $f\big((x_1+\gamma, x_2, \ldots, x_n)\big)$ and $f\big((x_1-\gamma, x_2, \ldots, x_n)\big)$ lie on opposite faces of the cube $\big\{p \mid \varrho\big(p, f(x)\big) = \gamma\big\}$, and so, for $i \neq 1$, the point $f\big((x_1, \ldots, x_i \pm \gamma, x_n)\big)$ lies in a hyperplane parallel to this pair of faces and passing through $f(x)$. By the induction hypothesis the lemma holds for f restricted to the $(n-1)$-dimensional γ-star

$$\{x\} \cup \{(x_1, \ldots, x_i \pm \gamma, \ldots, x_n) \mid i = 2, \ldots, n\},$$

and now it is easy to check that f is Euclidean on S and that $f(S)$ is a γ-star.

By way of a puzzle, we mention that it seems as if γ-stars are the minimal configurations on which ϱ-isometries must be Euclidean. To be precise, CONJECTURE. If a subset S of $\boldsymbol{R}^n$ has no more than $2n+1$ points and is not a γ-star, then there is a ϱ-isometry from S into $\boldsymbol{R}^n$ which is not Euclidean. (**P 1265**)

Returning to the business at hand,

LEMMA 2. *Let S be the γ-star at $x \in \boldsymbol{R}^n$ and $E \supseteq S$ a subset of the γ-dual-cube at x. Then any ϱ-isometry $f\colon S \to \boldsymbol{R}^n$ has a unique extension to E. In particular* (by Lemma 1), *any ϱ-isometry on E is Euclidean.*

Proof. It suffices to show that any point in the γ-dual-cube at x is uniquely determined among all points of $\boldsymbol{R}^n$ by its ϱ-distances to the points of the γ-star at x. In other words, it suffices to show that if for some $d_1, d_1', d_2, d_2', \ldots, d_n, d_n' \geqslant 0$ the intersection of n-cubes

$$I = \bigcap_{i=1}^{n} \{p \mid \varrho(p, (x_1, \ldots, x_i+\gamma, \ldots, x_n)) = d_i\} \cap \cap \bigcap_{i=1}^{n} \{p \mid \varrho(p, (x_1, \ldots, x_i-\gamma, \ldots, x_n)) = d_i'\}$$

contains a point p_0 in the γ-dual-cube at x, then, in fact, $I = \{p_0\}$.

Now the intersection of any n-cube of the form

$$\{p \mid \varrho(p, (x_1, \ldots, x_i \pm \gamma, \ldots, x_n)) = d\}$$

with the γ-dual-cube at x is a part of one of the faces of the n-cube, perpendicular to the x_i-axis. So if I contains a point p_0 of the γ-dual-cube at x, then the intersection

$$\{p \mid \varrho(p, (x_1, \ldots, x_i+\gamma, \ldots, x\ )) = d_i\} \cap \cap \{p \mid \varrho(p, (x_1, \ldots, x\ -\gamma, \ldots, x_n)) = d_i'\}$$

must be contained in a hyperplane perpendicular to the x_i-axis for each i. In other words, there is only one possible value for the i-th coordinate of any $p \in I$, for $i = 1, \ldots, n$, and so I must contain only a single point.

3. Construction of the partition. We are now ready to begin construction of the P_i of Theorem 1.

LEMMA 3. *Let F be a closed subset of $\boldsymbol{R}^n$. There exists $F' \subseteq F$, of Lebesgue measure 0, such that for every $\delta > 0$ and every $p \in F \setminus F'$ there is a γ-star at p contained in F, with $0 < \gamma < \delta$.*

Proof. Since a countable union of sets of Lebesgue measure zero has also measure zero, it suffices to consider bounded sets F.

Language and notation are cumbersome for the ideas we need in $\boldsymbol{R}^n$. We will give the proof for $\boldsymbol{R}^2$ — the way to generalize it will be clear.

Let $F_x = \{(u, v) \in F \mid u = x\}$ be the vertical section of F at the point (x, y), and $F^y = \{(u, v) \in F \mid v = y\}$ the horizontal one. By the Lebesgue density theorem ([5], p. 17), almost every (linear Lebesgue measure) $p \in F_x$ is a linear density point on F_x. By the Fubini theorem ([6], p. 77) almost every (plane Lebesgue measure) $(x, y) \in F$ is a linear density point on its section F_x. Similarly, almost every point (x, y) of F is a linear density point on F^y. Let F' be the set of exceptional points, so that every $(x, y) \in F \setminus F'$ is a linear density point on both F_x and F_y.

For given $\delta > 0$ and $(x, y) \in F$ pick η, $0 < \eta < \delta$, so that all four of the sets

$$(x, x+\eta) \times \{y\} \cap F_y, \qquad (x-\eta, x) \times \{y\} \cap F_y,$$

$$\{x\} \times (y, y+\eta) \cap F_x, \qquad \{x\} \times (y-\eta, y) \cap F_x$$

have linear measure greater than $3\eta/4$. Rotate the first three of these linear sets about (x, y) to coincide with the fourth and intersect the four resulting sets. By the choice of η the intersection has positive measure, and so contains a point $(x, y-\gamma)$ with $0 < \gamma < \eta$. The γ-star at (x, y) belongs to F.

Now we can put the pieces together.

Proof of Theorem 1. Again by countable additivity of Lebesgue measure we may restrict our attention to bounded sets E.

Let $F = \bar{E}$, and let F' be as in Lemma 3. For each $x \in F \setminus F'$ and each γ such that there is a γ-star at x contained in F let $P(x, \gamma)$ be the γ-dual-cube at x. From Lemma 3 it follows that the $P(x, \gamma)$'s form a Vitali covering of $F \setminus F'$ ([6], p. 109). Consequently, by the Vitali covering theorem, we may pick a countable set of the $P(x, \gamma)$'s, say $P_1, P_2, \ldots$, pairwise disjoint, such that

$$\lambda\Big((F \setminus F') \setminus \bigcup_{i=1}^{\infty} P_i\Big) = 0,$$

where λ is Lebesgue measure. Then also

$$\lambda\Big(F \setminus \bigcup_{i=1}^{\infty} P_i\Big) = 0.$$

Now ϱ, and hence f, is uniformly continuous. So f extends to a ϱ-isometry $\bar{f}\colon F \to R^n$. Since $\bar{f}$ is defined on the star corresponding to each P_i, $i \geqslant 1$, it follows from Lemma 2 that $\bar{f}$, and hence f, is Euclidean on $P_i \cap E$.

COROLLARY (see also [3]). *Lebesgue measure is ϱ-invariant.*

Proof. Let E, E' be Lebesgue measurable and $f\colon E \to E'$ a ϱ-isometry. Set $E_i = E \cap P_i$ for $i \geqslant 1$, $E_0 = E \setminus \bigcup_{i=1}^{\infty} E_i$, and $E'_i = f(E_i)$ for $i \geqslant 0$. Theorem 1 gives $\lambda(E'_i) = \lambda(E_i)$ for $i \geqslant 1$. Since $\lambda(E_0) = 0$, we get $\lambda(E'_0) \geqslant \lambda(E_0)$ by default. (Note that $E'_0 = E' \setminus \bigcup_{i=1}^{\infty} E'_i$ is measurable.) So $\lambda(E) \leqslant \lambda(E')$. By symmetry, $\lambda(E) = \lambda(E')$.

Actually, we do not need to assume that E' is measurable. f is $\sqrt{n}$-Lipschitz, and this together with the measurability of E implies that E' is measurable.

Proof of Theorem 2. Let a bounded subset E of $\boldsymbol{R}^2$ and a ϱ-isometry $f\colon E \to \boldsymbol{R}^2$ be given. Let $P_1, P_2, \ldots$ be given as in Theorem 1, set $E_i = E \cap P_i$ for $i \geqslant 1$, and set

$$E_0 = E \setminus \bigcup_{i=1}^{\infty} E_i.$$

Since E is bounded and the P_i are pairwise disjoint, we have

$$\lambda\Big(\bigcup_{i=n}^{\infty} P_i\Big) \to 0 \quad \text{as } n \to \infty.$$

Hence given $\varepsilon > 0$ we can pick n so that

$$\lambda^*\Big(\bigcup_{i=n}^{\infty} E_i\Big) < \varepsilon$$

(where λ^* is Lebesgue outer measure).

Let μ be a finitely additive universal extension of Lebesgue measure, invariant under Euclidean isometries. We get

$$\mu\big(f(E)\big) + \varepsilon \geqslant \sum_{i=1}^{n-1} \mu\big(f(E_i)\big) + \varepsilon > \sum_{i=1}^{n-1} \mu\big(f(E_i)\big) + \mu\Big(\bigcup_{i=n}^{\infty} E_i\Big)$$
$$= \sum_{i=1}^{n-1} \mu(E_i) + \mu\Big(\bigcup_{i=n}^{\infty} E_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \mu\Big(\bigcup_{i=0}^{\infty} E_i\Big) = \mu(E).$$

Since ε is arbitrary, $\mu\big(f(E)\big) \geqslant \mu(E)$. By symmetry, $\mu\big(f(E)\big) = \mu(E)$.

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