THE MAX NORM IN $\mathbb{R}^n$-ISOMETRIES AND MEASURE

by

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In this paper* we show that any isometry $f$ between two subsets $E$ and $F$ of $\mathbb{R}^n$, relative to the metric

\[(*) \quad \rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|,\]

is locally Euclidean, in the sense that $E$ (minus a Lebesgue null set) can be partitioned into countably many pieces (each measurable if $E$ is), on each of which $f$ is an ordinary Euclidean isometry.

It follows easily that if $E$ and $F$ are measurable, then they have the same Lebesgue measure. More interestingly, we also show, for $n = 2$, that if $E$ and $F$ are bounded sets (measurable or not), then they are assigned the same measure by all Banach universal extensions of Lebesgue measure.

1. Introduction and statement of results. Let $\rho$ be the metric in $\mathbb{R}^n$ given by $(*)$. For $E, F \subseteq \mathbb{R}^n$, $f: E \to F$ is a $\rho$-isometry, and $E$ and $F$ are $\rho$-isometric, if $f$ is surjective and

$$\rho(f(x), f(y)) = \rho(x, y) \quad \text{for all } x, y \in E.$$  

(Isometries relative to the usual metric will now be called Euclidean to distinguish them clearly from $\rho$-isometries.)

This paper is a study of $\rho$-isometries and their relation to measure.

The $\rho$-isometries of the whole space form a (rather small) subgroup of the group of Euclidean isometries (see Lemma 1 below), but $\rho$-isometries on smaller sets can be quite unlike Euclidean isometries:

Example. Let $E = [-1/2, 1/2] \times \{0\}$ and define $f: E \to \mathbb{R}^2$ by

$$(x, 0) \mapsto (x, x^2)$$

* By permission of the first-named author, these results first appeared in the Ph. D. Thesis (University of Colorado, 1979) of the second-named author.
(Fig. 1). Since for \(-1/2 \leq x, y \leq 1/2\) we have \(|x + y| \leq 1\), and hence also
\[ \varrho((x, x^2), (y, y^2)) = \max(|x - y|, |x - y| |x + y|) = |x - y| \]
\[ = \varrho((x, 0), (y, 0)), \]
f is a \(\varrho\)-isometry.

Our main result is

**Theorem 1.** Let \(E\) be any subset of \(\mathbb{R}^n\) and \(\overline{E}\) the closure of \(E\). There exist pairwise disjoint closed polyhedral regions \(P_1, P_2, \ldots\) in \(\mathbb{R}^n\) such that

(a) for any \(\varrho\)-isometry \(f: E \rightarrow \mathbb{R}^n\) and any \(i \geq 1\), the restriction of \(f\) to \(E \cap P_i\) is Euclidean;

(b) \(\overline{E} \setminus \bigcup_{i=1}^{\infty} P_i\) has Lebesgue measure zero.

Condition (b) implies that the "bad" part \(E \setminus \bigcup_{i=1}^{\infty} P_i\) of \(E\) has measure zero and is nowhere dense.

Theorem 1 yields easily that Lebesgue measure is \(\varrho\)-invariant, i.e. whenever \(E\) and \(F\) are \(\varrho\)-isometric measurable sets, they have the same Lebesgue measure. This is a special case of the main theorem of [3], where it is proved that Lebesgue measure is \(\sigma\)-invariant for any "reasonable" translation invariant metric \(\sigma\) in \(\mathbb{R}^n\) consistent with the usual topology.

A theorem proved by Banach and Tarski in 1924 should be recalled here (see [2]): Two measurable subsets \(E\) and \(F\) of \(\mathbb{R}^n\) have the same Lebesgue measure iff \(E\) can be partitioned into measurable parts \(E_0, E_1, \ldots\) and \(F\) into \(F_0, F_1, \ldots\) in such a way that \(E_0\) and \(F_0\) have Lebesgue measure 0 and, for each \(i \geq 1\), \(E_i\) is isometric (in the Euclidean sense) to \(F_i\).

If the set \(E\) in Theorem 1 is bounded, then the outer measure of \(\bigcup_{i=n}^{\infty} E \cap P_i\) tends to 0 as \(n\) tends to \(\infty\). Thus this theorem allows us to study \(\varrho\)-invariance for some finitely additive measures.

**Theorem 2.** Any Banach measure in the plane, if restricted to bounded sets, is \(\varrho\)-invariant. That is, if \(E, F \subseteq \mathbb{R}^2\) are bounded and \(\varrho\)-isometric (but not necessarily measurable), and \(\mu\) is a universal, finitely additive extension of Lebesgue measure in \(\mathbb{R}^2\), invariant under Euclidean isometries, then \(\mu(E) = \mu(F)\).

(The first Banach measure was described in [1]. See [4] for a survey.)
By using linear transformations in a routine way we can extend Theorem 1 (and hence Theorem 2) to metrics arising from norms whose unit balls are parallelepipeds (parallelograms for Theorem 2). And it is not hard to adapt the proof of Theorem 1 given below to norms whose unit balls are some other common polyhedral shapes.

**Conjecture.** Theorem 1 holds for any translation invariant metric consistent with the usual topology on \( \mathbb{R}^n \). (P 1264)

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2. **Sets on which \( \varrho \)-isometries are Euclidean.** The strategy for proving Theorem 1 is first to find a large class of simple sets on which every \( \varrho \)-isometry must be Euclidean, and then to show how to divide up an arbitrary set into countably many such subsets (modulo a very "thin" set).

**Definitions.** For \( x \in \mathbb{R}^n \) and \( \varrho \geq 0 \) we define the \( \varrho \)-star at \( x \) to be the set \( \{x\} \cup \{(x_1, \ldots, x_i \pm \varrho, \ldots, x_n) \mid i = 1, \ldots, n\} \) (Fig. 2).

![Fig. 2. The \( \varrho \)-star at (0, 0) in \( \mathbb{R}^2 \)](image)

The \( \varrho \)-dual-cube at \( x \) is the convex hull of the \( \varrho \)-star at \( x \).

**Lemma 1.** Let \( S \) be a \( \varrho \)-star at \( x \in \mathbb{R}^n \) and \( f: S \to \mathbb{R}^n \) a \( \varrho \)-isometry. Then \( f \) is Euclidean and \( f(S) \) is a \( \varrho \)-star at \( f(x) \).

**Proof.** The proof is by induction on \( n \). The lemma is trivial in \( \mathbb{R} \). In \( \mathbb{R}^n \), \( n > 1 \), \( f((x_1 + \varrho, x_2, \ldots, x_n)) \) and \( f((x_1 - \varrho, x_2, \ldots, x_n)) \) lie on opposite faces of the cube \( \{p \mid \varrho(p, f(x)) = \varrho\} \), and so, for \( i \neq 1 \), the point \( f((x_1, \ldots, x_i \pm \varrho, x_n)) \) lies in a hyperplane parallel to this pair of faces and passing through \( f(x) \). By the induction hypothesis the lemma holds for \( f \) restricted to the \((n-1)\)-dimensional \( \varrho \)-star

\[
\{x\} \cup \{(x_1, \ldots, x_i \pm \varrho, \ldots, x_n) \mid i = 2, \ldots, n\},
\]

and now it is easy to check that \( f \) is Euclidean on \( S \) and that \( f(S) \) is a \( \varrho \)-star.

By way of a puzzle, we mention that it seems as if \( \varrho \)-stars are the minimal configurations on which \( \varrho \)-isometries must be Euclidean. To be precise,
CONJECTURE. If a subset $S$ of $\mathbb{R}^n$ has no more than $2n + 1$ points and is not a $\gamma$-star, then there is a $\epsilon$-isometry from $S$ into $\mathbb{R}^n$ which is not Euclidean. (P 1265)

Returning to the business at hand,

**Lemma 2.** Let $S$ be the $\gamma$-star at $x \in \mathbb{R}^n$ and $E \supseteq S$ a subset of the $\gamma$-dual-cube at $x$. Then any $\epsilon$-isometry $f: S \to \mathbb{R}^n$ has a unique extension to $E$. In particular (by Lemma 1), any $\epsilon$-isometry on $E$ is Euclidean.

**Proof.** It suffices to show that any point in the $\gamma$-dual-cube at $x$ is uniquely determined among all points of $\mathbb{R}^n$ by its $\epsilon$-distances to the points of the $\gamma$-star at $x$. In other words, it suffices to show that if for some $d_1, d_1', d_2, d_2', \ldots, d_n, d_n' \geq 0$ the intersection of $n$-cubes

$$I = \bigcap_{i=1}^{n} \{ p \mid \epsilon(p, (x_1, \ldots, x_i + \gamma, \ldots, x_n)) = d_i \} \cap \bigcap_{i=1}^{n} \{ p \mid \epsilon(p, (x_1, \ldots, x_i - \gamma, \ldots, x_n)) = d_i' \}$$

contains a point $p_0$ in the $\gamma$-dual-cube at $x$, then, in fact, $I = \{p_0\}$.

Now the intersection of any $n$-cube of the form

$$\{ p \mid \epsilon(p, (x_1, \ldots, x_i \pm \gamma, \ldots, x_n)) = d \}$$

with the $\gamma$-dual-cube at $x$ is a part of one of the faces of the $n$-cube, perpendicular to the $x_i$-axis. So if $I$ contains a point $p_0$ of the $\gamma$-dual-cube at $x$, then the intersection

$$\{ p \mid \epsilon(p, (x_1, \ldots, x_i + \gamma, \ldots, x_n)) = d_i \} \cap \{ p \mid \epsilon(p, (x_1, \ldots, x_n - \gamma, \ldots, x_n)) = d_i' \}$$

must be contained in a hyperplane perpendicular to the $x_i$-axis for each $i$. In other words, there is only one possible value for the $i$-th coordinate of any $p \in I$, for $i = 1, \ldots, n$, and so $I$ must contain only a single point.

3. **Construction of the partition.** We are now ready to begin construction of the $P_i$ of Theorem 1.

**Lemma 3.** Let $F$ be a closed subset of $\mathbb{R}^n$. There exists $F' \subseteq F$, of Lebesgue measure 0, such that for every $\delta > 0$ and every $p \in F \setminus F'$ there is a $\gamma$-star at $p$ contained in $F'$, with $0 < \gamma < \delta$.

**Proof.** Since a countable union of sets of Lebesgue measure zero has also measure zero, it suffices to consider bounded sets $F$.

Language and notation are cumbersome for the ideas we need in $\mathbb{R}^n$. We will give the proof for $\mathbb{R}^2$ — the way to generalize it will be clear.

Let $F_x = \{(u, v) \in F \mid u = x\}$ be the vertical section of $F$ at the point $(x, y)$, and $F_y = \{(u, v) \in F \mid v = y\}$ the horizontal one. By the Lebesgue density theorem ([5], p. 17), almost every (linear Lebesgue
measure) $p \in F_x$ is a linear density point on $F_x$. By the Fubini theorem ([6], p. 77) almost every (plane Lebesgue measure) $(x, y) \in F$ is a linear density point on its section $F_x$. Similarly, almost every point $(x, y)$ of $F$ is a linear density point on $F^y$. Let $F''$ be the set of exceptional points, so that every $(x, y) \in F \setminus F'$ is a linear density point on both $F_x$ and $F^y$.

For given $\delta > 0$ and $(x, y) \in F$ pick $\eta$, $0 < \eta < \delta$, so that all four of the sets

$$(x, x + \eta) \times \{y\} \cap F_y, \quad (x - \eta, x) \times \{y\} \cap F_y,$$

$$\{x\} \times (y, y + \eta) \cap F_x, \quad \{x\} \times (y - \eta, y) \cap F_x$$

have linear measure greater than $3\eta/4$. Rotate the first three of these linear sets about $(x, y)$ to coincide with the fourth and intersect the four resulting sets. By the choice of $\eta$ the intersection has positive measure, and so contains a point $(x, y - \gamma)$ with $0 < \gamma < \eta$. The $\gamma$-star at $(x, y)$ belongs to $F$.

Now we can put the pieces together.

**Proof of Theorem 1.** Again by countable additivity of Lebesgue measure we may restrict our attention to bounded sets $E$.

Let $F = \overline{F}$, and let $F'$ be as in Lemma 3. For each $x \in F \setminus F'$ and each $\gamma$ such that there is a $\gamma$-star at $x$ contained in $F$ let $P(x, \gamma)$ be the $\gamma$-dual-cube at $x$. From Lemma 3 it follows that the $P(x, \gamma)$'s form a Vitali covering of $F \setminus F'$ ([6], p. 109). Consequently, by the Vitali covering theorem, we may pick a countable set of the $P(x, \gamma)$'s, say $P_1, P_2, \ldots$, pairwise disjoint, such that

$$\lambda((F \setminus F') \setminus \bigcup_{i=1}^{\infty} P_i) = 0,$$

where $\lambda$ is Lebesgue measure. Then also

$$\lambda(F \setminus \bigcup_{i=1}^{\infty} P_i) = 0.$$

Now $g$, and hence $f$, is uniformly continuous. So $f$ extends to a $g$-isometry $\tilde{f}: F \to \mathbb{R}^n$. Since $\tilde{f}$ is defined on the star corresponding to each $P_i$, $i \geq 1$, it follows from Lemma 2 that $\tilde{f}$, and hence $f$, is Euclidean on $P_i \cap E$.

**Corollary** (see also [3]). *Lebesgue measure is $g$-invariant.*

**Proof.** Let $E, E'$ be Lebesgue measurable and $f: E \to E'$ a $g$-isometry.

Set $E_i = E \cap P_i$ for $i \geq 1$, $E_0 = E \setminus \bigcup_{i=1}^{\infty} E_i$, and $E_i' = f(E_i)$ for $i \geq 0$.

Theorem 1 gives $\lambda(E_i') = \lambda(E_i)$ for $i \geq 1$. Since $\lambda(E_0) = 0$, we get $\lambda(E_0') \geq \lambda(E_0)$ by default. (Note that $E_0' = E' \setminus \bigcup_{i=1}^{\infty} E_i'$ is measurable.) So $\lambda(E) \leq \lambda(E')$. By symmetry, $\lambda(E) = \lambda(E')$. 

Actually, we do not need to assume that $E'$ is measurable. $f$ is $Vn$-Lipschitz, and this together with the measurability of $E$ implies that $E'$ is measurable.

Proof of Theorem 2. Let a bounded subset $E$ of $R^2$ and a $\varphi$-isometry $f: E \to R^3$ be given. Let $P_1, P_2, \ldots$ be given as in Theorem 1, set $E_i = E \cap P_i$ for $i \geq 1$, and set

$$E_0 = E \setminus \bigcup_{i=1}^{\infty} E_i.$$ 

Since $E$ is bounded and the $P_i$ are pairwise disjoint, we have

$$\lambda(\bigcup_{i=n}^{\infty} P_i) \to 0 \quad \text{as} \quad n \to \infty.$$ 

Hence given $\varepsilon > 0$ we can pick $n$ so that

$$\lambda^*\left(\bigcup_{i=n}^{\infty} E_i\right) < \varepsilon$$

(where $\lambda^*$ is Lebesgue outer measure).

Let $\mu$ be a finitely additive universal extension of Lebesgue measure, invariant under Euclidean isometries. We get

$$\mu(f(E)) + \varepsilon \geq \sum_{i=1}^{n-1} \mu(f(E_i)) + \varepsilon > \sum_{i=1}^{n-1} \mu(f(E_i)) + \mu\left(\bigcup_{i=n}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{n-1} \mu(E_i) + \mu\left(\bigcup_{i=n}^{\infty} E_i\right) = \mu\left(\bigcup_{i=n}^{\infty} E_i\right) = \mu(E).$$

Since $\varepsilon$ is arbitrary, $\mu(f(E)) \geq \mu(E)$. By symmetry, $\mu(f(E)) = \mu(E)$.

REFERENCES


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