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ON RICCI-RECURRENT MANIFOLDS

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1. Let M be a Riemannian manifold with (possibly indefinite) metric g. By R_{hijk} and R_{ij} we denote local coordinates of its curvature tensor and Ricci tensor, respectively, and by R its scalar curvature. The Weyl conformal tensor of M ($\dim M = n$) is defined by

$$(1)\qquad C_{hijk} = R_{hijk} - \frac{1}{n-2}(R_{hk}g_{ij} - R_{hj}g_{ik} + R_{ij}g_{hk} - R_{ik}g_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}).$$

A tensor field T of type (p, q) on M is said to be *recurrent* if

$$(2)\qquad T^{h_1\ldots h_p}{}_{i_1\ldots i_q} T^{j_1\ldots j_p}{}_{k_1\ldots k_q,l} = T^{j_1\ldots j_p}{}_{k_1\ldots k_q} T^{h_1\ldots h_p}{}_{i_1\ldots i_q,l},$$

where the comma indicates covariant differentiation with respect to the metric g. Relation (2) states that at any point $x \in M$ at which $T(x) \neq 0$ there is a (unique) covariant vector u (called the *recurrence vector* of T) which satisfies the condition

$$(3)\qquad T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k}(x) = T^{i_1\ldots i_p}{}_{j_1\ldots j_q}(x)\, u_k(x).$$

M is said to be *Ricci-recurrent* (*conformally recurrent*) if its Ricci tensor (Weyl conformal tensor) is recurrent (cf. [1], [3] and [5]).

2. Roter [4] has proved that in a Ricci-recurrent space whose recurrence vector does not vanish and is locally gradient the Ricci tensor fulfils the identity

$$(4)\qquad R_{is}R_j^{\ s} = \frac{R}{2}R_{ij}.$$

As we shall show below, this identity remains true even though the recurrence vector should not be locally a gradient. But first we state a certain new example of a Ricci-recurrent manifold whose recurrence vector is not locally a gradient.

Example 1. Let M be the Euclidean n-space endowed with the metric g given by

$$g_{ij}\,dx^i\,dx^j = Q(dx^1)^2 + k_{\alpha\beta}\,dx^\alpha\,dx^\beta + 2dx^1\,dx^n,$$

where $\alpha,\ \beta = 2,\ 3, \ldots, n-1$ $(n \geqslant 4)$, $[k_{\alpha\beta}]$ is a non-singular indefinite symmetric matrix consisting of constants, $Q = 2A_\alpha x^\alpha x^n$, and $A_2, \ldots, A_{n-1}$ are constants such that $A_2^2 + \ldots + A_{n-1}^2 > 0$ and $k^{\alpha\beta} A_\alpha A_\beta = 0$, $[k^{\alpha\beta}]$ being the reciprocal of $[k_{\alpha\beta}]$. We see that the non-vanishing Christoffel symbols of the metric are the following

$$\left\{ {1 \atop 1\ 1} \right\} = -A_\alpha x^\alpha, \quad \left\{ {\alpha \atop 1\ 1} \right\} = -k^{\alpha\beta} A_\beta x^n, \quad \left\{ {n \atop 1\ 1} \right\} = 2(A_\alpha x^\alpha)^2 x^n,$$

$$\left\{ {n \atop 1\ \alpha} \right\} = A_\alpha x^n, \quad \left\{ {n \atop 1\ n} \right\} = A_\alpha x^\alpha.$$

Therefore, the non-vanishing components of the curvature tensor, the Ricci tensor, and its covariant derivative are

$$R_{1\alpha n1} = A_\alpha, \qquad R_{1\alpha} = -A_\alpha, \qquad R_{1\alpha,1} = -A_\alpha A_\beta x^\beta.$$

Thus the Ricci tensor does not vanish everywhere on M and satisfies the condition $R_{ij,k} = R_{ij}\psi_k$ with $\psi_1 = A_\beta x^\beta$, $\psi_\alpha = \psi_n = 0$.

PROPOSITION 1. *Let M be a Ricci-recurrent manifold such that the set $U = \{x \in M \mid \psi(x) \neq 0\}$ is non-empty, ψ being the recurrence vector of the Ricci tensor. Then identity* (4) *is fulfilled on M.*

Proof. First we restrict ourselves to the set U. We have

$$R_{ij,k} = R_{ij}\psi_k. \tag{5}$$

The conditions of integrability of equations (5) are

$$R_{ij}(\psi_{k,l} - \psi_{l,k}) = -R_i^{\ s} R_{sjkl} - R_j^{\ s} R_{sikl}.$$

Hence, by the covariant differentiation and (5), we get

$$R_{ij}(\psi_{k,lm} - \psi_{l,km}) = -R_i^{\ s} R_{sjkl,m} - R_j^{\ s} R_{sikl,m}.$$

Contracting this equality with g^{lm}, putting $\xi_k = g^{lm}(\psi_{k,lm} - \psi_{l,km})$, and using (5) and the identity $R^s{}_{ijk,s} = R_{ij,k} - R_{ik,j}$, we obtain

$$R_{ij}\xi_k = -R_i^{\ s}(R_{kj}\psi_s - R_{ks}\psi_j) - R_j^{\ s}(R_{ki}\psi_s - R_{ks}\psi_i). \tag{6}$$

But by (5) and the identity $\frac{1}{2}R_{,i} = R_i^{\ s}{}_{,s}$ we have $R_i^{\ s}\psi_s = \frac{1}{2}R\psi_i$, which reduces (6) to the form

$$R_{ij}\xi_k = \left(R_{is}R_k^{\ s} - \frac{R}{2}R_{ik}\right)\psi_j + \left(R_{js}R_k^{\ s} - \frac{R}{2}R_{jk}\right)\psi_i. \tag{7}$$

We shall prove that $\xi_k = 0$ on U. To do this, assume that $\xi_k \neq 0$ at some point of U and restrict considerations to this point. From (7) we can easily deduce that the Ricci tensor takes the form $R_{ij} = \sigma_i \psi_j + \sigma_j \psi_i$ for a certain non-zero covariant vector σ, and, consequently, $R = 2\sigma_s \psi^s$. Therefore, (7) yields

$$(\sigma_i \sigma_k \psi_s \psi^s + \psi_i \psi_k \sigma_s \sigma^s - \sigma_i \xi_k)\psi_j + (\sigma_j \sigma_k \psi_s \psi^s + \psi_j \psi_k \sigma_s \sigma^s - \sigma_j \xi_k)\psi_i = 0,$$

whence

$$\sigma_j \sigma_k \psi_s \psi^s + \psi_j \psi_k \sigma_s \sigma^s - \sigma_j \xi_k = 0. \tag{8}$$

The last relation leads immediately to $\sigma_j \xi_k = \sigma_k \xi_j$. Hence $\sigma_j = \lambda \xi_j$, $\lambda \neq 0$. Therefore, $R_{ij} = \lambda(\xi_i \psi_j + \xi_j \psi_i)$ and $R = 2\lambda \xi_s \psi^s$, which used in the identity $R_i^s \psi_s = (R/2)\psi_i$ gives $\psi_s \psi^s = 0$. Formula (8) in virtue of the above equality implies $\sigma_i = \mu \psi_i$. Consequently, $\sigma_s \sigma^s = 0$ and (8) takes finally the form $\sigma_j \xi_k = 0$, which is a contradiction. Thus $\xi_k = 0$ on U.

Define a tensor field S on M by $S_{ij} = R_{is} R_j^s - (R/2) R_{ij}$. By (7) and the equality $\xi_k = 0$, the tensor S vanishes on U. Consequently, S is covariantly constant on U. Clearly, it is covariantly constant at the remaining points of M. In this case, therefore, $S = 0$ everywhere on M, which completes the proof.

Remark. The assertion of Proposition 1 can also be deduced from Theorems 1–3 of Patterson [3]. However, our proof is simpler.

We say that the Ricci tensor of M is of *constant rank* p if the matrix $[R_{ij}]$ has the rank p at any point of M. Note that if the Ricci tensor of a Ricci-recurrent manifold is non-zero everywhere on M, then it has a constant positive rank. Indeed, in this case the recurrence vector is defined on the whole M and we can use [2], p. 153. In our Example 1 and Example 2 below the Ricci tensor is of constant rank.

THEOREM 1. *Let M be a Ricci-recurrent manifold with vanishing scalar curvature and Ricci tensor non-vanishing everywhere on M. Assume additionally that the recurrence vector of the Ricci tensor does not vanish at certain points of M. Then the assignment to each $x \in M$ of the set D_x of all vectors $X \in T_x M$ given by $X^i = R_s^i Y^s$, where $Y \in T_x M$, defines a p-dimensional parallel and isotropic distribution D on M, where $p \leqslant n/2$ is the rank of the Ricci tensor.*

Proof. It remains to prove that D is parallel and isotropic. By the equality $R = 0$ and Proposition 1 it is easy to check that $R_{is} R_j^s = 0$, which implies the isotropy of D and $p \leqslant n/2$. On the other hand, using the relation $R_{ij,k} = R_{ij} \psi_k$ we derive

$$Z^r (R_s^i Y^s)_{,r} = R_s^i (Z^r \psi_r Y^s + Z^r Y^s_{\ ,r})$$

for arbitrary vector fields Y and Z on M. This gives the parallelity of D and completes the proof.

Example 2. Let n and p be natural numbers such that $n \geq 4$ and $1 \leq p < n/2$; and indices i, j run over $1, 2, \ldots, n$, indices α, β over $p+1, p+2, \ldots, n-p$, and indices a, b over $1, 2, \ldots, p$. Let M denote the Euclidean n-space endowed with the metric g given by

$$g_{ij}\,dx^i\,dx^j = \sum_a dx^a(Q dx^a + 2dx^{n-p+a}) + e_\alpha (dx^\alpha)^2,$$

where $e_\alpha = -1$ or $+1$, Q is a function of $x^{p+1}, \ldots, x^{n-p}$ only and such that $\delta^{\alpha\beta} e_\alpha Q_{.\alpha\beta}$ is non-constant and non-zero everywhere on M (for instance, $Q = \exp(x^{p+1})$), the dot being the partial differentiation. The non-vanishing Christoffel symbols of our metric are

$$\left\{ \begin{matrix} \alpha \\ a\ a \end{matrix} \right\} = -\frac{1}{2}\delta^{\alpha\beta} e_\alpha Q_{.\beta}, \qquad \left\{ \begin{matrix} n-p+a \\ a\ \alpha \end{matrix} \right\} = \frac{1}{2} Q_{.\alpha}$$

(no summation over a). Therefore, the non-vanishing components of the curvature tensor are

$$R_{a\alpha\beta a} = \tfrac{1}{2} Q_{.\alpha\beta}, \qquad R_{abba} = \tfrac{1}{4}\delta^{\alpha\beta} e_\alpha Q_{.\alpha} Q_{.\beta} \ (b \neq a),$$

and, consequently, the non-vanishing components of the Ricci tensor and its covariant derivative are those related to

$$R_{aa} = \tfrac{1}{2}\delta^{\alpha\beta} e_\alpha Q_{.\alpha\beta}, \qquad R_{aa,\alpha} = R_{aa.\alpha}.$$

Thus on M we have $R_{ij,k} = R_{ij}\psi_k$ with $\psi_k = (\log|\delta^{\alpha\beta} e_\alpha Q_{.\alpha\beta}|)_{,k}$.

3. As we know, in a Ricci-recurrent manifold the recurrence vector need not be locally a gradient. However, it must be locally a gradient if the manifold is additionally conformally recurrent. To see this, the following lemma will be necessary:

LEMMA. *Let $\sigma_1, \ldots, \sigma_N$ and $\omega_1, \ldots, \omega_N$ be two sequences of numbers which are linearly independent as elements of the Cartesian space R^N. Let T_{AB} and S_{AB} $(A, B = 1, 2, \ldots, N)$ be numbers such that $T_{BA} = T_{AB}$, $S_{AB} = S_{BA}$, and*

$$(9) \qquad T_{AB}\sigma_C + T_{BC}\sigma_A + T_{CA}\sigma_B + S_{AB}\omega_C + S_{BC}\omega_A + S_{CA}\omega_B = 0.$$

Then there are numbers $\theta_1, \ldots, \theta_N$ for which

$$(10) \qquad T_{AB} = -\omega_A\theta_B - \omega_B\theta_A, \qquad S_{AB} = \sigma_A\theta_B + \sigma_B\theta_A.$$

Proof. Take numbers $X^1, \ldots, X^N$ and $Y^1, \ldots, Y^N$ so that $\sigma_C X^C = 1$, $\sigma_C Y^C = 0$, $\omega_C X^C = 0$, and $\omega_C Y^C = 1$. Transvecting (9) with $X^A X^B X^C$ we obtain $T_{AB} X^A X^B = 0$. Therefore, the transvection of (9) with $X^B X^C$ gives $T_{AC} X^C = \lambda\omega_A$ for a certain number λ. Consequently, if we transvect (9) with X^C, we find

$$(11) \qquad T_{AB} = \omega_A \xi_B + \omega_B \xi_A$$

for certain numbers $\xi_1, \ldots, \xi_N$. In a similar manner, but using transvections of (9) with Y's, we conclude that

(12) $$S_{AB} = \sigma_A \theta_B + \sigma_B \theta_A$$

for certain numbers $\theta_1, \ldots, \theta_N$. Now, transvecting (9) with X^C we get, by (11) and (12),

$$\omega_A \{\xi_B + \theta_B + (\xi_C + \theta_C) X^C \sigma_B\} + \omega_B \{\xi_A + \theta_A + (\xi_C + \theta_C) X^C \sigma_A\} = 0.$$

Hence $\xi_B + \theta_B + (\xi_C + \theta_C) X^C \sigma_B = 0$. On the other hand, the transvection of (9) with Y^C and the use of (11) and (12) lead immediately to $\xi_B + \theta_B + (\xi_C + \theta_C) Y^C \omega_B = 0$. By the linear independence, the above formulae imply $\xi_A = -\theta_A$, which together with (11) and (12) completes the proof.

The following well-known fact is a consequence of our lemma:

COROLLARY. *Let σ_A and T_{AB} $(A, B = 1, 2, \ldots, N)$ be numbers satisfying the conditions $\sigma_1^2 + \ldots + \sigma_N^2 > 0$, $T_{AB} = T_{BA}$, and*

(13) $$T_{AB} \sigma_C + T_{BC} \sigma_A + T_{CA} \sigma_B = 0.$$

Then each T_{AB} is zero.

Now we are in a position to prove the following

PROPOSITION 2. *Let M be a Riemannian manifold of dimension greater than or equal to 4 and let U_1 (respectively, U_2) be the subset of M consisting of points at which the Ricci tensor (respectively, Weyl conformal tensor) is non-zero. Assume that on U_1*

(14) $$R_{ij,[kl]} = R_{ij,kl} - R_{ij.lk} = R_{ij} a_{kl}$$

for a certain tensor field a, and on U_2

(15) $$C_{hijk,[lm]} = C_{hijk} b_{lm}$$

for a certain tensor field b. Then $a_{ij} = 0$ everywhere on U_1 and $b_{ij} = 0$ everywhere on U_2.

Proof. The assertion is clear if the metric of M is definite. Indeed, it is sufficient to transvect (14) (respectively, (15)) with R^{ij} (respectively, C^{hijk}) and apply the Ricci identity. Thus, in the sequel, the metric is assumed to be indefinite.

The following identity, valid in any Riemannian manifold, will be necessary in this proof:

(16) $$R_{hijk,[lm]} + R_{jklm,[hi]} + R_{lmhi,[jk]} = 0.$$

First, consider the set $U_1 \setminus U_2$. We have $C_{hijk} = 0$, and by the Ricci identity we obtain $C_{hijk,[lm]} = 0$. Therefore, by (1) and (14) we get $R_{hijk,[lm]} = R_{hijk} a_{lm}$, which used in (16) gives

$$R_{hijk} a_{lm} + R_{jklm} a_{hi} + R_{lmhi} a_{jk} = 0.$$

This identity, for a fixed point of $U_1 \setminus U_2$, is of the form (13), indices A, B, C being replaced by the pairs hi, jk, lm, respectively. Thus from Corollary 1 we obtain $a_{ij} = 0$ everywhere on $U_1 \setminus U_2$, since the curvature tensor is non-zero at any point of this set.

In the second step, consider the set $U_2 \setminus U_1$. We have $R_{ij} = 0$, and by the Ricci identity we obtain $R_{ij,[kl]} = 0$. This together with (15) and (1) leads to $R_{kijh,[lm]} = C_{kijh} b_{lm}$, which substituted to (16) gives

$$C_{hijk} b_{lm} + C_{jklm} b_{hi} + C_{lmhi} b_{jk} = 0.$$

This identity is of the form (13). Hence $b_{ij} = 0$ everywhere on $U_2 \setminus U_1$.

Now we are in the set $U_1 \cap U_2$ if it is non-empty. By (1), (14), and (15) we obtain

$$R_{hijk,[lm]} = (R_{hijk} - C_{hijk}) a_{lm} + C_{hijk} b_{lm},$$

which together with (16) gives

$$\text{(17)} \quad (R_{hijk} - C_{hijk}) a_{lm} + (R_{jklm} - C_{jklm}) a_{hi} + (R_{lmhi} - C_{lmhi}) a_{jk} + \\ + C_{hijk} b_{lm} + C_{jklm} b_{hi} + C_{lmhi} b_{jk} = 0.$$

Fix an arbitrary point of $U_1 \cap U_2$ and restrict considerations to this point. Remark that (17) is of the form (9). First we show that a and b must be linearly dependent. Indeed, otherwise, by our Lemma we would have

$$C_{hijk} - R_{hijk} = b_{hi} c_{jk} + b_{jk} c_{hi}$$

for a certain skew-symmetric tensor c, which in view of (1) can be written as

$$\text{(18)} \quad R_{hk} g_{ij} - R_{hj} g_{ik} + R_{ij} g_{hk} - R_{ik} g_{hj} - \frac{R}{n-1}(g_{hk} g_{ij} - g_{hj} g_{ik}) \\ = (n-2)(b_{hi} c_{jk} + b_{jk} c_{hi}).$$

Assume that $R_{sr} X^s X^r = 0$ for any isotropic vector X. Then we have $R_{ij} = (R/n) g_{ij}$, which reduces (18) to

$$\frac{R}{n(n-1)}(g_{hk} g_{ij} - g_{hj} g_{ik}) = b_{hi} c_{jk} + b_{jk} c_{hi}.$$

Hence it is easy to check that $R = 0$, whence $R_{ij} = 0$, a contradiction. On the contrary, let X be an isotropic vector such that $R_{sr} X^s X^r \neq 0$. Transvecting (18) with $X^h X^k$ we obtain

$$R_{sr} X^s X^r g_{ij} \\ = \left\{R_{is} X^s - \frac{R}{2(n-1)} X_i\right\} X_j + \left\{R_{js} X^s - \frac{R}{2(n-1)} X_j\right\} X_i - 2(n-2) b_{is} X^s c_{jr} X^r,$$

a contradiction. Thus a and b are linearly dependent. Suppose that b is non-zero and $a_{ij} = \lambda b_{ij}$. From (17) it follows that

$$\{\lambda R_{hijk} + (1-\lambda) C_{hijk}\} b_{lm} + \{\lambda R_{jklm} + (1-\lambda) C_{jklm}\} b_{hi} + \{\lambda R_{lmhi} + (1-\lambda) C_{lmhi}\} b_{jk} = 0.$$

By the Corollary, the last relation leads to $\lambda R_{hijk} + (1-\lambda) C_{hijk} = 0$, which simply yields a contradiction. Similarly, the tensor a cannot be non-zero, which completes the proof of our proposition.

As a consequence of Proposition 2 we obtain the following

THEOREM 2. *In a conformally recurrent and Ricci-recurent manifold of dimension greater than or equal to* 4 *both the recurrence vectors are locally gradients.*

In [5] Roter has found, in points of a general position, the local structure of a Ricci-recurrent and conformally recurrent manifold with the recurrence vectors being locally gradients. Our Theorem 2 enables us to omit the local gradient assumptions in his main theorem.

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