On the existence and uniqueness of convex solutions of a functional equation in the indeterminate case

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Abstract. Under assumptions (i)-(iv) it is proved, in Theorem 1, by Helly's theorem, that equation (1) has exactly one convex and continuous solution fulfilling condition (4).

In Theorems 2, 3, 4 it is proved that (under different assumptions) equation (1) possesses at least one convex solution.

1. In the present paper we are concerned with the functional equation

\[ \varphi(x) = h(x, \varphi(f(x))), \]

where \( f(x) \) and \( h(x, y) \) are known real-valued functions of real variables and \( \varphi(x) \) is unknown.

We shall seek convex solution of equation (1) in the indeterminate case:

\[ p = |f'(\xi)h_\xi(\xi, \eta)| = 1, \]

where \( \xi \) is the fixed point of the function \( f(x) \) in the interval considered and \( (\xi, \eta) \) fulfills the condition \( h(\xi, \eta) = \eta \). A theory of the existence and uniqueness of convex solution of equation (1) has been developed in [2], [4], [5], [6] essentially under the \( p < 1 \).

2. We assume the following hypotheses:

(i) \( f(x) \) is defined and continuous in an interval \( I, \mathcal{E} f(\xi) = \xi \)

\[ 0 < \frac{f(x) - \xi}{x - \xi} < 1 \quad \text{for} \quad x \in I, x \neq \xi. \]

(ii) \( h(x, y) \) is defined and continuous in a domain \( \Omega, \mathcal{I}(\xi, \eta) \in \Omega \)

\[ h(\xi, \eta) = \eta; \quad \text{moreover, for every} \quad x \in I, \quad \text{the set} \quad \Omega_x \mathcal{A} \{y : (x, y) \in \Omega\} \quad \text{is a non-empty open interval and} \quad h(f(x), \Omega(x)) \subset \Omega_x. \]

(iii) There exist positive constants \( \alpha \) and \( \beta \) such that

\[ |h(x, y_1) - h(x, y_2)| \leq |y_1 - y_2| \]

for \( (x, y_1), (x, y_2) \in \Omega \cap \langle \xi - \alpha, \xi + \alpha \rangle \times \langle \eta - \beta, \eta + \beta \rangle \).
(iv) $f(x)$ is strictly increasing and convex (1) in $I$, $h(x, y)$ is increasing with respect to either variable and convex in two variables in $\Omega$, $\Omega$ is a convex domain.

At first we shall prove the following

**Theorem 1.** Let hypotheses (i)-(iv) be fulfilled. If, moreover, for every $x \in I$, $\Omega_x$ is bounded, then equation (1) has exactly one convex and continuous solution $\varphi$ in $I$ fulfilling the condition

$$\varphi(\xi) = \eta. \tag{4}$$

**Proof.** We may assume that $\xi$ is the left end point of $I$, so that $I = (\xi, b)$ (if $\xi$ is the right end point or inside point of $I$, the proof is analogous). Together with (1) we shall consider the sequence of equations

$$\varphi_n(x) = t_n h(x, \varphi_n[f(x)]), \quad t_n < 1, \quad t_n \to 1 \text{ as } n \to \infty, \quad n = 1, 2, \ldots \tag{5}$$

From Theorem 3.4 in [3] for every $n$ there exists exactly one function $\varphi_n(x)$, continuous in $I$, satisfying equation (5) in $I$ and fulfilling the condition

$$\varphi_n(\xi) = \eta. \tag{6}$$

It is given by the formula

$$\varphi_n(x) = \lim_{v \to \infty} \varphi_{n,v}(x), \tag{7}$$

where

$$\varphi_{n,v}(x) = \eta, \quad \varphi_{n,v+1}(x) = t_n h(x, \varphi_{n,v}[f(x)]), \quad v = 0, 1, 2, \ldots$$

In view of (iv), we easily verify that every function $\varphi_{n,v}(x)$ is convex and increasing in $I$ and consequently the function (7) is also convex and increasing in $I$.

We have also (6) for $n = 1, 2, \ldots$ and, for every $n$ and every $x \in I$, $\varphi_n(x) \in \Omega_{I - \xi}$. Hence, in view of the boundedness of $\Omega_x$, for every $x \in I$ there exists a number $M_x$ such that $|\varphi_n(x)| \leq M_x$. It follows from Helly's theorem (see [7], p. 372) that we can choose from $\{\varphi_n\}$ a subsequence $\{\varphi_{n_k}\}$ convergent in $I$ to a function $\varphi_0$. Evidently, $\varphi_0$ is also convex and increasing in $I$. Moreover, we have

$$\varphi_{n_k}(x) = t_{n_k} h(x, \varphi_{n_k}[f(x)]), \quad k = 1, 2, \ldots \tag{8}$$

From the convexity of $\varphi_0$ its continuity in $(\xi, b)$ results. From the convexity of $\varphi_0$ in $(\xi, b)$ in view of the conditions $\varphi_0(\xi) = \eta$, $\varphi_0(x) \geq \eta$ in $(\xi, b)$, we infer that $\varphi_0$ is continuous in $(\xi, b)$.

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(1) $g$ is convex in $\Omega \subset \mathbb{R}^n$, if for every $\hat{x}, \hat{y} \in \Omega$ and $\lambda \in (0, 1)$

$$g(\lambda \hat{x} + (1 - \lambda) \hat{y}) \leq \lambda g(\hat{x}) + (1 - \lambda) g(\hat{y}).$$
Since $\Omega$ is open, there exists an interval $I_0 = \langle \frac{\xi}{\eta}, a \rangle \subset I$ such that $I_0 \times \langle \eta, \varphi_0(a) \rangle \subset \Omega$. From (i) it follows that $f(x) < x$ and because $\varphi_0$ is increasing in $I$, for every $x \in I_0$ we have $\varphi_0[f(x)] \in \Omega$.

Passing the limit as $n \to \infty$, according to (8), we obtain

$$\varphi_0(x) = h(x, \varphi_0[f(x)]), \quad x \in I_0,$$

i.e. $\varphi_0(x)$ is a convex solution of equation (1) in $I_0$. The function $\varphi_0(x)$, $x \in I_0$ has a unique extension $\varphi$ onto the whole interval $I$ and the solution $\varphi$ is continuous in $I$ (cf. [3], p. 70, Theorem 3.2). It is easily verified (as in [5]) that is also convex in $I$.

The uniqueness of such solutions follows from Theorem 1 in [1]. This completes the proof.

3. We shall prove the following

**Theorem 2.** Let hypotheses (i), (ii), (iv) be fulfilled. If, moreover, there exists an $h_\eta(\xi, \eta)$ and

$$|f'(\xi_1) - h_{\eta}(\xi, \eta)| = 1,$$

where $\xi$ is the left end point of $I$ and $\Omega$ is bounded, then equation (1) possesses at least one convex solution in $I$ fulfilling condition (4).

**Proof.** It follows from [2] that for every $n$ there exists a function $\varphi_n(x)$, convex, continuous and increasing in $I$, satisfying equation (5) in $I$ and fulfilling condition (6).

Similarly as in Theorem 1, making use of Helly's theorem, we can prove that there exists a convex solution of equation (1) fulfilling condition (4), which was to be proved.

Suppose that:

(v) There exist positive constants $a, \beta, s, k, l$ such that:

$$|f(x) - f(x_1)| \leq s|x - x_1| \quad \text{for } x, x_1 \in \langle \xi - a, \xi + a \rangle \cap I,$$

$$|h(x, y) - h(x_1, y_1)| \leq k|x - x_1| + l|y - y_1|$$

for $(x, x_1) \in \langle \xi - a, \xi + a \rangle$, $(y, y_1) \in \langle \eta - \beta, \eta + \beta \rangle$,

$$(x, y), (x_1, y_1) \in \Omega, s \cdot l = 1.$$

(vi) Let $\xi = \eta = 0$ and $h(x, y) = Ax + By + o(|x| + |y|), (x, y) \to (0, 0)$ and $|h(x, y_1) - h(x, y_2)| \leq L|y_1 - y_2|$ in a neighbourhood of $(0, 0)$, $L \cdot f'(0) = 1$.

Then we have

**Theorem 3.** Let hypotheses (i), (ii), (iv), (v) be fulfilled. If, moreover, $\Omega$ is bounded, then equation (1) possesses at least one convex solution in $I$ fulfilling condition (4).

**Theorem 4.** If hypotheses (i), (ii), (iv), (vi) are fulfilled, then equation (1) possesses at least one convex solution in $I$ fulfilling condition $\varphi(0) = 0$. 
The proof of Theorems 3 and 4 in view of [6] and [4] respectively does not differ from that given in Theorems 1 and 2, and is therefore omitted.

Remark. In [4] it is proved that \( \varphi_n \) is continuous and convex in \( I \) but it is easily seen that \( \varphi_n \) is also increasing in \( I \).

References

[2] Z. Kominek and J. Matkowski, On the existence of a convex solution of the functional equation, \( \varphi(x) = h(x, \varphi(f(x))) \), this volume, p. 1–4.

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