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BANACH SPACES ADMITTING ESSENTIALLY INFINITE-DIMENSIONAL REPRESENTATION OF A COMPACT GROUP

BY

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0. Let X be a complex Banach space, and let G be a compact group. Representation $\varrho\colon g \to A_g$ of G in X is a homomorphism of G into $GL(X)$, the group of automorphisms of X. Representation ϱ is continuous if for each $x \in X$ the function $g \to A_g(x)$ is continuous.

Representation ϱ is called *cyclic* if there exists $x \in X$ such that finite linear combinations of vectors $A_g(x)$ with $g \in G$ form a dense subspace of X.

A representation ϱ is called *essentially infinite-dimensional* if its restriction to some infinite-dimensional ϱ-invariant subspace of X is cyclic.

Let $\boldsymbol{B}$ be a family of all separable Banach spaces (we identify isomorphic spaces).

In the present paper we study properties of following subfamilies of $\boldsymbol{B}$: $\boldsymbol{B}_\infty$ — the subfamily of $\boldsymbol{B}$ consisting of all spaces admitting essentially infinite-dimensional continuous representation of some compact group G; $\boldsymbol{B}_{\infty a}$ — the family of all spaces admitting essentially infinite-dimensional representation of some compact abelian group G; $\boldsymbol{B}_c$ — all spaces admitting cyclic continuous representation of some compact group G. Obviously, $\boldsymbol{B}_{\infty a} \subset \boldsymbol{B}_\infty$ and $\boldsymbol{B}_c \subset \boldsymbol{B}$.

It is natural to ask whether these inclusions are proper. We shall prove (Theorem 2) that if $X \in \boldsymbol{B}_c$ and X^{**} is separable, then X is reflexive. This allows examples of spaces which belong to $\boldsymbol{B}_{\infty a} \setminus \boldsymbol{B}_c$ (e.g. the space of James).

We do not know the solutions of the following problems.

PROBLEM 1. Does $\boldsymbol{B}_{\infty a} = \boldsymbol{B}_\infty$? (**P 867**)

This problem is connected with the question whether each infinite compact group has an infinite abelian subgroup.

PROBLEM 2. Does $\boldsymbol{B} = \boldsymbol{B}_\infty$? (**P 868**)

Positive answers to these questions would have strong consequences, since (Theorem 3) if $X \in \boldsymbol{B}_{\infty a}$, then X has an unconditional basic sequence.

We recall that (e_n) is called an *unconditional basic sequence* in X if each x in the closed linear span of the set $\{e_n\}_{n=1}^{\infty}$ can be represented in a unique way in the form $x = \sum_n c_n e_n$ and for each bounded sequence (η_n) the series $\sum_n c_n \eta_n e_n$ converges.

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1. In the following letters X, Y will be used for Banach spaces' and letters G, H for compact groups. The short form "let $\varrho: G \ni g \to A_g \in GL(X)$" will mean "let $\varrho: g \to A_g$ be a continuous representation of a compact group G in a Banach space X". We will denote the value of a linear functional y^* at a point x by $\langle x, y^* \rangle$.

Let $M(G)$ denote the convolution algebra of all finite, complex valued Borel measures on G, and let $L^1(G)$ be its ideal of all measures absolutely continuous with respect to the Haar measure on G. Let $\varrho: G \ni g \to A \in GL(X)$. To each $\mu \in M(G)$ we assign a linear operator $A_\mu^\varrho: X \to X$ by the formula

$$A_\mu^\varrho(x) = \int_G A_g(x)\mu(dg). \tag{1.1}$$

The mapping $\mu \to A_\mu^\varrho$ is a continuous homomorphism of $M(G)$ into $L(X)$, the algebra of all bounded operators on X (cf. [1], p. 335). With no loss of generality we may assume that $\|A_g\| = 1$ for $g \in G$ and, therefore, that the homomorphism $\mu \to A_\mu^\varrho$ has the norm 1.

For a compact group G let $\hat{G}$ denote the set of all normalized continuous characters of G. Let $\chi_1, \chi_2 \in \hat{G}$. Since

$$\chi_1 * \chi_2 = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ \chi_1 & \text{if } \chi_1 = \chi_2, \end{cases}$$

the operators A_χ^ϱ for $\chi \in \hat{G}$ form the family of projections such that $\|A_\chi^\varrho\| = 1$, and

$$A_{\chi_1}^\varrho \circ A_{\chi_2}^\varrho = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ A_{\chi_1}^\varrho & \text{if } \chi_1 = \chi_2. \end{cases}$$

2. Definition 2.1. Let $\varrho: G \ni g \to A_g \in LG(X)$. We define the *spectrum* S_ϱ of the representation ϱ as the subset of $\hat{G}$ containing all those χ for which $A_\chi^\varrho \neq 0$.

PROPOSITION 2.2. *For any representation ϱ the spectrum S_ϱ is not empty* (*we assume here that* $\dim X > 0$).

Proof. Let $x_0 \in X$, $x_0 \neq 0$. Define the mapping $Q\colon L^1(G) \to X$ by putting $Q(f) = A_f^\varrho(x_0)$. The range of Q is not $\{0\}$. By the Peter-Weyl theorem ([5], p. 74) we can find continuous function, a matrix coefficient $m_{i,j}^\tau$ of some irreducible representation τ of G, such that $Q(m_{i,j}^\tau) \neq 0$.

Let χ_τ be the character of τ. Since $\chi_\tau * m_{i,j}^\tau = \lambda m_{i,j}^\tau$, where $\lambda = \dim \tau$ is a positive integer, we have

$$A_{\chi_\tau}^\varrho \circ A_{m_{i,j}^\tau}^\varrho (x_0) = A_{\chi_\tau * m_{i,j}^\tau}^\varrho (x_0) = \lambda A_{m_{i,j}}(x_0) = \lambda Q(m_{i,j}^\tau) \neq 0.$$

Hence $A_{\chi_\tau}^\varrho \neq 0$ and therefore $\chi_\tau \in S_\varrho$, q.e.d.

Let $\varrho\colon G \ni g \to A_g \in GL(X)$. We define X_ϱ as the smallest closed linear subspace of X, containing all ranges of the operators A_χ for $\chi \in \hat{G}$, i.e.,

$$X_\varrho = \operatorname{span}\{A_\chi^\varrho(x) \text{ for } x \in X \text{ and } \chi \in \hat{G}\}.$$

PROPOSITION 2.3. $X_\varrho = X$.

Proof. Since X_ϱ is a ϱ-invariant closed subspace of X, the representation ϱ induces, by the formula $\tilde{A}_g([x]) = [A_g(x)]$, a continuous representation $\tilde{\varrho}$ in the quotient space X/X_ϱ ($[x]$ denotes the class of vector x in X/X_ϱ).

Suppose that $X \neq X_\varrho$. Then, by Proposition 2.2, $S_{\tilde{\varrho}}$ is not empty, and hence, for some $\chi_0 \in \hat{G}$,

$$\int_G \chi_0(g)\tilde{A}_g([x])\,dg = \Big[\int_G \chi_0(g)A_g(x)\,dg\Big] \neq 0.$$

But this would mean that $A_{\chi_0}^\varrho(x) \notin X_\varrho$, a contradiction.

Definition 2.4. Let $\varrho\colon G \ni g \to A_g \in GL(X)$. The *contragradient representation* ϱ^* induced by the representation ϱ is the mapping $G \ni g \to A_{g^{-1}}^* \in GL(X^*)$.

We shall need the following well-known (cf. [1], p. 335) equivalence:

PROPOSITION 2.5. *Let ϱ be a representation of a compact group G in a separable Banach space. The following two conditions are equivalent*:

(a) *ϱ is continuous*;

(b) *ϱ is weak measurable* (*i.e. for each $x \in X$ and $x^* \in X^*$ the function $g \to \langle A_g x, x^*\rangle$ is measurable*).

PROPOSITION 2.6. *Let $\varrho\colon G \in g \to A_g \in GL(X)$, and let X^* be separable. Then ϱ^* is a continuous representation.*

Proof. Let B, B^*, B^{**} be the unit balls in X, X^* and X^{**}, respectively. Let (x_n^*) be a dense countable subset of B^*. For each $x^{**} \in B^{**}$ and each positive integer n there exists $x_n \in B$ such that

$$|\langle x_k^*, x^{**}\rangle - \langle x_n, x_k^*\rangle| < \frac{1}{n} \quad \text{for } k = 1, 2, \ldots, n.$$

Since $\|x_n\| \leqslant 1$, we get

$$\langle x_n, x^*\rangle \xrightarrow[n\to\infty]{} \langle x^*, x^{**}\rangle \quad \text{for } x^* \in X^*.$$

Putting here $A^*_{g-1}(x^*)$ for x^* we get

$$\langle A^*_{g-1}(x^*), x^{**}\rangle = \lim_{n\to\infty}\langle A_{g-1}(x_n), x^*\rangle.$$

Thus the function $g \to \langle A^*_{g-1}(x^*), x^{**}\rangle$ is measurable, and by Proposition 2.5 the representation ϱ^* is continuous.

Remark. Let G be the circle group (the group of the reals mod 2π) acting via translations in the space $L^1(G)$. This is obviously a continuous representation. Operators of a contragradient representation are also translations in $L^\infty(G)$, the space of all measurable essentially bounded functions. The contragradient representation is not continuous nor even weakly measurable. This points out that the condition of separability of X^* in Proposition 2.6 is necessary.

Let ϱ be a continuous representation of G in X and ϱ^* be the continuous conjugate representation in X^*.

Let A^ϱ_μ and $A^{\varrho^*}_\mu$ be operators assigned to a measure μ by formula (1.1) and corresponding to representations ϱ and ϱ^* in X and X^*, respectively.

For $\mu \in M(G)$ define $\mu^* \in M(G)$ by $\mu^*(A) = \overline{\mu(A^{-1})}$ for any Borel subset A of G. The $*$-operation is isometric involution of $M(G)$.

PROPOSITION 2.7. $A^{\varrho^*}_{\mu^*} = (A^\varrho_\mu)^*$.

Proof is standard.

COROLLARY 2.8. *If ϱ is a continuous representation with the continuous representations ϱ^* and ϱ^{**}, then $S_\varrho = S_{\varrho^{**}}$. Moreover,*

$$A^{\varrho^{**}}_\chi = (A^\varrho_\chi)^{**} \quad \textit{for } \chi \in \hat{G}.$$

Proof. Let $\chi \in \hat{G}$. Since $A^{\varrho^{**}}_\chi = (A^{\varrho^*}_{\chi^*})^* = (A^\varrho_\chi)^{**}$, we get $A^{\varrho^{**}}_\chi = 0$ iff $A^\varrho_\chi = 0$.

THEOREM 1. *Let $\varrho: G \ni g \to A_g \in GL(X)$ and let X^{**} be separable. If for each $\chi \in \hat{G}$ the space $A^\varrho_\chi(X)$ is reflexive, then X is reflexive.*

The proof is based on the following

LEMMA 2.9. *Let Y be a reflexive subspace of a B-space X. Let π: $X \to X$ be a bounded projection from X onto Y. Let π^{**}: $X^{**} \to X^{**}$ be the projection, second conjugate to π, and let i: $X \to X^{**}$ be canonical embedding. Then $iY = \pi^{**}(X^{**})$.*

Proof. Let π^*: $X^* \to X^*$ be the operator conjugate to π. Let $x^* \in X^*$ and $y \in Y$. Since

$$\langle x^*, iy\rangle = \langle y, x^*\rangle = \langle \pi y, x^*\rangle = \langle y, \pi^* x^*\rangle = \langle y, Rx^*\rangle,$$

where $R: X^* \to Y^*$ denotes the restriction of functionals from X^* to the space Y, the weak * topology of $iY \cap B^{**}$ is the same as the weak topology of $Y \cap B$ transported to $iY \cap B^{**}$ via embedding i. Therefore, since Y is reflexive, $iY \cap B^{**}$ is weak * compact.

Hence to complete the proof it is sufficient to show that for each $x^* \in X^*$, $x^{**} \in B^{**} \cap \pi^{**}(X^{**})$ and $\varepsilon > 0$ there is $x \in B \cap Y$ such that

$$|\langle x^*, x^{**}\rangle - \langle x^*, ix\rangle| < \varepsilon.$$

Since iB is weak * dense in B^{**}, there is $x_1 \in B$ such that

$$|\langle \pi^*(x^*), x^{**}\rangle - \langle x_1, \pi^*(x^*)\rangle| < \varepsilon.$$

Therefore, since $x^{**} = \pi^{**}x^{**}$, we get for $x = \pi x_1$

$$\begin{aligned}|\langle x^*, x^{**}\rangle - \langle x^*, ix\rangle| &= |\langle x^*, \pi^{**}x^{**}\rangle - \langle \pi x_1, x^*\rangle| \\ &= |\langle \pi^* x^*, x^{**}\rangle - \langle x_1, \pi^*(x^*)\rangle| < \varepsilon.\end{aligned}$$

Proof of Theorem 1. By Proposition 2.7, the representation ϱ is continuous. Clearly, iX is ϱ^{**}-invariant. By Corollary 2.8 and Lemma 2.9, $X^{**}_{\varrho^{**}} \subset iX$. Hence, by Proposition 2.3, $iX = X^{**}$.

Let $\varrho: G \ni g \to A_g \in GL(X)$. It is known that if a subspace Y of X is minimal ϱ-invariant (i.e., if Y is ϱ-invariant and has no proper ϱ-invariant subspaces), then $\dim Y < \infty$. (The proof of this fact may be reduced to the Hilbert space case by defining a ϱ-invariant continuous Hilbert norm on X.)

LEMMA 2.10. *Let $\varrho: G \ni g \to A_g \in GL(X)$ be a cyclic representation. Then $\dim A^{\varrho}_{\chi}(X) < \infty$ for each $\chi \in \hat{G}$.*

Proof. Let x_0 be a cyclic vector for ϱ. Since

$$A^{\varrho}_{\chi}(X) = A^{\varrho}_{\chi}\big(\operatorname{span}(A_g(x_0))_{g\in G}\big) = \operatorname{span}\big(A^{\varrho}_{\chi} \circ A_g(x_0)\big)_{g\in G} = \operatorname{span}\big(A^{\varrho}_{\chi_g}(x_0)\big)_{g\in G},$$

where $\chi_g(h) = \chi(hg^{-1})$, and since characters are finite-dimensional (i.e, $\dim\operatorname{span}(\chi_g)_{g\in G} < \infty$), we get $\dim\operatorname{span}\big(A^{\varrho}_{\chi_g}(x_0)\big)_{g\in G} < \infty$.

THEOREM 2. *Let $\varrho: G \ni g \to A_g \in L(X)$ be a cyclic representation and let X be separable. Then X is reflexive.*

Proof follows by Theorem 1 and Lemma 2.10.

COROLLARY 2.11. *Let i: $X \to X^{**}$ be the canonical embedding. If dimension of the quotient space X^{**}/iX is finite and X is separable, then X does not admit any cyclic representation of a compact group.*

Remark. Let Y be a subspace of a B-space X, let Y have an unconditional basis (e_n) with coordinate functionals (f_n), and let π be a bounded projection from X onto Y. Then X admits a non-trivial representation of any compact abelian group G, which is cyclic when restricted to Y. In fact, let $(\chi_n)_{n=1}^{\infty}$ be any countable subset of $\hat{G}$ (in the case of G abelian continuous characters of irreducible representations are multiplicative, i.e., they are homomorphisms of G into T, the multiplicative group of complex numbers with module 1). The formula

$$A_g(x) = \sum_n \chi_n(g) f_n(x) e_n + x - \pi(x)$$

defines a continuous representation $g \to A_g$ of G in X having required properties.

Let J be the space of James (cf. [2]), i.e., the space of all complex sequences $\xi = (\xi_n)$ such that $\lim_n \xi_n = 0$ and

$$\|\xi\| = \sup_{\{n_1, \ldots, n_k\}} \left\{ |\xi_{n_k} - \xi_{n_1}|^2 + \sum_{i=1}^{k-1} |\xi_{n_{i+1}} - \xi_{n_i}|^2 \right\}^{1/2},$$

with the supremum taken over all finite sets $\{n_1, \ldots, n_k\}$ of positive integers.

Since J is separable and $\dim J^{**}/iJ = 1$, we have, by Corollary 1.14, $J \notin B_c$. It is known that J contains a complemented subspace with an unconditional basis; hence J admits a non-trivial representation of any infinite compact abelian group.

3. Let X be a Banach space, let $x_m \in X, f_m \in X^*$, $m = 1, 2, \ldots$, be a biorthogonal sequence, i.e. $f_m(x_n) = \delta_n^m$.

Let X_0 be the subspace (not closed) spanned by $(x_m)_{m=1}^{\infty}$. For a given sequence (η_m) of complex numbers let A_η be the linear (in general, unbounded) operator on X_0 defined as

$$A_\eta(x) = \sum_m \eta_m f_m(x) x_m.$$

If for each bounded sequence $\eta = (\eta_m)$ the operator A_η is bounded, then (x_m) is an unconditional basic sequence in X (cf. [3]).

Let G be a compact abelian group. A subset S of the dual group $\hat{G}$ is called a *Sidon set* if for each bounded function f on G there is a measure $\mu \in M(G)$ such that $f(\chi) = \hat{\mu}(\chi)$ for $\chi \in S$ ($\hat{\mu}$, as usually, denotes the Fourier transform of μ).

We recall the following theorem ([4], p. 126):

If G is a compact abelian group, then each infinite subset of $\hat{G}$ contains an infinite Sidon set.

PROPOSITION 3.1. *Let G be a compact abelian group and let $\varrho: G \ni g \to A_g \in GL(X)$. Then, for each $\mu \in M(G)$ and $\chi \in \hat{G}$,*

$$A_\mu^\varrho \circ A_\chi^\varrho = \hat{\mu}(\chi) A_\chi^\varrho.$$

Proof. This is a consequence of the formula $\chi(gh) = \chi(g)\chi(h)$, valid for characters of abelian group, and the fact that the mapping $\mu \to A_\mu^\varrho$ is a homomorphism.

PROPOSITION 3.2. *Let $\varrho\colon G\ni g\to A_g\in GL(X)$. The following two conditions are equivalent*:

(i) *the representation ϱ is essentially infinite-dimensional*;

(ii) *S_ϱ is infinite.*

Proof. The proof is similar to the case of the Hilbert space.

THEOREM 3. *Let X be a Banach space, G a compact abelian group, and let $\varrho\colon G\ni g\to A_g\in GL(X)$ be essentially infinite-dimensional. Then X has an unconditional basic sequence.*

Proof. By Proposition 3.2, S_ϱ is an infinite subset of G, and hence it contains an infinite Sidon set S.

Let $x_m\in A^\varrho_{\chi_m}(X)$ for some sequence χ_m of characters from S. By Proposition 3.1, for each $\mu\in M(G)$ we have

$$A^\varrho_\mu(x_m) = \hat{\mu}(\chi_m)\cdot x_m.$$

Since S is a Sidon set, for each bounded sequence (η_m) of complex numbers there is $\mu\in M(G)$ such that $\eta_m = \mu(\chi_m)$ for all m, and hence the operator A_η, assigned to the sequence (η_m), is the restriction of A^ϱ_μ to the space X_0. Thus A_η is bounded and therefore (x_m) is an unconditional basic sequence.

Remark. Theorem 3 can be extended to other classes of compact groups, e.g., to the class of compact Lie groups. (In this case for a given essentially infinite-dimensional representation there is a compact abelian subgroup such that restriction of A to this subgroup remains essentially infinite-dimensional.)

We do not know whether the same is true for an arbitrary compact infinite group. (**P 869**)

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